# Convergence of Dirichlet Forms and monotonicity on non-smooth Spaces: the Riemann-Hilbert problem on extension domains


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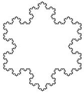

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# Riemann–Hilbert method: classical theory

- **Cauchy, 1820s–1830s:** Cauchy integral formula and Cauchy-type integrals become the basic tools for recovering holomorphic functions from boundary data. **Riemann, 1850s:** analytic functions are studied through boundary values, analytic continuation, monodromy, and Riemann surfaces. **Sokhotski, 1868:** boundary limits of Cauchy-type integrals are described, leading to the Sokhotski–Plemelj jump formulas. **Hilbert, 1900:** Hilbert's 21st problem asks for linear differential equations with prescribed singularities and monodromy. **Plemelj, 1908:** the scalar Riemann–Hilbert boundary value problem is solved using Cauchy integrals:

$$Cf(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta, \qquad C_+ f - C_- f = f.$$

- **M. Riesz, 1920s:** the Hilbert transform is developed as a central singular integral operator, with $L^p$ boundedness for $1 < p < \infty$.
- **1940s–1960s:** the Riemann–Hilbert problem leads to the theory of singular integral equations and boundary value problems.

# Riemann–Hilbert method: modern analysis

**1970s–1980s:** Calderon's program connects Cauchy integrals, commutators, and singular integral operators on Lipschitz curves and surfaces. **1981, Peter Jones:** uniform domains are linked to Sobolev extension, giving a geometric language for rough domains. **1982, Coifman–McIntosh–Meyer:** the Cauchy integral is proved to be bounded on $L^2$ for Lipschitz curves. **1984, Verchota:** layer potentials are used to solve the Dirichlet and regularity problems for Laplace's equation in Lipschitz domains. **1980s–1990s:** Dahlberg, Jerison, Kenig, Fabes, David, Semmes and others develop harmonic measure, boundary value problems, and uniformly rectifiable geometry. **1990s–2020s:** layer potentials and singular integral methods are extended to more general elliptic systems and rough geometric settings. **Mitrea school:** Dorina, Irina, and Marius Mitrea, often with Taylor, Hofmann, Martell, Morris and others, develop layer potentials, traces, function spaces, and boundary problems on nonsmooth, uniformly rectifiable, and geometric domains. **Deift–Zhou:** nonlinear steepest descent connects Riemann–Hilbert problems to inverse scattering, orthogonal polynomials, random matrices.

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# Framework: two-sided extension domains [always assumed]

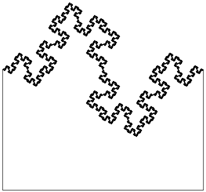
## Definition

$\Omega$ is a **two-sided extension domain** if:

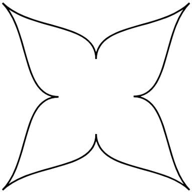
(i) $\Omega$ is an $H^1$-Sobolev extension domain[1];

(ii) $\overline{\Omega}^c := \mathbb{R}^n \backslash \overline{\Omega}$ is an extension domain;

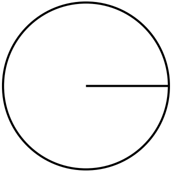
(iii) $\partial\Omega = \partial(\overline{\Omega}^c)$ et $\lambda^{(n)}(\partial\Omega) = 0$.



Two-sided extension domain



(i) and (ii) fail



(iii) fails

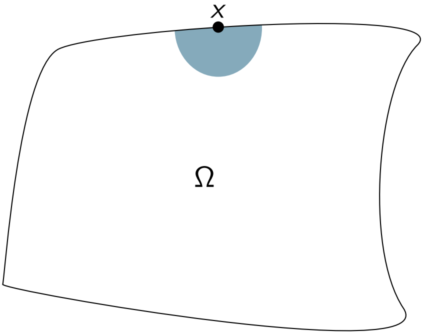
**No specified boundary measure.**

[1] [Hajłasz, Koskela et Tuominen, 2008]

# Boundary conditions on extension domains

- *Dirichlet BC:* **trace operator** [Biegert, 2009]

$$\operatorname{Tr}_i u(x) = \lim_{r \to 0^+} \frac{1}{\lambda^{(n)}(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u \, \mathrm{d}x, \quad u \in H^1(\Omega), \; x \in \partial\Omega \text{ q.e.}$$



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**Theorem (Trace theorem)**

- $\mathcal{B}(\partial\Omega) := \operatorname{Tr}_i(H^1(\Omega))$ *is a Hilbert space for*

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid \operatorname{Tr}_i v = f\} = \|\tilde{v}\|_{H^1(\Omega)},$$

*where* $(-\Delta + 1)\tilde{v} = 0$ *weakly on* $\Omega$ *(i.e.* $\tilde{v} \in V_1(\Omega)$*) and* $\operatorname{Tr}_i \tilde{v} = f$*.*

- $\operatorname{Ker} \operatorname{Tr}_i = H^1_0(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$.
- $V_1(\Omega) = (H^1_0(\Omega))^\perp$.

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We consider $\mathrm{Tr}_i : H^1(\Omega)\to\mathcal{B}(\partial\Omega)$.

- *Neumann BC:* **weak normal derivative** e.g. Doob 1962, Constantinescu-Cornea 1963, Maeda 1964, 1980

$$\forall v\in H^1(\Omega),\quad \left\langle \frac{\partial_i u}{\partial\nu}, \mathrm{Tr}_i\, v\right\rangle_{\mathcal{B}',\mathcal{B}} = \int_\Omega (\Delta u) v\,\mathrm{d}x + \int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x,$$

where $u\in H^1(\Omega)$ with $\Delta u\in L^2(\Omega)$.

We consider $\frac{\partial_i}{\partial\nu} : H^1(\Omega)\cap\{\Delta\cdot\in L^2(\Omega)\}\to\mathcal{B}'(\partial\Omega)$.

- *Dirichlet BC:* **trace operator** [Biegert, 2009]

$$\mathrm{Tr}_e\, u(x) = \lim_{r\to 0^+} \frac{1}{\lambda^{(n)}(\overline{\Omega}^c \cap B_r(x))} \int_{\overline{\Omega}^c \cap B_r(x)} u \,\mathrm{d}x, \ u \in H^1(\overline{\Omega}^c), \ x \in \partial\Omega \text{ q.p.}$$

We consider $\mathrm{Tr}_e : H^1(\overline{\Omega}^c) \to \mathcal{B}(\partial\Omega)$.

- *Neumann BC:* **weak normal derivative**

$$\forall v \in H^1(\overline{\Omega}^c), \quad \left\langle \frac{\partial_e u}{\partial \nu}, \mathrm{Tr}_e\, v \right\rangle_{\mathcal{B}', \mathcal{B}} = -\int_{\overline{\Omega}^c} (\Delta u) v \,\mathrm{d}x - \int_{\overline{\Omega}^c} \nabla u \cdot \nabla v \,\mathrm{d}x,$$

where $u \in H^1(\overline{\Omega}^c)$ with $\Delta u \in L^2(\overline{\Omega}^c)$.

We consider $\frac{\partial_e}{\partial \nu} : H^1(\overline{\Omega}^c) \cap \{\Delta \cdot \in L^2(\overline{\Omega}^c)\} \to \mathcal{B}'(\partial\Omega)$.

