

# Spectral decimation of piecewise centrosymmetric Jacobi operators on graphs

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A. Teplyaev

UCONN

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# Outline

- 1 Introduction
- 2 Orthogonal polynomials and centrosymmetric Jacobi matrices
- 3 Spectral decimation of piecewise centrosymmetric Jacobi matrices on graphs
- 4 The one-dimensional path graphs case
- 5 Applications to almost Mathieu-type operators

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# Jacobi operators on graphs

## Definition

A Jacobi operator on a graph  $G = (V(G), E(G))$  is a matrix  $\mathbf{J} = (\mathbf{J}(x, y))_{x, y \in V(G)}$  indexed by the vertices  $V(G)$ , such that  $\mathbf{J}(x, y) = 0$  whenever  $(x, y) \notin E(G)$  and  $x \neq y$ . We denote the set of all Jacobi matrices on graphs by

$$\mathcal{J} := \{ \mathbf{J} \mid \mathbf{J} \text{ is a Jacobi matrix on a graph} \}. \quad (1)$$

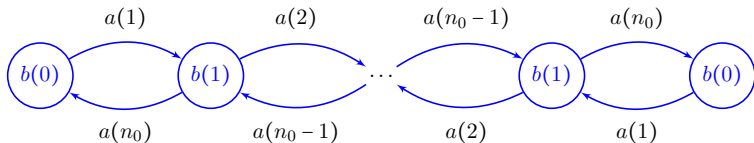
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<sup>0</sup>N. Avni, J. Breuer, and B. Simon. Periodic Jacobi matrices on trees. *Adv. Math.*, 370:107241, 42, 2020.

# The building block graph $G_{CS}$

- $\mathbf{J}_{CS}$  finite tridiagonal matrix, centrosymmetric, i.e.  
 $\mathbf{J}_{CS} = R \mathbf{J}_{CS} R$ , where  $R$  is the anti-diagonal identity matrix.
- $G_{CS} = (V(G_{CS}), E(G_{CS}))$  is a finite directed weighted path graph.
- $\mathbf{J}_{CS}$  as the weight matrix of  $G_{CS}$ .

$$\mathbf{J}_{CS} := \begin{pmatrix} b(0) & a(1) & 0 & \dots & 0 \\ a(n_0) & b(1) & a(2) & \dots & 0 \\ 0 & a(n_0 - 1) & b(2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a(n_0) \\ 0 & 0 & 0 & a(1) & b(0) \end{pmatrix}$$



# The model graph $G_p$

- $G_p = (V(G_p), E(G_p))$  is a locally finite connected directed graph.
- Random walk on  $G_p$ , i.e. the weight matrix of  $G_p$  is given by a *probabilistic graph Laplacian*

$$\Delta_{\mathbf{p}} f(x) = f(x) - \sum_{y:(x,y) \in E(G_p)} p(x,y) f(y),$$

where  $\mathbf{p} = \{p(x,y)\}_{(x,y) \in E(G_p)}$  is a given sequence of transition probabilities.

- Symmetric random walk on  $G_p$ , then  $\Delta_{\mathbf{p}}$  becomes the standard probabilistic graph Laplacian.

## Substitution graph and substitution operators

- We fix the graph  $G_p$ . The collection of all probabilistic Laplacians on  $G_p$  is denoted by  $\mathcal{L}_{G_p}$ .
- We denote the class of centrosymmetric matrices by  $\mathcal{CSJ}$ .
- Given a building block  $G_{cs}$ , we construct a new graph  $G = (V(G), E(G))$  by substituting copies of  $G_{cs}$  between adjacent vertices in  $G_p$ .
- $G$  is called a substitution graph.
- For a choice of the probabilistic Laplacian  $\Delta_p$  of  $G_p$ , the weight matrix of  $G$  is given by a *substitution operator*

