# Processes on the Sierpinski gasket and Basilica graphs

Peidong Wang

University of Connecticut, Department of Mathematics

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# Roadmap

- First: the Sierpinski gasket and three processes on it – Brownian motion, the self-avoiding limit, and the self-repelling interpolation.
- Second: how the discrete models on graph approximations converge to continuous processes on the gasket.
- Third: the self-similar Basilica graph point of view for analysis on the Basilica Julia set: cell structure, resistance forms, Laplacians, spectral / walk dimensions.
- Finally: the Schreier-graph self-repelling model, its penalty rules, one-shot kernels, renormalization, and the walk dimension in the SRW endpoint.

# Notation on the gasket: compact versus unbounded models

**Compact unit gasket.**

- $K$ denotes the standard unit Sierpinski gasket with corner points $O, a, b$.
- $V_n \subset K$ is the level-$n$ vertex set inside one compact cell.

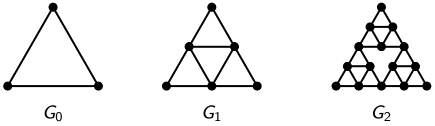
**Unbounded pre-gasket / infinite gasket.**

- An increasing family

$$F_0 \subset F_1 \subset F_2 \subset \cdots, \qquad F = \bigcup_{n \geq 0} F_n,$$

  with $a_n = 2^n a$ and $b_n = 2^n b$.
- Discrete paths go from $O$ to $a_n$ in this unbounded graph; after rescaling by $2^{-n}$ they become paths in the compact unit gasket.
- For Brownian motion, local renormalization is still computed inside one compact cell even when the limit diffusion is viewed on the unbounded gasket.

# The compact pre-gaskets inside one cell




- Inside one unit cell, start from the equilateral triangle with vertices $O$, $a$, $b$ and set

$$F_{n+1} = \tfrac{1}{2}F_n \cup \left(\tfrac{1}{2}F_n + a\right) \cup \left(\tfrac{1}{2}F_n + b\right), \qquad G_n = \text{vertex set of } F_n.$$

- The compact limit $K = \overline{\bigcup_{n\geq 0} F_n}$ is the unit gasket. These are the graphs used for level-by-level decimation and one-cell computations.
- In the unbounded notation one glues rescaled copies of this same cell across larger and larger scales; the renormalization formulas are identical once the path is viewed inside each cell.

# Path space and decimation on the gasket

Let

$$\mathcal{C} = \left\{ w \in C([0,\infty) \to F) : w(0) = O,\ \lim_{t\to\infty} w(t) = a_0 \right\}.$$

For $w \in \mathcal{C}$, define the hitting times of level-$m$ vertices by

$$T_0^m(w) = 0, \qquad T_i^m(w) = \inf\{t > T_{i-1}^m(w) : w(t) \in G_m \setminus \{w(T_{i-1}^m(w))\}\}.$$

The *decimation map* $Q_m : \mathcal{C} \to \mathcal{C}$ is the piecewise-linear path through the visited vertices

$$(Q_m w)(i) = w(T_i^m(w)), \qquad Q_k \circ Q_m = Q_k \quad (k \le m).$$

- $Q_m$ forgets the microscopic details below scale $2^{-m}$.
- The convergence theorems on the gasket are proved by showing projective consistency of the laws under $Q_m$ and then controlling the random time change between levels.

# Three processes on the gasket: the discrete pictures

| Process | Discrete model on $G_n$ | What is weighted / forbidden? |
|---|---|---|
| Brownian motion | Simple random walk | No self-interaction; only the graph structure matters. |
| Self-avoiding process | Self-avoiding paths from $O$ to $a_n$ | Self-intersections are forbidden already at the discrete level. |
| Self-repelling family | Weighted nearest-neighbor paths | Backtracks, sharp turns, and selected returns are penalized by a factor $u \in [0, 1]$. |

**Takeaway.** On the gasket the key question is always the same: choose the right discrete law, coarse-grain by $Q_m$, identify the time-scaling factor, and pass to a continuous limit.

# Brownian motion on the gasket: the discrete walk

For $x \in G_n$, let $X(n,x)$ be simple random walk on $G_n$ started at $x$. After coupling the walks across levels, one gets the nesting property

$$X(m,x)(i) = X(n,x)\left(T_i^m(X(n,x))\right), \qquad x \in G_m,\ m < n.$$

Thus the coarse skeleton of the finer walk is exactly the coarser walk.

If we define the rescaled path by

$$X_n(x)(j5^{-n}) = X(n,x)(j)$$

and interpolate linearly in time, then the candidate limit process is obtained by proving that the random crossing times between level-$m$ vertices stabilize as $n \to \infty$.

**Main idea.**

- Space scaling is fixed by the gasket: each level shrinks lengths by 2.
- Time scaling is not obvious; it comes from the mean number of fine traversals needed to realize one coarse traversal.

# One-cell computation for Brownian motion

Take one level-0 edge and look at its realization inside the level-1 graph. If $N$ is the number of level-1 traversals needed to cross one level-0 edge, then

$$f(u) = \mathbb{E}(u^N) = \frac{u^2}{4-3u}, \qquad \mathbb{E}[N] = 5.$$

Similarly, if $H$ counts returns to the starting vertex before breakout, then

$$h(u) = \mathbb{E}(u^H) = \frac{3u}{5-2u}, \qquad \mathbb{E}[H] = \frac{5}{3}.$$

**Interpretation.**

- Each coarse edge is replaced by a random cluster of finer edges.
- The number of offspring has mean 5; this is exactly the time renormalization factor.