# Some important isometries and estimates

$$\begin{tikzcd}
V_1(\Omega) = H_0^1(\Omega)^{\perp} \arrow[r, "\mathrm{Tr}_i"] \arrow[d, bend right, "\iota"'] \arrow[dr, "\frac{\partial_i}{\partial \nu}"] & \mathcal{B}(\partial\Omega) \arrow[d, bend left, "\Lambda"] \\
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V_1'(\Omega) & & \mathcal{B}'(\partial\Omega) \arrow[ll, "\mathrm{Tr}_i^*"]
\end{tikzcd}$$

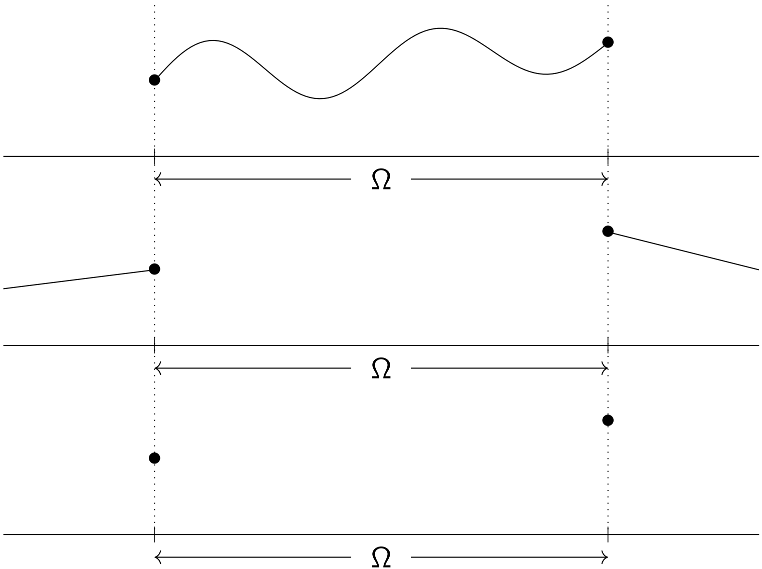
## Proposition

*Let $\Omega$ be a two-sided extension domain. Then, for $v \in V_1(\overline{\Omega}^c)$, it holds*

$$(\|\mathrm{E}_{\overline{\Omega}^c}\|^2 - 1)^{-\frac{1}{2}} \|\mathrm{Tr}_e\, v\|_{\mathcal{B}(\partial\Omega)} \leq \|v\|_{H^1(\overline{\Omega}^c)} \leq (\|\mathrm{E}_\Omega\|^2 - 1)^{\frac{1}{2}} \|\mathrm{Tr}_e\, v\|_{\mathcal{B}(\partial\Omega)},$$

*where the constants are optimal.*

# Specific extensions



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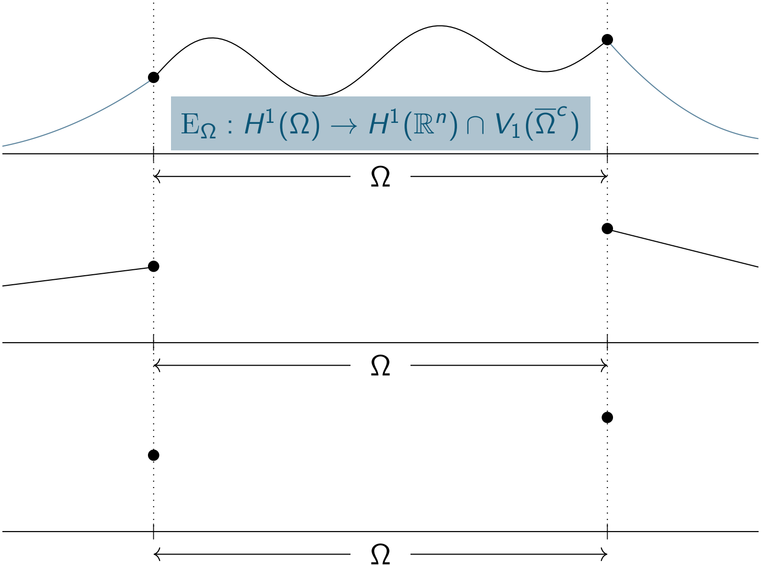

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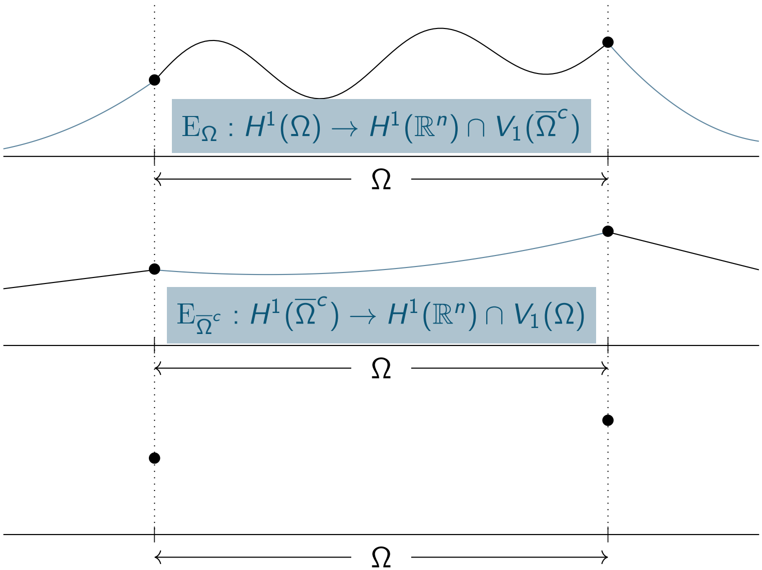

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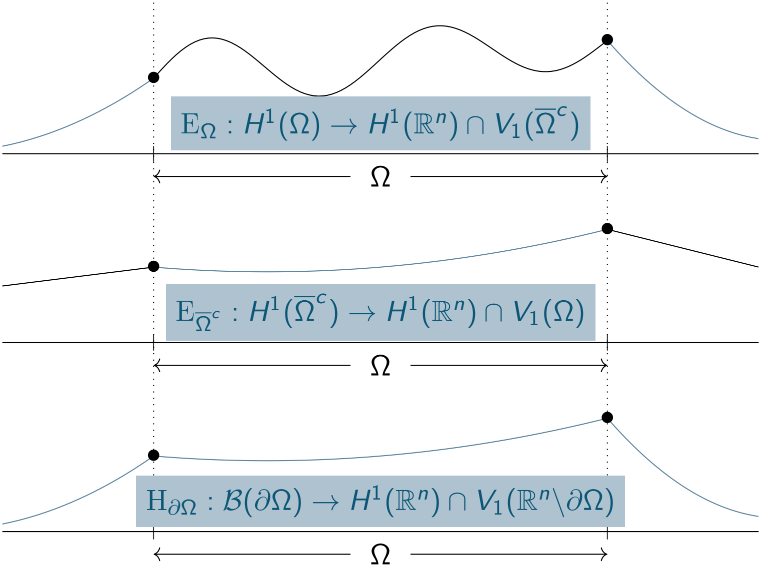

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**Riemann-Hilbert problem:**

$$\begin{cases} u \text{ is holomorphic on } \mathbb{C}\backslash\partial\Omega, \\ \mathrm{Tr}_i^{\partial\Omega} u - \mathrm{Tr}_e^{\partial\Omega} u = f, \\ u(z) \to 0 \text{ as } |z| \to +\infty. \end{cases}$$

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[2] [Muskhekishvili, 1977]

[3] [Chapman and Vanden-Broeck, 2006]

# The Riemann-Hilbert problem on Lipschitz domains

**Riemann-Hilbert problem:**

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If $\Omega$ is Lipschitz, then $u = \Phi_{\partial\Omega} f$ where

$$\Phi_{\partial\Omega} f(z) = \frac{1}{2\mathrm{i}\pi} \int_{\partial\Omega} \frac{f(y)}{y-z}\, \lambda(\mathrm{d}y), \quad z \in \mathbb{C},$$

is the Cauchy integral[2].

Used in signal processing and for the computation of gravitational waves[3].