$$F_{G_p} : \mathcal{L}_{G_p} \times \mathcal{CSJ} \rightarrow \mathcal{J}$$

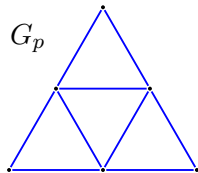
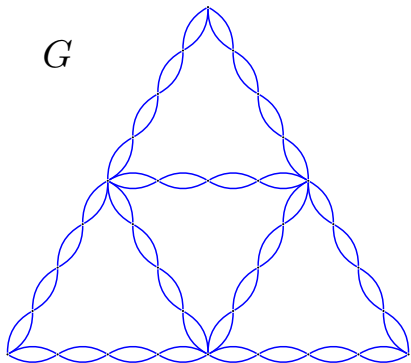
$$(\Delta_p, \mathbf{J}_{cs}) \mapsto F_{G_p}(\Delta_p, \mathbf{J}_{cs})$$

- $F_{G_p}(\Delta_p, \mathbf{J}_c)$  is a Jacobi operator on the graph  $G$ .

# Jacobi operators on a Sierpinski lattice

- $G_{cs}$  is a finite directed weighted path graph associated with a centrosymmetric Jacobi matrix  $\mathbf{J}_{cs}$
- $G_p$  is a finite graph approximation of a Sierpinski lattice.
- The substitution graph  $G$  is constructed by substituting copies of the graph  $G_{cs}$  between adjacent vertices in  $G_p$ .

 $G_{cs}$ 

 $G_p$ 

 $G$ 


# Substitution Operator

How is the weight matrix  $F_{G_p}(\Delta_p, \mathbf{J}_{cs})$  defined?

## Definition (Substitution Operator)

- The substitution procedure naturally induces a *covering map*, which we denote by  $\varphi : V(G) \rightarrow V(G_{cs})$
- We denote shortly  $\mathbf{J} := F_{G_p}(\Delta_p, \mathbf{J}_c)$ . The diagonal entries are given by

$$\mathbf{J}(x, x) := \mathbf{J}_{cs}(\varphi(x), \varphi(x)), \quad x \in V(G).$$

- Each  $(x, y) \in E(G)$  is associated with an edge  $(\varphi(x), \varphi(y))$  in a copy of  $G_{cs}$  and suppose that this copy replaced the edges between the vertices, say  $u, v \in V(G_p)$ . Then we set

$$\begin{cases} \mathbf{J}(x, y) := p(u, v) \mathbf{J}_{cs}(0, 1), & \text{if } x = u, \\ \mathbf{J}(x, y) := \mathbf{J}_{cs}(\varphi(x), \varphi(y)), & \text{if } x \notin V(G_p). \end{cases}$$

# Piecewise centrosymmetric Jacobi operators

- We call the operator  $\mathbf{J} := F_{G_p}(\Delta_{\mathbf{p}}, \mathbf{J}_c)$  a piecewise centrosymmetric Jacobi operator on  $G$ .
- The goal of this project is to study the spectral theory of these piecewise centrosymmetric Jacobi operators.
- Our main result proves that the spectral theory of a piecewise centrosymmetric Jacobi operator  $\mathbf{J}$  can be explicitly related to the spectral theory of the probabilistic Laplacian  $\Delta_{\mathbf{p}}$  using certain orthogonal polynomials.

# Main result

## Theorem

The following statements hold:

- (1)  $R_{\mathbf{J}_{cs}}^{-1}(\sigma(\Delta_{\mathbf{p}}) \setminus \{0, 2\}) \subset \sigma(\mathbf{J}) \subset R_{\mathbf{J}_{cs}}^{-1}(\sigma(\Delta_{\mathbf{p}}) \cup \{0, 2\})$ .
- (2) The resolvent operators satisfy the following renormalization identity

$$U^*(zI - \mathbf{J})^{-1}U = -\frac{P_{n_0-1}^D(z)}{\prod_{i=1}^{n_0} a(i)} (R_{\mathbf{J}_{cs}}(z)I - \Delta_{\mathbf{p}})^{-1}$$

for  $z \in \rho(\mathbf{J})$  and  $z \notin \mathcal{E}_{\mathbf{J}_{cs}} = \sigma(\mathbf{J}_{cs}^D)$ .