# Brownian motion: branching time structure behind the limit

For a coarse path segment between two consecutive hits of $G_m$, let

$$S_i^m(X_{m+r}) = T_i^m(X_{m+r}) - T_{i-1}^m(X_{m+r})$$

be its crossing time measured on the finer graph $G_{m+r}$. Then, conditionally on the level-$m$ skeleton,

$$\{5^{-r} S_i^m(X_{m+r})\}_{r\geq 0}$$

is a supercritical branching process with offspring law $N$ and mean 5. Hence, for each fixed $m, i$,

$$5^{-r} S_i^m(X_{m+r}) \longrightarrow S_i^{*m} \quad \text{a.s. and in } L^2.$$

Moreover the variables $S_i^{*m}$ are i.i.d. over different coarse edges and are independent of the level-$m$ skeleton.

# Brownian motion: proof idea for convergence to the continuous limit

**Step 1.** The nesting relation gives a consistent family of coarse skeletons. For every $m$, the sequence $Q_m X_n$ stabilizes to a path on $G_m$.

**Step 2.** The branching-process limit gives stable coarse crossing times $T_i^{*m}$. These are the random traversal times attached to the edges of the level-$m$ skeleton.

**Step 3.** Fix a macroscopic time interval $[0, M]$ and choose $m$ large. The path can only move along a finite number of level-$m$ edges on this interval, and each such edge has a limiting traversal time.

**Step 4.** Because the mesh size is $2^{-m}$, the difference between two paths that agree on the level-$m$ skeleton is at most $O(2^{-m})$. This yields almost sure convergence in $C([0, \infty), F)$.

$$X_n(x) \longrightarrow X(x),$$

where $X$ is Brownian motion on the gasket.

# Brownian motion: what the limit remembers

The limit process $X$ is the canonical diffusion on the gasket. Its basic scaling quantity is

$$d_{\mathrm{w}} = \frac{\log 5}{\log 2}.$$

Equivalently, the graph Laplacians and transition kernels inherit the same renormalization.

**Consequences.**

- the transition density satisfies sub-Gaussian bounds with the time scale $t \asymp r^{d_{\mathrm{w}}}$;
- Green and resolvent kernels inherit the same scaling;
- this Brownian motion becomes one endpoint of the self-repelling family ($u = 1$).

# Discrete self-avoiding paths on the pre-gasket

A self-avoiding path is a map $w : \mathbb{Z}_{\geq 0} \to G_0$ such that

- there exists a terminal time $L(w)$ with $w(i) = w(L(w))$ for all $i \geq L(w)$;
- $w(i_1) \neq w(i_2)$ for $0 \leq i_1 < i_2 \leq L(w)$;
- $\|w(i) - w(i+1)\| = 1$ and each edge lies in the pre-gasket.

For paths from $O$ to $a_n$, two local step types are recorded:

$$S_1(w) = \{\text{single-triangle traversals}\}, \qquad S_2(w) = \{\text{two-step bends inside one small triangle}\},$$

with cardinalities $s_1(w), s_2(w)$ satisfying

$$L(w) = s_1(w) + 2s_2(w).$$

# Generating polynomials for self-avoiding paths

Let $W^{(n)}$ be the self-avoiding paths from $O$ to $a_n$ that avoid $b_n$, and let $\widetilde{W}^{(n)}$ be the paths that do hit $b_n$ before exiting. Define

$$\Phi_n(x,y) = \sum_{w \in W^{(n)}} x^{s_1(w)} y^{s_2(w)}, \qquad \Theta_n(x,y) = \sum_{w \in \widetilde{W}^{(n)}} x^{s_1(w)} y^{s_2(w)}.$$

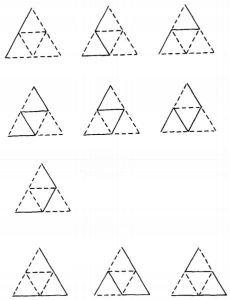
At level 1 one obtains the explicit polynomials

$$\Phi_1(x,y) = (x+y)^2 + x^2(x+2y), \qquad \Theta_1(x,y) = xy(x+2y).$$

The key recursion is

$$\Phi_{n+1} = \Phi_1(\Phi_n, \Theta_n), \qquad \Theta_{n+1} = \Theta_1(\Phi_n, \Theta_n).$$

# Enumerating the paths



- These are the admissible level-1 self-avoiding path types used to compute the first-step generating polynomials.
- The upper seven configurations contribute to $\Phi_1(x, y)$, corresponding to paths from $O$ to $a_1$ that avoid $b_1$.
- The lower three configurations contribute to $\Theta_1(x, y)$, corresponding to paths from $O$ to $a_1$ that pass through $b_1$.

# What the self-avoiding recursion tells us

The pair $(\Phi_n, \Theta_n)$ is a two-dimensional renormalization map.
At criticality the scaled length has a non-trivial law whose Laplace transform $g$ satisfies

$$g(\lambda\xi) = \frac{3-\sqrt{5}}{2}\, g(\xi)^3 + \frac{\sqrt{5}-1}{2}\, g(\xi)^2, \qquad \lambda = \frac{7-\sqrt{5}}{2}.$$

The corresponding displacement exponent is

$$\nu = \frac{\log 2}{\log \lambda}, \qquad \dim_H(\text{range}) = \frac{\log \lambda}{\log 2}.$$

# How the discrete self-avoiding model leads to a continuous process

At the endpoint $u = 0$, every path containing a penalized event receives zero weight. What survives are exactly the admissible self-avoiding trajectories.

The continuum construction follows the same four steps as in the general self-repelling model:

1. choose the critical fixed point $x_0$ so that the laws are consistent under decimation;
2. build a projective family $\{Y_n\}$ with $Q_m Y_n = Y_m$;
3. show that each coarse crossing time is a supercritical branching process with mean

$$\lambda_0 = \frac{7 - \sqrt{5}}{2};$$

4. rescale time by $\lambda_0^{-n}$ and prove uniform convergence of the rescaled paths.