[2] [Muskhekishvili, 1977]

[3] [Chapman and Vanden-Broeck, 2006]

**Transmission problem:**

$$\begin{cases} -\Delta u = 0 \quad \text{on } \mathbb{R}^n \backslash \partial\Omega, \\ \mathrm{Tr}_i^{\partial\Omega} u - \mathrm{Tr}_e^{\partial\Omega} u = f \in \mathcal{B}(\partial\Omega), \\ \dfrac{\partial_i u}{\partial \nu} - \dfrac{\partial_e u}{\partial \nu} = g \in \mathcal{B}'(\partial\Omega). \end{cases}$$

[4] Classical case: [Verchota, 1984]
Extension domains: [C., Hinz, Rozanova-Pierrat and Teplyaev, 2024]

# Connection to the harmonic transmission problem

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By the superposition principle,

$$u = u_{\mathcal{S}} - u_{\mathcal{D}} \qquad \text{where} \quad \begin{cases} \mathrm{Tr}_i^{\partial\Omega} u_{\mathcal{S}} - \mathrm{Tr}_e^{\partial\Omega} u_{\mathcal{S}} = 0, \\ \dfrac{\partial_i u_{\mathcal{D}}}{\partial \nu} - \dfrac{\partial_e u_{\mathcal{D}}}{\partial \nu} = 0. \end{cases}$$

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We introduce the single and double layer potential operators[4]

$$\mathcal{S}_{\partial\Omega} : g \in \mathcal{B}'(\partial\Omega) \longmapsto u_{\mathcal{S}} \in V_1(\mathbb{R}^n \backslash \partial\Omega),$$
$$\mathcal{D}_{\partial\Omega} : f \in \mathcal{B}(\partial\Omega) \longmapsto u_{\mathcal{D}} \in V_1(\mathbb{R}^n \backslash \partial\Omega).$$

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# Outline of the method

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- For Lipschitz boundaries, $\Phi_{\partial\Omega} f$ solves a transmission problem:

$$\Phi_{\partial\Omega} f = \mathcal{S}_{\partial\Omega} g_f - \mathcal{D}_{\partial\Omega} f.$$

- $\mathcal{S}_{\partial\Omega}$ and $\mathcal{D}_{\partial\Omega}$ are well-defined for two-sided extension domains.

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**Approximate $\Omega$ with smoother $(\Omega_k)_{k\in\mathbb{N}}$**, to prove that the solutions for $\partial\Omega_k$, $\Phi_{\partial\Omega_k} f_k$, **will converge** to a solution for $\partial\Omega$, $\Phi_{\partial\Omega} f$.

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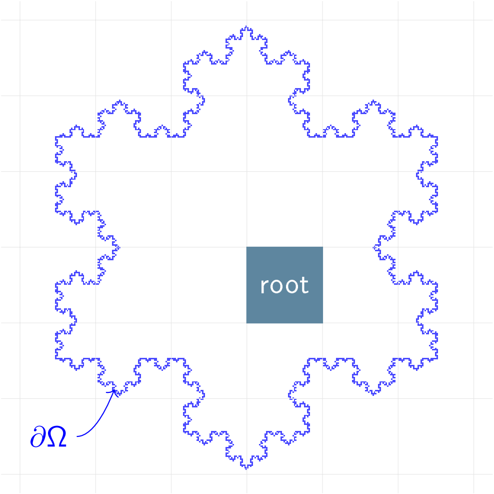
**How to approximate $\Omega$?**
**What does *converge* mean for functions defined on different spaces?**

# Dyadic approximation



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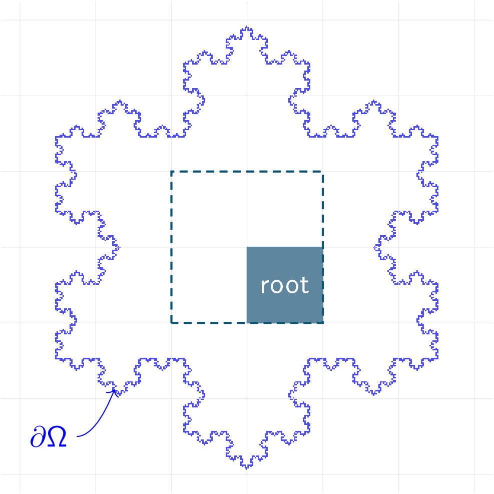

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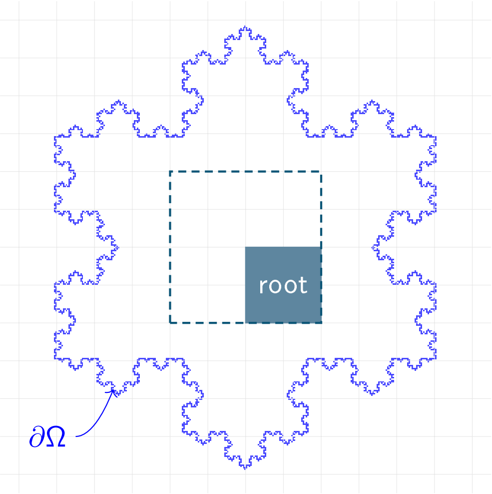

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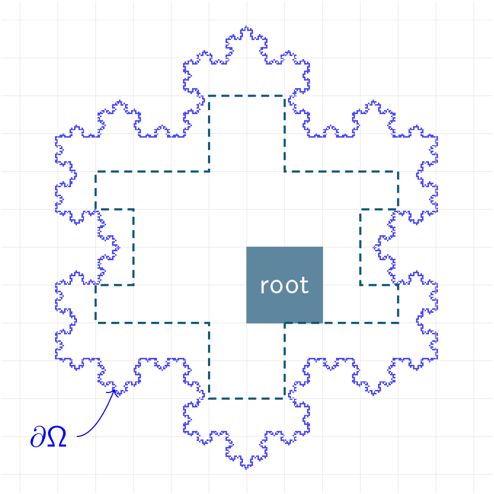

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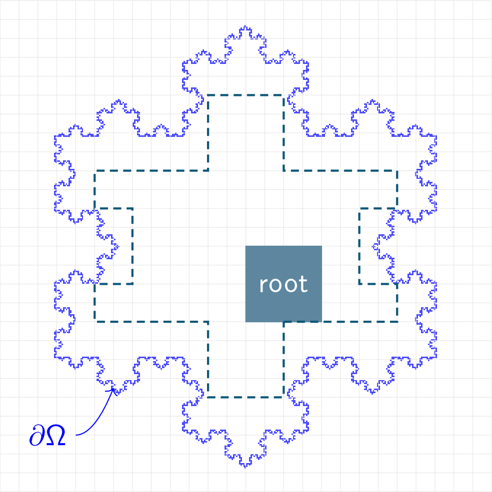

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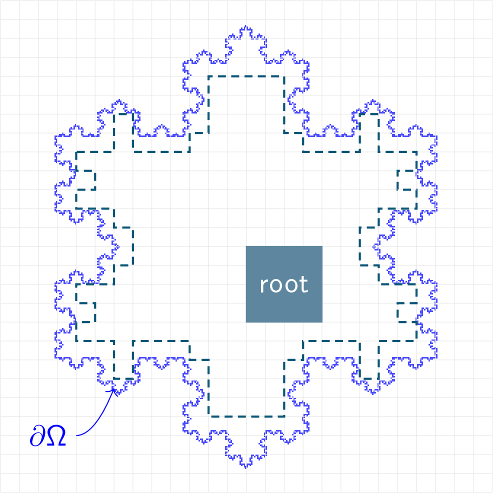

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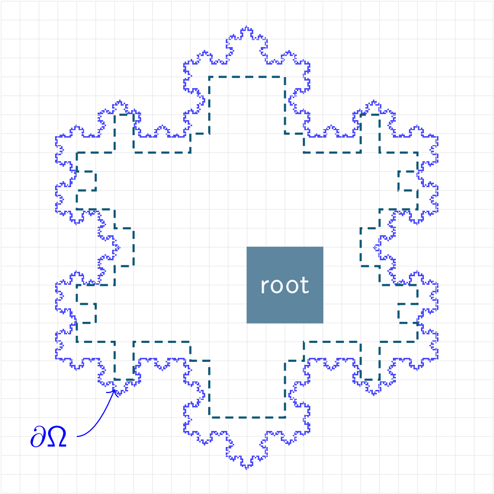


**Proposition**

*Let $(\Omega_k)_{k\in\mathbb{N}}$ be a dyadic approximation of $\Omega$. It holds $\Omega_k \nearrow \Omega$.*

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# Convergence of and along a sequence of Hilbert spaces

**Definition (Convergence of Hilbert spaces [Kuwae-Shioya, 2003])**

A sequence $(H_k)$ of Hilbert spaces converges to a Hilbert space $H$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$, where $(\mathcal{T}_k \in \mathcal{L}(H, H_k))_{k\in\mathbb{N}}$, if it holds

$$\forall u \in H, \quad \|\mathcal{T}_k u\|_{H_k} \xrightarrow[k\to\infty]{} \|u\|_H.$$

Morally, $u \in H$ is represented in $H_k$ by $\mathcal{T}_k u$.