- (3) Polynomial  $R_{\mathbf{J}_{cs}}$  gives an isomorphism of the types of the spectral measures of the operators  $\mathbf{J}$  and  $\Delta_{\mathbf{p}}$  outside of the finite exceptional set  $R_{\mathbf{J}_{cs}}^{-1}(\{0, 2\})$ .

The polynomials  $R_{\mathbf{J}_{cs}}$  and  $P_{n_0-1}^D(z)$  are given explicitly below.

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## Three-term recurrence relations

- $\mathbf{J}_{CS}$  is a Jacobi matrix, orthogonal polynomials naturally arise through the associated three-term recurrence relations.
- Two cases are relevant,  $\mathbf{J}_{CS}$  and the restriction of  $\mathbf{J}_{CS}$  to the interior vertices of  $G_{CS}$ , which we denote by  $\mathbf{J}_{CS}^D$ .

$$\mathbf{J}_{CS} = \left( \begin{array}{c|ccc|c} b(0) & a(1) & 0 & \dots & 0 \\ \hline \vdots & & \mathbf{J}_{CS}^D & & \vdots \\ \hline 0 & \dots & 0 & a(1) & b(0) \end{array} \right).$$

## Three-term recurrence relations

- We initialize  $P_0(z) := 1$  and  $P_1(z) := z - b(0)$ .
- For  $k \in \{2, \dots, n_0 + 1\}$ , we define  $P_k(z)$  to be the determinant of the leading principal  $k \times k$  submatrix of  $zI - \mathbf{J}_{cs}$ , i.e.

$$P_k(z) := \det \begin{pmatrix} z - b(0) & -a(1) & 0 & \dots & 0 \\ -a(n_0) & z - b(1) & -a(2) & \dots & 0 \\ 0 & -a(n_0 - 1) & z - b(2) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -a(k-1) \\ 0 & 0 & 0 & -a(n_0 + 2 - k) & z - b(k-1) \end{pmatrix}.$$

## Three-term recurrence relations

- The polynomials  $P_0(z), P_1(z), \dots, P_{n_0+1}(z)$  satisfy the following recurrence relations

$$\begin{cases} P_0(z) = 1, & P_1(z) = z - b(0) \\ P_k(z) = (z - b(k-1))P_{k-1}(z) - a(k-1)a(n_0+2-k)P_{k-2}(z), & k \in \{2, \dots, n_0+1\}. \end{cases}$$

- Similarly we define polynomials  $P_k^D(z)$  corresponding to the Jacobi matrix  $\mathbf{J}_{CS}^D$

Schur complement of  $\mathbf{J}_{cs}$ 

We decompose  $\mathbf{J}_{cs}$  on  $\ell(\{0, n_0\}) \oplus \ell(\{1, \dots, n_0 - 1\})$  in the block form

$$\begin{pmatrix} S_0 & \bar{X}_0 \\ X_0 & Q_0 \end{pmatrix} = \left( \begin{array}{cc|cccc} b(0) & 0 & a(1) & 0 & \dots & 0 \\ 0 & b(0) & 0 & 0 & \dots & a(1) \\ \hline a(n_0) & 0 & b(1) & a(2) & \dots & 0 \\ 0 & 0 & a(n_0 - 1) & b(2) & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & a(n_0 - 1) \\ 0 & a(n_0) & 0 & 0 & a(2) & b(1) \end{array} \right). \quad (2)$$

Schur complement of  $\mathbf{J}_{cs}$

$$\text{Schur}_{\ell(\partial G_{cs})}(\mathbf{J}_{cs}) := zI - S_0 - \bar{X}_0(zI - Q_0)^{-1}X_0.$$

## Schur complement of $\mathbf{J}_{cs}$

Our first result states that the Schur complement of  $\mathbf{J}_{cs}$  can be expressed using the polynomials  $P_{n_0}(z)$  and  $P_{n_0-1}^D(z)$ .