Thus one gets a continuous self-avoiding process $X^{\mathrm{SA}}$ on the gasket.

# Why the self-avoiding limit is still self-avoiding

Each discrete path on $G_n$ is self-avoiding, so for any fixed coarse level $m$ the decimated path $Q_m Y_n$ never revisits a level-$m$ vertex before absorption.

After rescaling time, the limiting path $X^{\mathrm{SA}}$ inherits this property at every level $m$. Since the level-$m$ cells separate points and their diameters go to 0, a genuine self-intersection of the limit path would force a repeated visit at some finite level, which is impossible.

Hence the limit process is not merely continuous: it remains geometrically self-avoiding.

**Takeaway.** The self-avoiding limit is obtained by the same projective/branching mechanism as Brownian motion, but the admissible skeletons are much more rigid.

# Self-repelling walk on the gasket: what is penalized?

For a path $w$ on the level-$n$ graph, define for each scale $k \leq n$ two counters:

- $N_k(w)$ = number of U-turns or sharp turns made by the level-$k$ skeleton at vertices in $G_k \setminus G_{k-1}$;
- $M_k(w)$ = number of returns of the level-$k$ skeleton to a vertex of $G_{k-1}$.

These are the local reversals and revisits that are penalized.

With step weight $x$ and penalty parameter $u \in [0,1]$, the partition function is

$$\Phi_n(x,u) = \sum_{w \in W_n} \left( \prod_{k=1}^{n} u^{N_k(w)+M_k(w)} \right) x^{L(w)}.$$

Similarly, $\Theta_n$ and $\Psi_n$ are defined on the two auxiliary path classes used at level 1.

**Endpoints.**

$$u = 1: \text{ no penalty (Brownian endpoint)}, \qquad u = 0: \text{ penalized events forbidden (self-avoiding endpoint)}.$$

# The level-1 decomposition for self-repelling paths

There are three generating functions at scale 1:

- $\Theta_1(x,u)$: inner excursions from $O$ back to $O$ without reaching $a$ or $b$;
- $\Psi_1(x,u)$: terminal excursions from $O$ to $a$ without revisiting $O$ or touching $b$;
- $\Phi_1(x,u)$: full paths from $O$ to $a$.

The path decomposition is geometric:

$$\Phi_1(x,u) = \Psi_1(x,u)\sum_{m\geq 0}\big(2u\,\Theta_1(x,u)\big)^m = \frac{\Psi_1(x,u)}{1-2u\Theta_1(x,u)}.$$

Each return to $O$ contributes:

- the weight already encoded in $\Theta_1$ for the excursion itself, and
- an extra factor $u$ for the return to the previous coarse vertex,
- an extra factor 2 for the choice of the next branch.

# The four state equations for $\Theta(x, u)$

We work at level 1 and attach weight $x$ per step and weight $u$ for each backtrack or sharp turn. Define the states

$$A := \text{at } S \text{ with previous step from } O, \quad B := \text{at } M \text{ with previous step from some } S,$$

$$C := \text{at } S \text{ with previous step from } M, \quad D := \text{at } S \text{ with previous step from the other } S.$$

Enumerating one-step continuations gives exactly the four recurrences

$$\begin{aligned} A &= ux + xB + uxD, \\ B &= 2ux\, C, \\ C &= x + uxB + uxD, \\ D &= ux + uxB + uxD. \end{aligned}$$

These equations are the level-1 bookkeeping behind the explicit formula for $\Theta_1$.

# The eight state equations for $\Psi(x, u)$

We enumerate paths starting at $O$ and exiting at $a$, never revisiting $O$ nor touching $b$. The states are

$$A_a, A_b;\ B_a, B_b;\ C_a, C_b;\ D_a, D_b,$$

where the subscripts indicate which branch is being followed. The first-step rules are

$$\begin{aligned} A_a &= x + xB_a + uxD_b, & A_b &= xB_b + uxD_a, \\ B_a &= ux + ux(C_a + C_b), & B_b &= x + ux(C_a + C_b), \\ C_a &= ux + uxB_a + uxD_b, & C_b &= uxB_b + uxD_a, \\ D_a &= x + uxB_a + uxD_b, & D_b &= uxB_b + uxD_a. \end{aligned}$$

The target generating function is

$$\Psi(x, u) = xA_a + xA_b.$$

**Takeaway.** For $\Theta$ we need four states; for $\Psi$ we need eight states. The point is to encode the previous step, because the penalty depends on local geometry.

# Explicit level-1 formulas on the gasket

Solving the state equations gives the closed forms

$$\Theta(x,u) = \frac{2ux^2}{(1+ux)(1-2ux)}\Big\{1+2(1-u^2)x^2 - 2(1-u)^2ux^3\Big\},$$

$$\Psi(x,u) = \frac{x^2}{(1+ux)(1-2ux)}\Big\{1+(1+u)x - u(1-u^2)x^2 + 2(1-u)^2u^2x^3\Big\},$$

and therefore

$$\Phi(x,u) = \frac{x^2\{1+(1+u)x - u(1-u^2)x^2 + 2(1-u)^2u^2x^3\}}{(1+ux)(1-2ux) - 4u^2x^2\{1+2(1-u^2)x^2 - 2u(1-u)^2x^3\}}.$$

**Interpretation.** The level-1 map is already a rational renormalization map in the single variable $x$ once $u$ is fixed.