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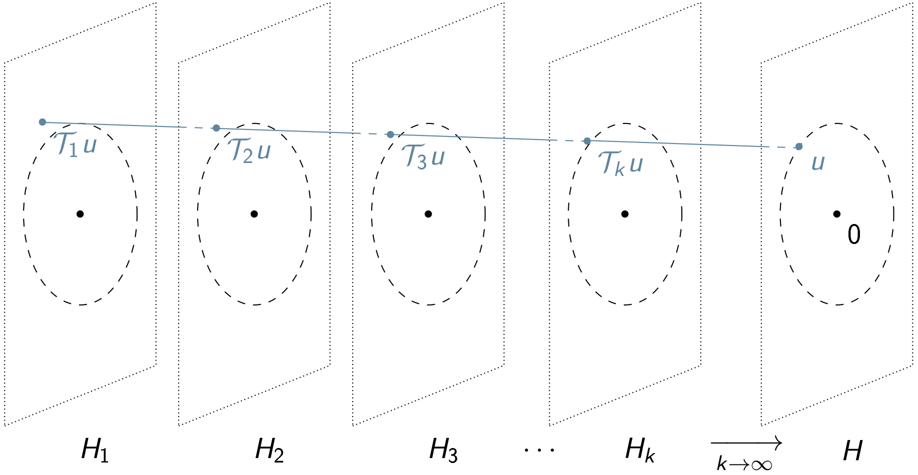
## Definition (Convergence of vectors)

Assume $H_k \to H$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$. A sequence $(u_k \in H_k)_{k\in\mathbb{N}}$ is said to converge to $u \in H$ if it holds

$$\|u_k - \mathcal{T}_k u\|_{H_k} \xrightarrow[k\to\infty]{} 0.$$

Converging to $u$ means becoming arbitrarily close to its representatives $\mathcal{T}_k u$.

# Illustration of the convergence of Hilbert spaces



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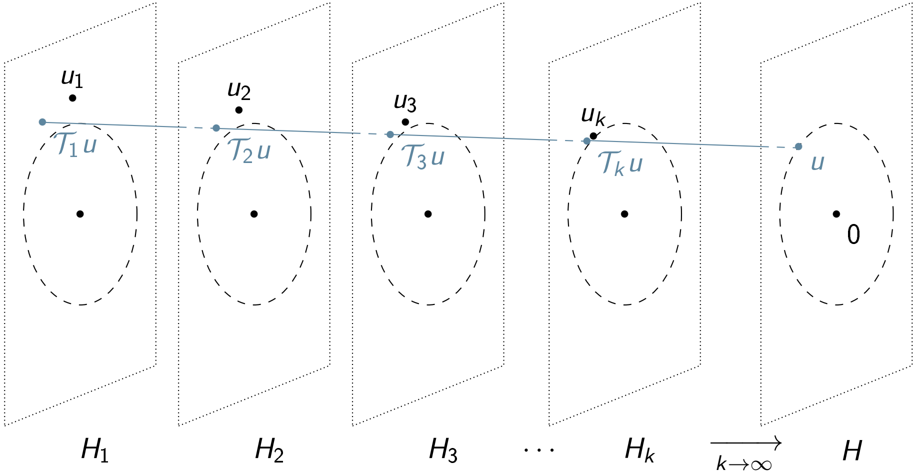


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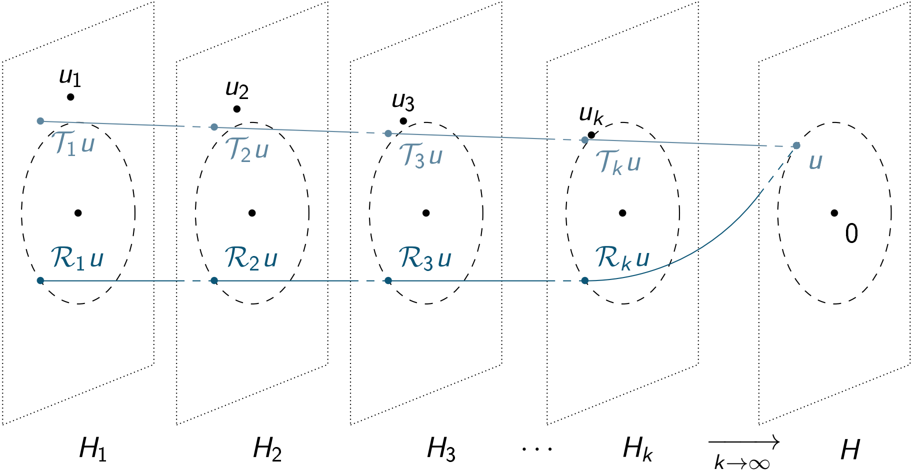


$u_k \to u$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$... but not through $(\mathcal{R}_k)_{k\in\mathbb{N}}$ !

# Decompositions of the space of solutions

The space of solutions to $(-\Delta + 1)u = 0$ on $\mathbb{R}^n \backslash \partial\Omega$ can be described as

$$\begin{aligned} V_1(\mathbb{R}^n \backslash \partial\Omega) &= V_1(\Omega) \quad \oplus \quad V_1(\overline{\Omega}^c) && \text{(geographic)} \\ &= \underbrace{V_{1,\mathcal{S}}(\mathbb{R}^n \backslash \partial\Omega)}_{\text{null jump in } \mathrm{Tr}} \quad \oplus \quad \underbrace{V_{1,\mathcal{D}}(\mathbb{R}^n \backslash \partial\Omega)}_{\text{null jump in } \frac{\partial}{\partial\nu}} && \text{(in potentials)} \end{aligned}$$

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The problem is connected to the decomposition in potentials...

but the geographic decomposition is more tangible.

# Convergence framework for $V_1$ functions inside

If $\Omega_k \nearrow \Omega$, which $u_k \in V_1(\Omega_k)$ represents a given $u \in V_1(\Omega)$ best?

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$$\|u|_{\Omega_k}\|^2_{H^1(\Omega_k)} = \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2)\mathbb{1}_{\Omega_k}\,\mathrm{d}x$$
$$\xrightarrow[k\to\infty]{} \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2)\mathbb{1}_{\Omega}\,\mathrm{d}x = \|u\|^2_{H^1(\Omega)}$$

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**Proposition**

*If $\Omega_k \nearrow \Omega$, then $V_1(\Omega_k) \longrightarrow V_1(\Omega)$ through $(\cdot|_{\Omega_k})_{k\in\mathbb{N}}$.*

# Convergence framework at the boundary: traces

If $\Omega_k \nearrow \Omega$, which $f_k \in \mathcal{B}(\partial\Omega_k)$ represents a given $f \in \mathcal{B}(\partial\Omega)$ best?