### Theorem

For each  $z \in \rho(\mathbf{J}_{cs}^D)$  we have

$$\text{Schur}_{\ell(\partial G_{cs})}(\mathbf{J}_{cs}) = \phi_{\mathbf{J}_{cs}}(z) \left( R_{\mathbf{J}_{cs}}(z) I - \Delta_0 \right), \quad (3)$$

where the functions  $R_{\mathbf{J}_{cs}}(z)$  and  $\phi_{\mathbf{J}_{cs}}(z)$  are given by

$$R_{\mathbf{J}_{cs}}(z) := 1 - \frac{P_{n_0}(z)}{\prod_{i=1}^{n_0} a(i)}, \quad \phi_{\mathbf{J}_{cs}}(z) := -\frac{\prod_{i=1}^{n_0} a(i)}{P_{n_0-1}^D(z)}.$$

Note  $\Delta_0$  denotes the trivial probabilistic graph Laplacian

$$\Delta_0 := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (4)$$

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## Spectral similarity of operators

- $\mathcal{H}, \mathcal{H}_0$  Hilbert spaces,  $U : \mathcal{H}_0 \rightarrow \mathcal{H}$  isometry.
- $H$  (resp.  $H_0$ ) bounded linear operator on  $\mathcal{H}$  (resp.  $\mathcal{H}_0$ )
- Let  $\phi, \psi$  be complex-valued functions.

### Definition (Malozemov, Teplyaev<sup>1</sup>)

We call the operator  $H$  *spectrally similar* to the operator  $H_0$  with functions  $\phi$  and  $\psi$  if

$$U^*(H - zI)^{-1}U = (\phi(z)H_0 - \psi(z)I)^{-1}, \quad (5)$$

for all  $z \in \mathbb{C}$  such that the two sides of the equation are well defined.

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<sup>1</sup>L. Malozemov and A. Teplyaev. Self-similarity, operators and dynamics. Math. Phys. Anal. Geom., 6(3):201–218, 2003.

# Spectral decimation function

- For  $z$  in the domain of both  $\phi$  and  $\psi$  and satisfying  $\phi(z) \neq 0$  we have  $z \in \rho(H)$  (the resolvent set of  $H$ ) if and only if

$$R(z) = \frac{\psi(z)}{\phi(z)} \in \rho(H_0), \text{ i.e.}$$

$$U^*(H - zI)^{-1}U = \frac{1}{\phi(z)} \left( H_0 - \frac{\psi(z)}{\phi(z)} I \right)^{-1}. \quad (6)$$

- We call  $R(z) = \frac{\psi(z)}{\phi(z)}$  the *spectral decimation function*.

## Characterization of spectral similarity

- Identify  $\mathcal{H}_0$  with a closed subspace of  $\mathcal{H}$  via the isometry  $U$ .
- $\mathcal{H}_1$  the orthogonal complement  $\mathcal{H}_0$ .
- Decompose  $H$  on  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  in the block form

$$H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}. \quad (7)$$

### Lemma (Malozemov, Teplyaev<sup>2</sup>)

For  $z \in \rho(H) \cap \rho(Q)$  the operators  $H$  and  $H_0$  are spectrally similar if and only if the Schur complement of  $H$  (with respect to the decomposition (7)), given by  $Schur_{\mathcal{H}_0}(H) := zI - S - \bar{X}(zI - Q)^{-1}X$ , satisfies

$$Schur_{\mathcal{H}_0}(H) = \psi(z)I - \phi(z)H_0. \quad (8)$$

<sup>2</sup>L. Malozemov and A. Teplyaev. Self-similarity, operators and dynamics. Math. Phys. Anal. Geom., 6(3):201–218, 2003.

## Why spectral similarity?

The *exceptional set* of  $H$  is given by  $\mathcal{E}_H := \{z \in \mathbb{C} \mid z \in \sigma(Q) \text{ or } \phi(z) = 0\}$

### Proposition (Malozemov, Teplyaev<sup>3</sup>)

Let  $H$  be spectrally similar to  $H_0$  with functions  $\phi$  and  $\psi$  and  $z \notin \mathcal{E}_H$ .  
Then

- ①  $R(z) \in \rho(H_0)$  if and only if  $z \in \rho(H)$ .
  - ②  $R(z)$  is an eigenvalue of  $H_0$  if and only if  $z$  is an eigenvalue of  $H$ .
- Moreover, there is a one-to-one map

$$f_0 \mapsto f := f_0 - (zI - Q)^{-1} X f_0 \quad (9)$$

from the eigenspace of  $H_0$  corresponding to  $R(z)$  onto the eigenspace of  $H$  corresponding to  $z$ .