# Self-repelling walk: renormalization across scales

The crucial composition rule is

$$\Phi_n(x,u) = \Phi_m\big(\Phi_{n-m}(x,u),u\big), \qquad m < n,$$

and in particular

$$\Phi_{n+1}(x,u) = \Phi\big(\Phi_n(x,u),u\big).$$

For each $u \in [0,1]$ there is a unique positive fixed point $x_u$ with

$$\Phi(x_u,u) = x_u, \qquad \lambda_u := \partial_x \Phi(x_u,u) > 2.$$

At the two endpoints,

$$x_1 = \frac{1}{4}, \quad \lambda_1 = 5, \qquad x_0 = \frac{\sqrt{5}-1}{2}, \quad \lambda_0 = \frac{7-\sqrt{5}}{2}.$$

**Takeaway.** The single number $\lambda_u$ is the time-scaling factor. It varies continuously from the self-avoiding endpoint to the Brownian endpoint.

# Projective consistency for self-repelling laws

Let $\widetilde{P}_n^u(x)$ be the weighted path law on $W_n$ defined by the partition function $\Phi_n(x,u)$. The decimation map satisfies

$$Q_m \widetilde{P}_n^u(x) = \widetilde{P}_m^u\big(\Phi_{n-m}(x,u)\big).$$

Hence, at the fixed point $x = x_u$,

$$Q_m \widetilde{P}_n^u(x_u) = \widetilde{P}_m^u(x_u), \qquad m \leq n.$$

So one may apply Kolmogorov's extension theorem and obtain a random projective family

$$Y_1, Y_2, \ldots, \qquad Q_m Y_n = Y_m.$$

**Meaning.** The level-$n$ path is a refinement of the level-$(n-1)$ path, not a different object. The remaining task is to understand the random traversal time attached to each refined edge.

# Branching time structure and convergence for the self-repelling family

For each coarse edge in the level-$m$ skeleton, let

$$S_i^m(Y_{m+n}) = T_i^m(Y_{m+n}) - T_{i-1}^m(Y_{m+n})$$

be the number of fine steps used to cross it. Conditionally on $Y_m$, the process

$$\{S_i^m(Y_{m+n})\}_{n\geq 0}$$

is a supercritical branching process with mean $\lambda_u$. Therefore,

$$\lambda_u^{-n} S_i^m(Y_{m+n}) \longrightarrow S_i^{*m} \quad \text{a.s. and in } L^2.$$

If we rescale time by

$$X_n^{(u)}(t) = Y_n(\lambda_u^n t),$$

then $X_n^{(u)}$ converges uniformly almost surely to a continuous limit process $X^{(u)}$.

**Endpoint picture.**

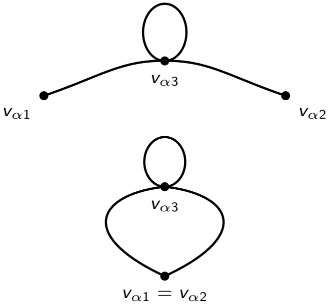
$$X^{(1)} = \text{Brownian motion}, \qquad X^{(0)} = \text{self-avoiding process}.$$

# A unified proof scheme for the three gasket processes

| Step | What happens |
|---|---|
| Choose discrete law | SRW, self-avoiding, or self-repelling weighted walk on $G_n$. |
| Projective consistency | Show the decimated law on $G_m$ matches the $m$-level law after renormalization; at the fixed point it becomes exact. |
| Random time | One coarse traversal breaks into a branching family of finer traversals. The mean offspring is 5 for Brownian motion, $\lambda_0$ for self-avoiding, and $\lambda_u$ for the general self-repelling model. |
| Uniform convergence | Once the coarse skeleton and coarse crossing times converge, the mesh size $2^{-m}$ forces the full paths to converge in $C([0,\infty), F)$. |

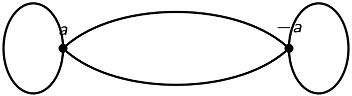
**Takeaway.** The difference between the three models is in the discrete weights. The convergence mechanism is the same: decimation + branching-time structure + time renormalization.

# The self-similar Basilica graph viewpoint: the underlying cell structure

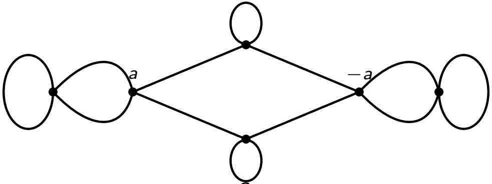



- An **arc-type cell** has two boundary points; a **loop-type cell** has one boundary point.
- Each arc-type cell is decomposed into two smaller arc-type cells meeting at a middle point and one loop-type cell attached at that same point.
- Each loop-type cell is decomposed into two arc-type cells plus one smaller loop-type cell.

# How the self-similar Basilica graph approximation is built




$G_0$




$G_1$

- For each arc-type $n$-cell, place one edge joining its two boundary vertices.
- For each loop-type $n$-cell, place one loop at its unique boundary vertex.
- The resulting finite multigraph is the self-similar Basilica graph $G_n$.

**General rule.** It remembers the cell structure but ignores the interior geometry of each cell; this is exactly what one needs for resistance forms.

# Graph energies and resistance forms on the self-similar Basilica graph

For a function $u$ on the vertex set $V_n$ of the self-similar Basilica graph, define the level-$n$ energy

$$\mathcal{E}_n(u) = \sum_{\alpha \in A_n} r_\alpha^{-1} \big( u(v_{\alpha 1}) - u(v_{\alpha 2}) \big)^2,$$

where $r_\alpha$ is the resistance assigned to the arc-type cell $J_\alpha$.

**Compatibility condition.** The trace of $\mathcal{E}_{n+1}$ on $V_n$ equals $\mathcal{E}_n$ if and only if

$$r_\alpha = r_{\alpha 1} + r_{\alpha 2}$$

for every arc-type cell.

This gives a compatible sequence of finite graph energies, hence a resistance form on

$$V_* = \bigcup_{n \geq 0} V_n.$$

After completion in the effective resistance metric, one obtains a local regular Dirichlet form and therefore a diffusion process.