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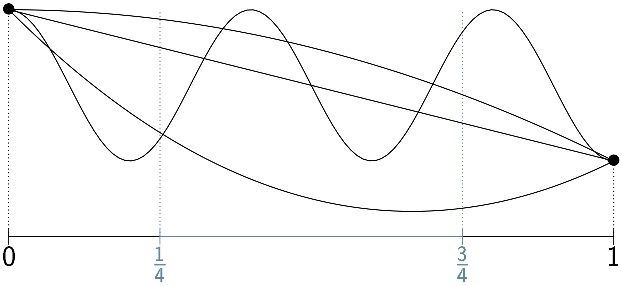


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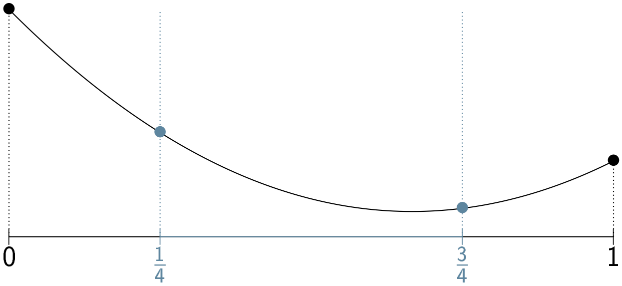


There is an infinite number of ways to connect the dots...

Extend using a **solution to the problem** (here, $(-\Delta + 1)u = 0$).

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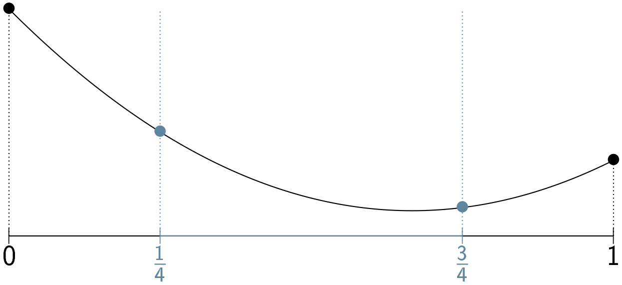


## Proposition

*If* $\Omega_k \nearrow \Omega$, *then* $\mathcal{B}(\partial\Omega_k) \longrightarrow \mathcal{B}(\partial\Omega)$ *through* $(\mathrm{Tr}^{\partial\Omega_k} \circ \mathrm{H}_{\partial\Omega})_{k\in\mathbb{N}}$.

# Did we do things right?

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$$u_k \xrightarrow[k\to\infty]{} u \quad \textit{through } (\cdot|_{\Omega_k})_{k\in\mathbb{N}}$$

$$\iff \quad \mathrm{Tr}_i^{\partial\Omega_k} u_k \xrightarrow[k\to\infty]{} \mathrm{Tr}_i^{\partial\Omega} u \quad \textit{through } (\mathrm{Tr}^{\partial\Omega_k} \circ \mathrm{H}_{\partial\Omega})_{k\in\mathbb{N}}.$$

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The convergence frameworks for $V_1$, $\mathcal{B}$ et $\mathcal{B}'$ are indeed **compatibles**.

# Are those convergence frameworks relevant?

A priori, a convergence across Hilbert spaces is very weak...

**Proposition**

*Assume* $\Omega_k \nearrow \Omega$ *and* $(\mathrm{E}_{\Omega_k})_{k\in\mathbb{N}}$ *is uniformly bounded. If* $(u_k \in V_1(\Omega_k))_{k\in\mathbb{N}}$ *and* $u \in V_1(\Omega)$*, then*

$$u_k \xrightarrow[k\to\infty]{} u \quad \text{through } (\cdot|_{\Omega_k})_{k\in\mathbb{N}} \qquad \Longleftrightarrow \qquad \|\mathrm{E}_\Omega u - \mathrm{E}_{\Omega_k} u_k\|_{H^1(\mathbb{R}^n)} \xrightarrow[k\to\infty]{} 0.$$

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The convergence across spaces for $V_1$ functions can be strengthened into a (standard) convergence on $H^1(\mathbb{R}^n)$: **we did things in the 'right way'!**

Consequently, the convergence framework for $\mathcal{B}$ is also 'right'.

# Convergence framework for $V_1$ functions outside

If $\Omega_k \nearrow \Omega$, which $v_k \in V_1(\overline{\Omega}_k^c)$ represents a given $v \in V_1(\overline{\Omega}^c)$ best?

Ex: $\Omega = ]0,1[$ and $\Omega_k = ]2^{-k}, 1-2^{-k}[$, $k \geq 2$.


## Proposition

*If $\Omega_k \nearrow \Omega$, then $V_1(\overline{\Omega}_k^c) \longrightarrow V_1(\overline{\Omega}^c)$ through ( $\qquad\qquad$ $)_{k\in\mathbb{N}}$.*

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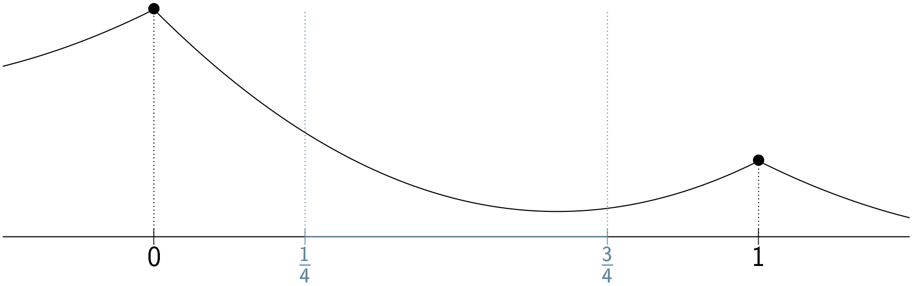


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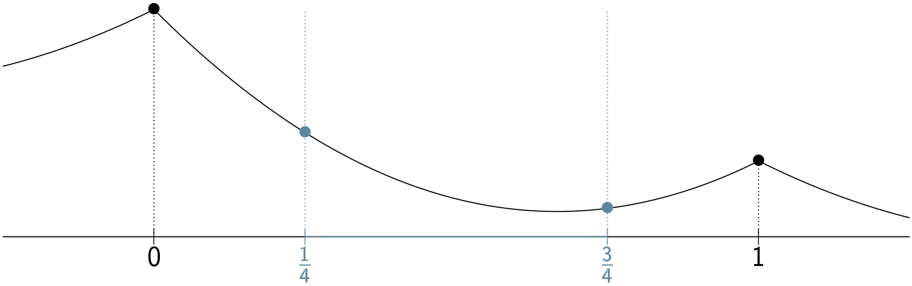


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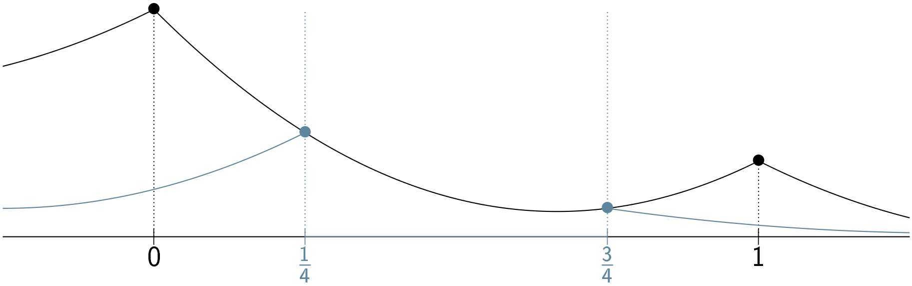


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*If $\Omega_k \nearrow \Omega$, then $V_1(\overline{\Omega}_k^c) \longrightarrow V_1(\overline{\Omega}^c)$ through $(\mathcal{E}_k)_{k\in\mathbb{N}}$.*

# Did we do things right? – part 2

Can we link the convergence of the solutions and their exterior boundary values as we did before?

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## Proposition

*Assume* $\Omega_k \nearrow \Omega$, *and* $(\mathrm{E}_{\Omega_k})$ *and* $(\mathrm{E}_{\overline{\Omega}_k^c})$ *are uniformly bounded. If* $(u_k \in V_1(\overline{\Omega}_k))_{k\in\mathbb{N}}$ *and* $u \in V_1(\overline{\Omega})$, *then*

$$
\begin{aligned}
& u_k \xrightarrow[k\to\infty]{} u \quad \textit{through } (\mathcal{E}_k)_{k\in\mathbb{N}} \\
\iff \quad & \mathrm{Tr}_e^{\partial\Omega_k} u_k \xrightarrow[k\to\infty]{} \mathrm{Tr}_e^{\partial\Omega} u \quad \textit{through } (\mathrm{Tr}^{\partial\Omega_k} \circ \mathrm{H}_{\partial\Omega})_{k\in\mathbb{N}} \\
\iff \quad & \|\mathrm{E}_{\overline{\Omega}^c} u - \mathrm{E}_{\overline{\Omega}_k^c} u_k\|_{H^1(\mathbb{R}^n)} \xrightarrow[k\to\infty]{} 0.
\end{aligned}
$$

# Back to the whole space

We built the convergence of the $V_1$ spaces along the geographic decomposition...

**Proposition**

*Assume $\Omega_k \nearrow \Omega$, and $(\mathrm{E}_{\Omega_k})$ and $(\mathrm{E}_{\overline{\Omega}_k^c})$ are uniformly bounded. If $(u_k \in V_1(\mathbb{R}^n \backslash \partial\Omega_k))_k \to u \in V_1(\mathbb{R}^n \backslash \partial\Omega)$ through $(\cdot|_{\Omega_k} \oplus \mathcal{E}_k)_k$, and*

$$u = \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f \quad \text{et} \quad u_k = \mathcal{S}_{\partial\Omega_k} g_k - \mathcal{D}_{\partial\Omega_k} f_k,$$

*then*

$$\begin{aligned} \mathcal{S}_{\partial\Omega_k} g_k &\longrightarrow \mathcal{S}_{\partial\Omega} g && \text{through } (\cdot|_{\Omega_k} \oplus \mathcal{E}_k), \\ \mathcal{D}_{\partial\Omega_k} f_k &\longrightarrow \mathcal{D}_{\partial\Omega} f && \text{through } (\cdot|_{\Omega_k} \oplus \mathcal{E}_k). \end{aligned}$$

... yet **the decomposition in potentials is also preserved**!

# The Riemann-Hilbert problem on extension domains

Under the same hypotheses, we can deduce

$$f_k \longrightarrow f \quad \text{through } (\mathrm{Tr}^{\partial\Omega_k}\,\mathrm{H}_{\partial\Omega_k}) \\ \qquad\qquad \Longrightarrow \quad \Phi_{\partial\Omega_k} f_k \longrightarrow \Phi_{\partial\Omega} f \quad \text{through } (\cdot|_{\partial\Omega} \oplus \mathcal{E}_k).$$

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Since the convergence through $(\cdot|_{\partial\Omega} \oplus \mathcal{E}_k)$ can be **strengthened** into an $H^1(\mathbb{R}^n)$ convergence, it also holds

$$\Phi_{\partial\Omega_k} f_k \text{ holomorphic on } \mathbb{C}\backslash\partial\Omega_k \quad \Longrightarrow \quad \Phi_{\partial\Omega} f \text{ holomorphic on } \mathbb{C}\backslash\partial\Omega.$$

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**Theorem**

*Let $\Omega$ be a two-sided extension domain of $\mathbb{C}$. Assume there exists a sequence of Lipschitz domains $(\Omega_k)_{k\in\mathbb{N}}$ such that $\Omega_k \nearrow \Omega$, and $(\mathrm{E}_{\Omega_k})$ and $(\mathrm{E}_{\overline{\Omega}_k^c})$ be uniformly bounded.*
*Then,* ***we can define a Cauchy integral*** $\Phi_{\partial\Omega} : \mathcal{B}(\partial\Omega) \to H^1(\mathbb{C}\backslash\partial\Omega)$ ***which solves the Riemann-Hilbert problem.***

# … end of the converegnce part

*To find out more:*

- G. Claret, A. Rozanova-Pierrat and A. Teplyaev, *Convergence of layer potentials and Riemann-Hilbert problem on extension domains* (2026).
- G. Claret, M. Hinz, A. Rozanova-Pierrat and A. Teplyaev, *Layer potential operators for transmission problems on extension domains*. Journal de Mathématiques Pures et Appliquées (9) 212 (2026), Paper No. 103888, 42pp.

# Link with the Poincaré-Steklov operator

Let $u \in V_1(\Omega)$. Then,

$$\frac{\partial_i u}{\partial \nu} = \Lambda_1(\mathrm{Tr}_i\, u) \quad \text{and} \quad \mathcal{S}_{\partial\Omega}\left(\frac{\partial_i u}{\partial \nu}\right) - \mathcal{D}_{\partial\Omega}(\mathrm{Tr}_i\, u) = u\mathbb{1}_{\Omega} \oplus 0\mathbb{1}_{\overline{\Omega}^c}.$$

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Hence,

$$\mathrm{Tr}_e\left[\mathcal{S}_{\partial\Omega}\frac{\partial_i u}{\partial \nu}\right] = \mathrm{Tr}_e\left[\mathcal{D}_{\partial\Omega}(\mathrm{Tr}_i\, u)\right] \quad \text{and} \quad \frac{\partial_e}{\partial \nu}\left[\mathcal{S}_{\partial\Omega}\frac{\partial_i u}{\partial \nu}\right] = \frac{\partial_e}{\partial \nu}\left[\mathcal{D}_{\partial\Omega}(\mathrm{Tr}_i\, u)\right].$$

**Proposition**

*Let $\Omega$ be a two-sided extension domain. Then,*

$$\Lambda_1 = \mathcal{V}_{\partial\Omega}^{-1}\left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right) = -\left(-\frac{1}{2}I + \mathcal{K}^*_{\partial\Omega}\right)^{-1}\mathcal{W}_{\partial\Omega}.$$

# Neumann-Poincaré operators for Lipschitz boundaries[5]

For $f \in L^2(\partial\Omega, \sigma)$,

$$\mathcal{K}_{\partial\Omega} f(x) := \frac{1}{\omega_n} \text{p.v.} \int_{\partial\Omega} \frac{\langle y - x, \nu_y \rangle}{|x - y|^n} f(y)\, \sigma(\mathrm{d}y),$$

with $\omega_n$ the area of the unit sphere in $\mathbb{R}^n$ and $(\nu_y)_{y \in \partial\Omega}$ the outgoing normal vector field.

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with $\omega_n$ the area of the unit sphere in $\mathbb{R}^n$ and $(\nu_y)_{y \in \partial\Omega}$ the outgoing normal vector field.

It holds, for $f, g \in L^2(\partial\Omega, \sigma)$:

$$\mathcal{K}_{\partial\Omega} f = \frac{1}{2}(\mathrm{Tr}_i + \mathrm{Tr}_e)\mathcal{D}_{\partial\Omega} f \qquad \text{and} \qquad \mathcal{K}^*_{\partial\Omega} g = \frac{1}{2}\left(\frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu}\right)\mathcal{S}_{\partial\Omega} g.$$

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# Neumann-Poincaré operators for extension domains

## Definition

Let $\Omega$ be a *two-sided extension domain* of $\mathbb{R}^n$.
We define

$$\mathcal{K}_{\partial\Omega} : f \in \mathcal{B}(\partial\Omega) \longmapsto \frac{1}{2}(\mathrm{Tr}_i + \mathrm{Tr}_e)\mathcal{D}_{\partial\Omega} f \in \mathcal{B}(\partial\Omega).$$

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It holds: $$\mathcal{K}^*_{\partial\Omega} : g \in \mathcal{B}'(\partial\Omega) \longmapsto \frac{1}{2}\left(\frac{\partial_i}{\partial\nu} + \frac{\partial_e}{\partial\nu}\right)\mathcal{S}_{\partial\Omega} g \in \mathcal{B}'(\partial\Omega).$$

The operator $\mathcal{K}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \to \mathcal{B}(\partial\Omega)$ is **linear and continuous**.