# The local resistance metric and the analytic limit

The cell resistances also define a geodesic metric

$$S(x,y) = \inf\Big\{ \sum_j r_{\alpha_j} : \text{a chain of cells joins } x \text{ to } y \Big\}.$$

One proves that the effective resistance metric $R$ and the local resistance metric $S$ are comparable:

$$\frac{1}{2} S(x,y) \leq R(x,y) \leq S(x,y).$$

Therefore the resistance completion agrees with the Basilica Julia set as soon as the cell diameters shrink to 0.

**Analytic output.**

- a local regular Dirichlet form $\mathcal{E}$,
- a weak Laplacian defined by

$$\mathcal{E}(u,v) = -\int (\Delta u)\, v \, d\mu,$$

- and hence a diffusion process on the Basilica Julia set.

# Self-similar harmonic structure on the self-similar Basilica graph

Choose the resistances and measure by

$$r_\alpha = 2^{-|\alpha|}, \qquad \mu_B(J_\alpha) = \frac{1}{4 \cdot 3^{n-1}} \quad (\alpha \in A_n).$$

Then the graph Laplacians on the self-similar Basilica graphs satisfy a renormalized limit of the form

$$-\Delta u = c \lim_{n \to \infty} 6^n \Delta_n u,$$

with a normalization constant $c > 0$.

So the time scale multiplies by 6 when the spatial scale multiplies by 2. This gives the walk dimension

$$d_{\mathrm{w}} = \frac{\log 6}{\log 2}.$$

The corresponding Weyl exponent is $\log 3 / \log 6$, hence

$$d_{\mathrm{s}} = 2\frac{\log 3}{\log 6} = \frac{\log 9}{\log 6}.$$

# Spectral decimation on the self-similar Basilica graph

For the self-similar random walk on one cell, the transition matrix has block Schur complement

$$S_n(z) = \phi(z)\big(M_{n-1} - R(z)\big).$$

For the symmetric choice $p = q = \frac{1}{4}$ one gets the scalar decimation map

$$R(z) = 6z - 4z^2.$$

The eigenfunction extension rule on one cell is

$$u(v_{\alpha 3}) = \frac{p}{2p - z}\big(u(v_{\alpha 1}) + u(v_{\alpha 2})\big).$$

Thus the spectrum on $G_n$ is generated by repeated inverse images under $R$.

**Consequences.**

- explicit recursive description of eigenvalues and eigenfunctions;
- pure point integrated density of states on the graph side;
- direct access to the Weyl exponent and therefore to $d_{\mathrm{s}}$ and $d_{\mathrm{w}}$.

# Conformally invariant harmonic structure on the self-similar Basilica graph

Now require the Dirichlet form to be compatible with the dynamics of

$$P(z) = z^2 - 1.$$

Self-similarity under $P$ forces the resistance scaling

$$r_{P(J_\alpha)} = \rho \, r_{J_\alpha}, \qquad \rho = \sqrt{2}.$$

The corresponding Laplacian scales by the factor

$$2\sqrt{2}.$$

The eigenvalues are no longer given by an explicit spectral-decimation formula, but the graph-directed renewal argument yields

$$N(\lambda) \asymp \lambda^{2/3}.$$

Therefore

$$d_{\mathrm{s}} = \frac{4}{3}, \qquad d_{\mathrm{w}} = 3.$$

The resistance metric dimension is 2, so in this structure the Laplacian behaves like an operator of order 3 rather than order 2.

# Self-similar Basilica graph: the main analytic quantities

| Structure | Main scaling relation | $d_{\mathrm{s}}$ | $d_{\mathrm{w}}$ |
|---|---|---|---|
| Self-similar Basilica graph | $-\Delta \sim 6^n \Delta_n$; spectral decimation $R(z) = 6z - 4z^2$ | $\log 9 / \log 6$ | $\log 6 / \log 2$ |
| Conformally invariant Basilica graph | $P$-invariant Dirichlet form; Laplacian scaling $2\sqrt{2}$ | $4/3$ | $3$ |

# A path model on the self-similar Basilica graph family

Besides the analytic Dirichlet-form picture, one can also place a penalized walk model on the same recursive self-similar Basilica graphs.

For $G_n$ with boundary vertices $L_n, R_n$, let $T_n$ be the first hitting time of $R_n$ starting from $L_n$, and let $N_n$ count penalized events. We set

$$\Phi_n(x,u) := \mathbb{E}_{L_n}\left[x^{T_n} u^{N_n}\right], \qquad x \geq 0, \quad u \in [0,1].$$

**Penalty convention.**

- A return to $L_n$ before hitting $R_n$ is penalized by a factor $u$.
- In the model, every traversal of a loop is also penalized by $u$.

$$\Phi_0(x,u) = x.$$

For $G_1$ the midpoint $U_1$ has a loop, and first-step decomposition gives

$$f_L = x f_U, \qquad f_U = x\left(\tfrac{1}{3}\cdot 1 + \tfrac{1}{3}\, u f_L + \tfrac{1}{3}\, u f_U\right),$$

so

$$\Phi_1(x,u) = \frac{x^2}{3 - ux - ux^2}.$$

# Self-similar Basilica graph: renewal identity and scalar decimation

For each $n$, define two one-shot kernels:

$$A_n(x,u) = \text{weight of paths from } L_n \text{ hitting } R_n \text{ before returning to } L_n,$$

$$B_n(x,u) = \text{weight of paths from } L_n \text{ returning to } L_n \text{ before hitting } R_n.$$

Then renewal at the left boundary gives

$$\Phi_n(x,u) = \frac{A_n(x,u)}{1 - B_n(x,u)}.$$

For the self-similar Basilica recursion, these two kernels collapse to a scalar iteration map:

$$\Phi_{n+1}(x,u) = \varphi_u\big(\Phi_n(x,u)\big), \qquad \varphi_u(z) = \frac{z^2}{4 - 2uz - uz^2}.$$

In particular,

$$\Phi_2 = \varphi_u(\Phi_1), \qquad \Phi_3 = \varphi_u(\Phi_2).$$

**Interpretation.** Passing from level $n$ to level $n+1$ means that a coarse crossing of the self-similar Basilica graph is replaced by two subcrossings, with the denominator recording the possibility of penalized returns inside the cell.