# Understanding Neumann-Poincaré operators

$$
\begin{tikzcd}
 & {[\![ \frac{\partial u}{\partial \nu} ]\!]} \arrow[r, "\mathcal{K}^*_{\partial\Omega}"] \arrow[d, bend right, "\mathcal{S}_{\partial\Omega}"'] & \frac{1}{2}\left(\frac{\partial_i}{\partial\nu}+\frac{\partial_e}{\partial\nu}\right)u_{\mathcal{S}} \\
 & \begin{matrix} u \\ \| \\ u_{\mathcal{S}} - u_{\mathcal{D}} \end{matrix} & \\
\frac{1}{2}(\mathrm{Tr}_i+\mathrm{Tr}_e)u_{\mathcal{D}} & -[\![\mathrm{Tr}\, u]\!] \arrow[l, "\mathcal{K}_{\partial\Omega}"'] \arrow[u, bend right, "\mathcal{D}_{\partial\Omega}"'] &
\end{tikzcd}
$$

# Calderón projectors

$$\mathcal{V}_{\partial\Omega} := \operatorname{Tr}\mathcal{S}_{\partial\Omega} : \mathcal{B}'(\partial\Omega) \to \mathcal{B}(\partial\Omega), \quad \mathcal{W}_{\partial\Omega} := -\frac{\partial}{\partial\nu}\mathcal{D}_{\partial\Omega} : \mathcal{B}(\partial\Omega) \to \mathcal{B}'(\partial\Omega).$$

## Theorem

*Let $\Omega$ be a two-sided extension domain of $\mathbb{R}^n$. Consider the operators*

$$\mathcal{C}_i := \frac{1}{2}I + \mathcal{M} \qquad \text{and} \qquad \mathcal{C}_e := \frac{1}{2}I - \mathcal{M},$$

*where*

$$\mathcal{M} := \begin{pmatrix} -\mathcal{K}_{\partial\Omega} & \mathcal{V}_{\partial\Omega} \\ \mathcal{W}_{\partial\Omega} & \mathcal{K}^*_{\partial\Omega} \end{pmatrix} : \mathcal{B}(\partial\Omega) \times \mathcal{B}'(\partial\Omega) \to \mathcal{B}(\partial\Omega) \times \mathcal{B}'(\partial\Omega).$$

*Then $\mathcal{C}_i$ and $\mathcal{C}_e$ are linear continuous* ***(Calderón) projectors****. In particular,*

$$\mathcal{K}_{\partial\Omega}\mathcal{V}_{\partial\Omega} = \mathcal{V}_{\partial\Omega}\mathcal{K}^*_{\partial\Omega} \qquad \text{and} \qquad \mathcal{W}_{\partial\Omega}\mathcal{K}_{\partial\Omega} = \mathcal{K}^*_{\partial\Omega}\mathcal{W}_{\partial\Omega}.$$

Generalization of: [Nédélec, 2011].

# Interpretation of the Calderón projectors

$u$ is a solution to the transmission problem:

$$\mathcal{C}_i \begin{pmatrix} [\![\operatorname{Tr} u]\!] \\ \left[\!\left[\frac{\partial u}{\partial \nu}\right]\!\right] \end{pmatrix} = \begin{pmatrix} \operatorname{Tr}_i u \\ \frac{\partial_i u}{\partial \nu} \end{pmatrix}.$$

$v = u\mathbb{1}_\Omega + 0\mathbb{1}_{\overline{\Omega}^c}$ is a solution with:

$$[\![\operatorname{Tr} v]\!] = \operatorname{Tr}_i u \qquad \text{and} \qquad \left[\!\left[\frac{\partial v}{\partial \nu}\right]\!\right] = \frac{\partial_i u}{\partial \nu},$$

hence

$$\mathcal{C}_i^2 \begin{pmatrix} [\![\operatorname{Tr} u]\!] \\ \left[\!\left[\frac{\partial u}{\partial \nu}\right]\!\right] \end{pmatrix} = \mathcal{C}_i \begin{pmatrix} [\![\operatorname{Tr} v]\!] \\ \left[\!\left[\frac{\partial v}{\partial \nu}\right]\!\right] \end{pmatrix} = \begin{pmatrix} \operatorname{Tr}_i u \\ \frac{\partial_i u}{\partial \nu} \end{pmatrix}.$$

# Understanding boundary value operators

$$
\begin{tikzcd}
\operatorname{Tr} u_{\mathcal{S}} & \left[\!\left[\dfrac{\partial u}{\partial \nu}\right]\!\right] \arrow[l, "\mathcal{V}_{\partial\Omega}"'] \arrow[r, "\mathcal{K}^*_{\partial\Omega}"] \arrow[d, bend right, "\mathcal{S}_{\partial\Omega}"'] & \dfrac{1}{2}\left(\dfrac{\partial_i}{\partial \nu} + \dfrac{\partial_e}{\partial \nu}\right) u_{\mathcal{S}} \\
& \begin{array}{c} u \\ \| \\ u_{\mathcal{S}} - u_{\mathcal{D}} \end{array} & \\
\dfrac{1}{2}(\operatorname{Tr}_i + \operatorname{Tr}_e) u_{\mathcal{D}} & -[\![\operatorname{Tr} u]\!] \arrow[l, "\mathcal{K}_{\partial\Omega}"] \arrow[r, "\mathcal{W}_{\partial\Omega}"'] \arrow[u, bend right, "\mathcal{D}_{\partial\Omega}"'] & \dfrac{\partial u_{\mathcal{D}}}{\partial \nu}
\end{tikzcd}
$$

# Well-posedness as boundary ~~integral~~ equations

A usual approach to one-sided problems ([Colton and Kress, 2013], [Chandler-Wilde et al., 2012], [Caetano et al., 2024]) is via boundary integral equations.

**Proposition**

*For all $h \in \mathcal{B}(\partial\Omega)$, the boundary equations*

$$\left(-\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right) f = h \qquad \text{and} \qquad \mathcal{V}_{\partial\Omega} g = h$$

*have unique solutions $f \in \mathcal{B}(\partial\Omega)$ and $g \in \mathcal{B}'(\partial\Omega)$ respectively.*

**Any interior Dirichlet solution is a restricted SLP/DLP.**

# Neumann series

**Equivalent norm on $\mathcal{B}(\partial\Omega)$:** $\|\cdot\|^2_{\mathcal{V}^{-1}_{\partial\Omega}} := \langle \mathcal{V}^{-1}_{\partial\Omega}\cdot,\cdot\rangle_{\mathcal{B}',\mathcal{B}(\partial\Omega)}$.

**Theorem**

*Let $\Omega$ be a two-sided extension domain of $\mathbb{R}^n$. Then, there exists $c\in]1/2,1[$ such that for all $f\in\mathcal{B}(\partial\Omega)$:*

$$(1-c)\|f\|_{\mathcal{V}^{-1}_{\partial\Omega}} \le \left\|\left(\frac{1}{2}I+\mathcal{K}_{\partial\Omega}\right)f\right\|_{\mathcal{V}^{-1}_{\partial\Omega}} \le c\|f\|_{\mathcal{V}^{-1}_{\partial\Omega}}.$$

Generalization of: [Steinbach and Wendland, 2001].