# First explicit iterates for the self-similar Basilica decimation

Starting from $\Phi_1(x,u) = \dfrac{x^2}{3 - ux - ux^2}$, the scalar map gives

$$\Phi_2(x,u) = \frac{x^4}{12 - 4ux - 8ux^2 + u^2x^2 + 2u^2x^3 - ux^4}.$$

A longer but still explicit computation yields

$$\Phi_3(x,u) = \frac{x^8}{D_3(x,u)},$$

where

$$\begin{aligned} D_3(x,u) = 5184 - u\,(6912x + 8640x^2 + 360x^4 + x^8) \\ + u^2(3456x^2 + 8640x^3 + 5328x^4 + 240x^5 + 300x^6 + 6x^8) \\ - u^3(768x^3 + 2880x^4 + 3552x^5 + 1480x^6 + 100x^7 + 60x^8) \\ + u^4(64x^4 + 320x^5 + 592x^6 + 480x^7 + 144x^8). \end{aligned}$$

The key structural point is not the exact denominator, but the fact that the same one-variable map $\varphi_u$ reproduces every level:

$$\Phi_{n+1} = \varphi_u(\Phi_n).$$

# Self-similar Basilica decimation: fixed point and walk dimension

The scalar map

$$\varphi_u(z) = \frac{z^2}{4 - 2uz - uz^2}$$

has a unique nonzero fixed point $x_u > 0$ for $u \in (0, 1]$, namely

$$x_u = \frac{-(2u+1) + \sqrt{4u^2 + 20u + 1}}{2u}.$$

Linearizing at $x_u$ gives the time-scaling factor

$$\lambda_u = \varphi_u'(x_u) = 1 + \sqrt{4u^2 + 20u + 1}.$$

Assuming the decimation picture determines the macroscopic time scaling, the predicted walk dimension is

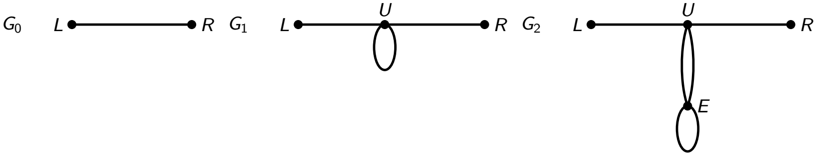
$$d_w(u) = \log_2 \lambda_u = \frac{\log\left(1 + \sqrt{4u^2 + 20u + 1}\right)}{\log 2}.$$

Hence

$$d_w(0) = 1, \qquad d_w(1) = \log_2 6 \approx 2.585.$$

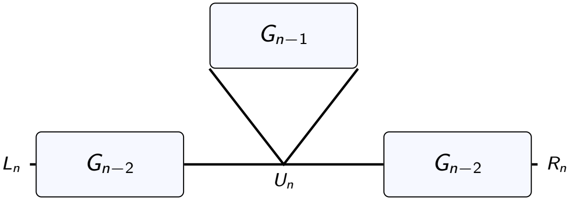
So even in this one-variable toy model, the fixed point of the renormalization map already encodes the large-scale time growth.

# Decomposed Schreier graphs: first few levels




- $G_0$: one edge from $L$ to $R$.
- $G_1$: the midpoint $U$ carries a level-1 loop.
- $G_2$: the lower vertex $E$ carries a level-1 loop and is connected to $U$ by two parallel edges.

# General recursive construction of the decomposed Schreier graph family




For $n \geq 3$, the graph $G_n$ is obtained by gluing

- one core copy of $G_{n-1}$, and
- two arms, each a copy of $G_{n-2}$,

at a single cut vertex $U_n$.

So one should think of

$$G_n = G_{n-2}^{\text{left}} \cup G_{n-1}^{\text{core}} \cup G_{n-2}^{\text{right}}$$

with all four interior boundary points identified at $U_n$. The geometry is still recursive.

# Loop-penalized walk on the decomposed Schreier graphs: definition of the weight

For a path $\gamma$ from $L_n$ to $R_n$, define
$$\Phi_n(x,u) = \sum_{\gamma: L_n \to R_n} x^{|\gamma|} u^{N(\gamma)},$$
where $|\gamma|$ is the length and $N(\gamma)$ counts all **penalized steps**.

In the model, the penalized steps are:

- every return step to $L_n$ after time 0 (restart penalty),
- every return to the currently tracked cut vertex during the same excursion,
- every step leaving a level-1 loop-vertex.

If a loop-vertex $v$ has parent $p$, two edges $v \to p$, and one self-loop $v \to v$, then
$$f_v(x,u) = x\left(\frac{2}{3}\, u f_p(x,u) + \frac{1}{3}\, u f_v(x,u)\right),$$
because *all three outgoing edges* from $v$ now carry the factor $u$.