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## Theorem

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$$(1-c)\|f\|_{\mathcal{V}^{-1}_{\partial\Omega}} \leq \left\| \left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right) f \right\|_{\mathcal{V}^{-1}_{\partial\Omega}} \leq c\|f\|_{\mathcal{V}^{-1}_{\partial\Omega}}.$$

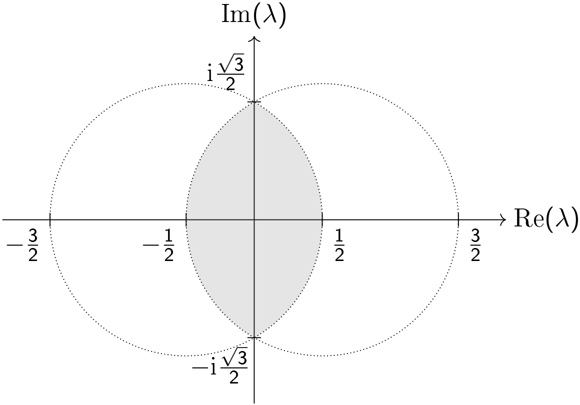
In particular,

$$-(\mathrm{Tr}_i\,\mathcal{D}_{\partial\Omega})^{-1} = \left(\frac{1}{2}I - \mathcal{K}_{\partial\Omega}\right)^{-1} = \sum_{k=0}^{+\infty}\left(\frac{1}{2}I + \mathcal{K}_{\partial\Omega}\right)^k = \sum_{k=0}^{+\infty}(\mathrm{Tr}_e\,\mathcal{D}_{\partial\Omega})^k.$$

Generalization of: [Steinbach and Wendland, 2001].

# Spectrum of the Neumann-Poincaré operator

With a perturbation result [Davies, 2007]...




... **the spectrum of $\mathcal{K}_{\partial\Omega}$ (and of $\mathcal{K}^*_{\partial\Omega}$) lies in the gray area.**

# To find out more:


- G. Claret, A. Rozanova-Pierrat and A. Teplyaev, *Convergence of layer potentials and Riemann-Hilbert problem on extension domains* (2026) arXiv:2411.03767
- G. Claret, M. Hinz, A. Rozanova-Pierrat and A. Teplyaev, *Layer potential operators for transmission problems on extension domains*. Journal de Mathématiques Pures et Appliquées (9) 212 (2026), Paper No.103888, 42pp
- M. Hinz, A. Rozanova-Pierrat and A. Teplyaev, *Boundary value problems on non-Lipschitz uniform domains: Stability, compactness and the existence of optimal shapes*. Asymptotic Analysis 134 (2023), no. 1–2, 25–61.
- M. Hinz, F. Magoulès, A. Rozanova-Pierrat, M. Rynkovskaya and A. Teplyaev, *On the existence of optimal shapes in architecture*. Applied Mathematical Modelling 94 (2021), 676–687.
- M. Hinz, A. Rozanova-Pierrat and A. Teplyaev, *Non-Lipschitz uniform domain shape optimization in linear acoustics*. SIAM Journal on Control and Optimization 59 (2021), no. 2, 1007–1032.

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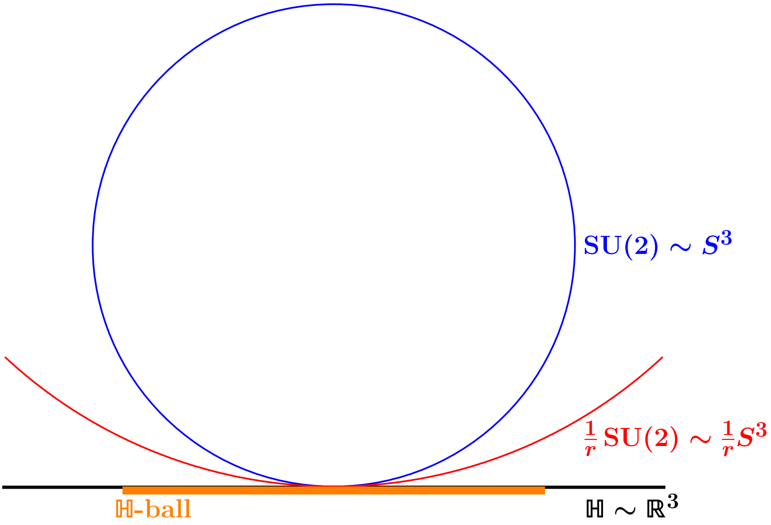
**Theorem** (Carfagnini, Gordina, T)

- $0 < \lambda_1^{\mathbb{H}} < \lambda_2^{\mathbb{H}} \leqslant \lambda_3^{\mathbb{H}} \leqslant \ldots$ Dirichlet eigenvalues in the unit ball $B_1^{\mathbb{H}}$ in $\mathbb{H}$, counted with multiplicity

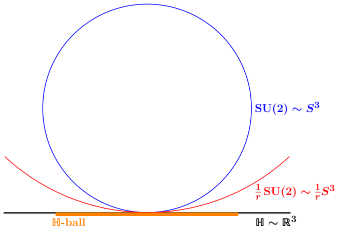
- $0 < \lambda_1^r < \lambda_2^r \leqslant \lambda_3^r \leqslant \ldots$ Dirichlet eigenvalues in the $r$-ball $B_r^{\mathrm{SU}(2)}$ in SU(2), counted with multiplicity

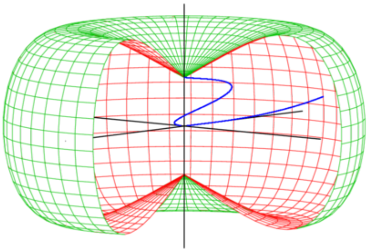
$$\Longrightarrow \quad \lim_{r\to 0} r^2 \lambda_n^r = \lambda_n^{\mathbb{H}} \qquad\qquad n \geqslant 1$$



$\mathbf{SU(2)} \sim S^3$
$\frac{1}{r}\,\mathbf{SU(2)} \sim \frac{1}{r}S^3$
$\mathbb{H}$-ball
$\mathbb{H} \sim \mathbb{R}^3$


1



the Heisenberg ball [picture made by Nate Eldredge]

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# Spectral Mass, Monotonicity, and Projective Limits (Patricia Alonso-Ruiz, Jun Kigami, AT)

**Strong Monotonicity Lemma:** Let $\mathcal{A}$ be a $\mathcal{E}_*$-closed subset of $\mathcal{F}$. Then conditions (M1), (M2) and (M3) are equivalent:

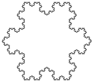
(M1) For any $u \in \mathcal{F}$,

$$(\mathrm{Pr}_{\mathcal{E}_*}(u), \mathrm{Pr}_{\mathcal{E}_*}(u))_{\mathcal{H}} \leqslant (u, v)_{\mathcal{H}}, \tag{1}$$

where $\mathrm{Pr}_{\mathcal{E}_*}(u) : \mathcal{F} \to \mathcal{A}$ is the $\mathcal{E}_*$-orthogonal projection onto $\mathcal{A}$.

(M2) $(H + I)(\mathcal{A} \cap \mathcal{D}) = \overline{\mathcal{A}}_{\mathcal{H}}$, where $\overline{\mathcal{A}}_{\mathcal{H}}$ denotes the $\mathcal{H}$-closure of $\mathcal{A}$.

(M3) $(u, v)_{\mathcal{H}} = 0$ for any $u \in \mathcal{A}$ and any $v \in \mathcal{A}^{\perp}_{\mathcal{E}_*}$, where $\mathcal{A}^{\perp}_{\mathcal{E}_*}$ is the $\mathcal{E}_*$-orthogonal complement of $\mathcal{A}$ in $\mathcal{F}$. Also, $\mathcal{A}^{\perp}_{\mathcal{E}_*}$ is the $\mathcal{E}_*$-orthogonal complement of $\mathcal{A}_{\mathcal{E}_*}$ in $\mathcal{F}$.



# Thank you for your attention!