# Base computations for $G_1$ and $G_2$

For $G_1$ the loop at $U$ is penalized:

$$f_L = xf_U, \qquad f_U = x\left(\frac{1}{3}\cdot 1 + \frac{1}{3}(uf_L) + \frac{1}{3}(uf_U)\right).$$

Hence

$$\Phi_1(x,u) = f_L(x,u) = \frac{x^2}{3 - ux - ux^2}.$$

For $G_2$ the lower loop-vertex $E$ has two penalized returns to $U$ and one penalized self-loop:

$$f_L = xf_U, \qquad f_U = x\left(\frac{1}{4}\cdot 1 + \frac{1}{4}(uf_L) + \frac{1}{2}f_E\right),$$

$$f_E = x\left(\frac{2}{3}(uf_U) + \frac{1}{3}(uf_E)\right).$$

Therefore

$$\Phi_2(x,u) = \frac{x^2(3 - ux)}{12 - 4ux - 7ux^2 + u^2x^3}.$$

# State generating functions on decomposed Schreier graphs

For a finite multigraph $G$ with left boundary $L$ and right boundary $R$, let

$$T = \inf\{t \geq 0 : X_t = R\}$$

be the first hitting time of $R$, and let $N$ be the number of penalized steps before time $T$. For a state $(v, s)$, define

$$f_{v,s}(x, u) = \mathbb{E}_{(v,s)}\left[x^T u^N\right], \qquad f_{R,\sigma}(x, u) = 1.$$

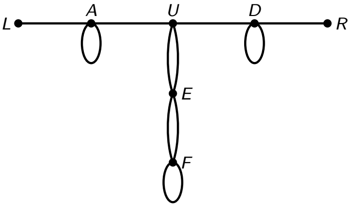
The left-to-right partition function is

$$\Phi_n(x, u) = f_{L_n,\mathrm{init}}(x, u).$$

The first-step decomposition is always

$$f_{v,s}(x, u) = \sum_{(w,s')} p((v, s), (w, s'))\, x\, u^{\mathbf{1}\{\text{step is penalized}\}} f_{w,s'}(x, u).$$

# Why $G_3$ already needs memory




For $G_3$ the penalty is no longer determined by the current vertex alone. We need a memory state $s \in \{0, 1\}$:

- $s = 0$: since the last visit to $L$, the walk has not yet visited $U$;
- $s = 1$: since the last visit to $L$, the walk has already visited $U$.

Penalized events are now:

1. arrivals to $L$ after time 0 (this also resets $s \leftarrow 0$),
2. arrivals to $U$ while $s = 1$, and
3. every step leaving the loop-vertex $F$.

In particular,

$$f_{F,1} = x\left(\frac{2}{3}\left(uf_{E,1}\right) + \frac{1}{3}\left(uf_{F,1}\right)\right).$$

# Key $G_3$ state equations in the all-loop-penalized model

The relevant states are $(L,0),(A,0)$ before the first visit to $U$, and then $(U,1),(A,1),(D,1),(E,1),(F,1)$. The first-step equations begin with

$$f_{L,0} = x f_{A,0}, \qquad f_{A,0} = x\left(\frac{1}{3}(u f_{L,0}) + \frac{1}{3} f_{U,1} + \frac{1}{3}(u f_{A,0})\right),$$

$$f_{U,1} = x\left(\frac{1}{4} f_{A,1} + \frac{1}{4} f_{D,1} + \frac{1}{2} f_{E,1}\right), \qquad f_{A,1} = x\left(\frac{1}{3}(u f_{L,0}) + \frac{1}{3}(u f_{U,1}) + \frac{1}{3}(u f_{A,1})\right),$$

$$f_{D,1} = x\left(\frac{1}{3}\cdot 1 + \frac{1}{3}(u f_{U,1}) + \frac{1}{3}(u f_{D,1})\right), \qquad f_{E,1} = x\left(\frac{1}{2}(u f_{U,1}) + \frac{1}{2} f_{F,1}\right),$$

$$f_{F,1} = x\left(\frac{2}{3}(u f_{E,1}) + \frac{1}{3}(u f_{F,1})\right).$$

These equations solve to a rational function $\Phi_3(x,u)$ and agree with the renormalization recursion below.

# One-shot kernels and the renewal identity

For the decomposed Schreier graph $G_n$, define the one-shot kernels

$$A_n(x,u) = \sum_{w:w_0=L_n} W_n(w;x,u)\,\mathbf{1}\{w \text{ hits } R_n \text{ before returning to } L_n\},$$

$$B_n(x,u) = \sum_{w:w_0=L_n} W_n(w;x,u)\,\mathbf{1}\{w \text{ returns to } L_n \text{ before hitting } R_n\},$$

where

$$W_n(w;x,u) = P_{\mathrm{SRW}}(w)\,x^{|w|}\,u^{N(w)}.$$

The full partition function satisfies the renewal identity

$$\Phi_n(x,u) = \frac{A_n(x,u)}{1 - B_n(x,u)}.$$

**Meaning.** Either the path succeeds on its first excursion from $L_n$ (weight $A_n$), or it fails and returns to $L_n$ once (weight $B_n$), after which the process restarts.

# Safe-return kernel at the cut vertex

At the cut vertex $U_n$, one macro-excursion can do three things:

- return safely to $U_n$,
- get absorbed at $R_n$,
- get absorbed at $L_n$.

The safe-return kernel is

$$Q_n(x,u) = \frac{B_{n-2}(x,u)}{2} + \frac{B_{n-1}(x,u) + uA_{n-1}(x,u)}{2}.$$

The one-shot absorption weights from $U_n$ are

$$P_{R,n}(x,u) = \frac{A_{n-2}(x,u)}{4}, \qquad P_{L,n}(x,u) = \frac{uA_{n-2}(x,u)}{4}.$$

Therefore repeated safe returns sum geometrically, exactly as in the previous model.

# Renormalization theorem: unchanged recursion, new base data

For $n \geq 3$, the one-shot kernels satisfy

$$Q_n(x,u) = \frac{B_{n-2}(x,u)}{2} + \frac{B_{n-1}(x,u) + uA_{n-1}(x,u)}{2},$$

$$A_n(x,u) = \frac{A_{n-2}(x,u)^2}{4\big(1 - Q_n(x,u)\big)}, \qquad B_n(x,u) = B_{n-2}(x,u) + uA_n(x,u),$$

so that

$$\Phi_n(x,u) = \frac{A_n(x,u)}{1 - B_n(x,u)}.$$

The initial data are

$$A_0 = x, \quad B_0 = 0, \qquad A_1 = \frac{x^2}{3 - ux}, \quad B_1 = uA_1,$$

$$A_2 = \frac{x^2(3 - ux)}{4(3 - ux - ux^2)}, \qquad B_2 = uA_2.$$

# Low-level rational functions and verification of the recursion

Solving the corrected state equations on $G_3$ gives

$$\Phi_3(x,u) = \frac{x^4}{D_3(x,u)},$$

with

$$D_3(x,u) = 36 - 24ux - 27ux^2 + 4u^2x^2 + 12u^2x^3 + (2u^2 - u - u^3)x^4.$$

For $G_4$ one again obtains an explicit rational function,

$$\Phi_4(x,u) = \frac{x^4(3-ux)^3 P_4(x,u)}{D_4(x,u)},$$

where $P_4, D_4$ are explicit polynomials.

**Check.** The state-equation solutions and the recursive formulas agree for $n = 0, 1, 2, 3, 4, 5$.

# SRW endpoint on decomposed Schreier graphs: the walk dimension

At $u = 1$ the new model coincides with ordinary simple random walk, so $a_n := A_n(1,1)$ and $b_n := B_n(1,1)$ satisfy

$$a_n + b_n = 1.$$

Differentiating at $x = 1$ and setting

$$\alpha_n = \partial_x A_n(x,1)|_{x=1}, \qquad \beta_n = \partial_x B_n(x,1)|_{x=1}, \qquad s_n = \alpha_n + \beta_n,$$

one obtains the linear recurrence

$$s_n = s_{n-1} + 2s_{n-2}, \qquad s_0 = 1, \quad s_1 = \frac{5}{2}.$$

Hence

$$s_n = \frac{7}{6}\,2^n - \frac{1}{6}(-1)^n.$$

Because the linear size doubles when $n \mapsto n+2$, while the mean crossing time grows by a factor 8, the SRW walk dimension is

$$d_{\mathrm{w}}(1) = \frac{\log 8}{\log 2} = 3.$$

# Positive-branch quantities that control the generating functions

The natural singularity and fixed-point quantities are

$$x_{B,n}(u): \; B_n(x,u) = 1, \qquad x_{Q,n}(u): \; Q_n(x,u) = 1, \qquad x_{f,n}(u): \; \Phi_n(x,u) = x.$$

We also define the $B$-admissible set

$$D_B(u) = \{x > 0 : B_n(x,u) < 1 \text{ for all } n\}, \qquad x_B^c(u) = \sup D_B(u).$$

The point of these definitions is:

- $B_n < 1$ keeps the renewal denominator $1 - B_n$ positive,
- on the positive branch, $B_n < 1$ also forces $Q_n < 1$,
- because the Basilica recursion is two-step, even and odd subsequences may behave differently.

So the analytic study of the model is really a study of how $x_{B,n}, x_{Q,n}, x_{f,n}$ move with $n$ and with $u$.

# A useful renormalized set of variables

For the two-step recursion it is convenient to write

$$E_k = uA_{2k}, \qquad O_k = uA_{2k+1}, \qquad S_k = 1 - \frac{B_{2k} + B_{2k+1}}{2}, \qquad \theta_k = \frac{E_{k+1}}{S_k}.$$

Then the recursion gives the exact identities

$$1 - Q_{2k+2} = S_k - \frac{O_k}{2}, \qquad 1 - Q_{2k+3} = S_k - E_{k+1} = S_k(1 - \theta_k).$$

Moreover,

$$\theta_{k+1} \geq F_u(\theta_k) := \frac{\theta_k^2}{u(2 - \theta_k)^2}.$$

The nonzero fixed point of $F_u$ is

$$\theta_*(u) = \frac{(4u+1) - \sqrt{1+8u}}{2u}, \qquad C_*(u) = \frac{1}{1 - \theta_*(u)}.$$

These renormalized variables measure how close the even and odd return kernels are to criticality; they are the key quantities in odd-gap estimates and in the search for ballistic scaling.

# Conjectural phase transition and the research strategy

The current picture suggests a sharp change at $u = 1$:

$$\textbf{Conjecture:} \qquad d_{\mathrm{w}}(u) = 1 \text{ for every } 0 < u < 1, \qquad d_{\mathrm{w}}(1) = 3.$$

The idea is that as soon as loops are penalized, the walk should become essentially ballistic across scales.

Natural quantities to control are

$$\epsilon_k(x,u) = 1 - B_{2k}(x,u), \qquad r_k(x,u) = \frac{A_{2k+1}(x,u)}{A_{2k}(x,u)}.$$

If one can prove

- a uniform odd gap $B_{2k+1}(x,u) \le 1 - \delta(u)$ on the admissible branch, and
- the decay $r_k(x,u) \to 0$,

then the even gaps should satisfy an asymptotically quadratic recursion, giving the time scaling factor

$$\lambda(u) = 2$$

and hence

$$d_{\mathrm{w}}(u) = \frac{\log 2}{\log 2} = 1.$$

# Big picture: what changes from one model to another?

| Model | Main object | Main renormalization variable |
|---|---|---|
| Gasket Brownian motion | random paths / diffusion | branching of coarse crossing times |
| Gasket self-avoiding / self-repelling | weighted paths | generating function $\Phi$ and its fixed point $x_u$ |
| Self-similar Basilica graph | Dirichlet form / Laplacian | cell resistances, graph Laplacians, spectral decimation |
| Loop-penalized Schreier-graph model | weighted SRW with memory | one-shot kernels $(A_n, B_n)$, safe-return kernel $Q_n$, and critical points |

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Thank you!