<sup>3</sup>L. Malozemov and A. Teplyaev. Self-similarity, operators and dynamics. Math. Phys. Anal. Geom., 6(3):201–218, 2003.

# Schur complement of centrosymmetric Jacobi matrices

- Theorem 4 implies that a centrosymmetric Jacobi matrix is spectral similar to the trivial probabilistic graph Laplacian  $\Delta_0$  wrt.

$$R_{\mathbf{J}_{cs}}(z) := 1 - \frac{P_{n_0}(z)}{\prod_{i=1}^{n_0} a(i)}, \quad \phi_{\mathbf{J}_{cs}}(z) := -\frac{\prod_{i=1}^{n_0} a(i)}{P_{n_0-1}^D(z)}.$$

- In particular,  $\phi_{\mathbf{J}_{cs}}(z) \neq 0$  for all  $z$ .
- The block form (2) of  $\mathbf{J}_{cs}$  shows that  $Q_0 = \mathbf{J}_{cs}^D$ .
- Hence, the exceptional set  $\mathcal{E}_{\mathbf{J}_{cs}}^e = \sigma(\mathbf{J}_{cs}^D)$ .

## Spectral similarity between $\mathbf{J}$ and $\Delta_{\mathbf{p}}$

- Recall, we started with a centrosymmetric Jacobi matrix  $\mathbf{J}_{cs}$  (or equivalently  $G_{cs}$ ) and with a probabilistic graph Laplacian  $\Delta_{\mathbf{p}}$  (or equivalently  $G_{\mathbf{p}}$ ).
- We construct a new graph  $G = (V(G), E(G))$  by substituting copies of  $G_{cs}$  between adjacent vertices in  $G_{\mathbf{p}}$ .
- The weight matrix of  $G$  is given by  $\mathbf{J} = F_{G_{\mathbf{p}}}(\Delta_{\mathbf{p}}, \mathbf{J}_{cs})$ .

### Proposition

$\mathbf{J} = F_{G_{\mathbf{p}}}(\Delta_{\mathbf{p}}, \mathbf{J}_{cs})$  is spectrally similar to  $\Delta_{\mathbf{p}}$  with the same functions as in the previous slide, i.e. associated spectral decimation function and exceptional set coincide with  $R_{\mathbf{J}_{cs}}(z)$  and  $\mathcal{E}_{\mathbf{J}_{cs}}$ , respectively.

## Proof idea

- Schur complement is preserved under the substitution operator  $F_{G_p}$ , in the sense

$$F_{G_p}(\Delta_{\mathbf{p}}, \text{Schur}_{\ell(\partial G_{cs})}(\mathbf{J}_{cs})) = \text{Schur}_{\ell(V(G_p))}(\mathbf{J}).$$

- Simple properties of the substitution operator imply

$$\begin{aligned} \text{Schur}_{\ell(V(G_p))}(\mathbf{J}) &= \phi_{\mathbf{J}_{cs}}(z) F_{G_p}(\Delta_{\mathbf{p}}, R_{\mathbf{J}_{cs}}(z)I - \Delta_0) \\ &= \phi_{\mathbf{J}_{cs}}(z) (R_{\mathbf{J}_{cs}}(z) F_{G_p}(\Delta_{\mathbf{p}}, I) - F_{G_p}(\Delta_{\mathbf{p}}, \Delta_0)) \\ &= \phi_{\mathbf{J}_{cs}}(z) (R_{\mathbf{J}_{cs}}(z)I - \Delta_{\mathbf{p}}). \end{aligned}$$

## Remarks

For the proof, we assumed:

- ① There exists a Hilbert space of  $\mathbb{C}$ -valued functions on  $V(G_p)$ , which we denote by  $\ell^2(G_p, d\pi_p)$ , such that the probabilistic graph Laplacian  $\Delta_p$  is bounded and self-adjoint.
- ② There exists a Hilbert space of  $\mathbb{C}$ -valued functions on  $V(G)$ , which we denote by  $\ell^2(G, d\pi_J)$ , such that  $\mathbf{J}$  is bounded and self-adjoint.
- ③ There exists an isometry  $U : \ell^2(G_p, d\pi_p) \rightarrow \ell^2(G, d\pi_J)$ .

We constructed such Hilbert spaces in the one-dimensional path graphs case. The key idea is to equip the set of vertices  $V(G_p)$  and  $V(G)$  with a measure satisfying a Kolmogorov's cycle type condition.

# Summary

## Theorem

The following statements hold:

- (1)  $R_{\mathbf{J}_{cs}}^{-1}(\sigma(\Delta_{\mathbf{p}}) \setminus \{0, 2\}) \subset \sigma(\mathbf{J}) \subset R_{\mathbf{J}_{cs}}^{-1}(\sigma(\Delta_{\mathbf{p}}) \cup \{0, 2\})$ .
- (2) The resolvent operators satisfy the following renormalization identity

$$U^*(zI - \mathbf{J})^{-1}U = -\frac{P_{n_0-1}^D(z)}{\prod_{i=1}^{n_0} a(i)} (R_{\mathbf{J}_{cs}}(z)I - \Delta_{\mathbf{p}})^{-1}$$

for  $z \in \rho(\mathbf{J})$  and  $z \notin \mathcal{E}_{\mathbf{J}_{cs}} = \sigma(\mathbf{J}_{cs}^D)$ .

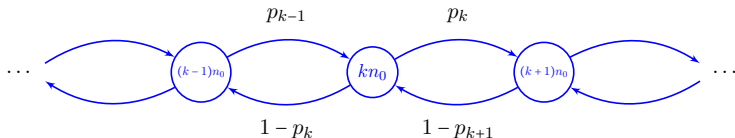
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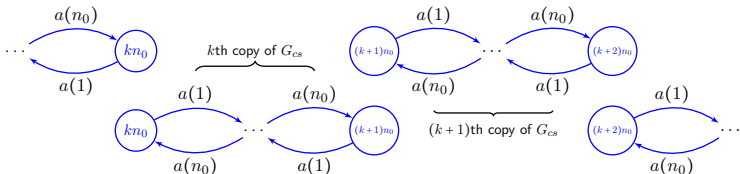
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# One-dimensional path graphs

- $\Delta_{\mathbf{p}}$  defined by the transition probabilities  $p_k$



- The substitution operator  $\mathbf{J} = F_{G_p}(\Delta_{\mathbf{p}}, \mathbf{J}_{cs})$



# Self-adjointness

- $\Delta_p$  is a bounded self-adjoint operator acting on

$$\left\{ \begin{array}{l} \ell^2(G_p, d\pi_p) := \left\{ f : V(G_p) \rightarrow \mathbb{C} \mid \sum_{x \in V(G_p)} |f(x)|^2 \pi_p(x) < \infty \right\}, \\ \langle f, g \rangle_p = \sum_{x \in V(G_p)} \overline{f(x)} g(x) \pi_p(x), \end{array} \right.$$

where the set of vertices  $V(G_p)$  is equipped with the following measures satisfying the Kolmogorov's cycle condition

$$\pi_p(0) = 1, \quad \pi_p(kn_0) = \pi_p((k-1)n_0) \frac{p_{k-1}}{p_k}, \quad kn_0 \in V(G_p) \setminus \{0\}. \quad (10)$$

- The same is done for  $\mathbf{J}$ .

# Spectrum of $\mathbf{J}$

## Proposition

Assume  $G_p$  is a finite path graph. Then

$$\sigma(\mathbf{J}) = (R_{\mathbf{J}_{cs}}^{-1}(\sigma(\Delta_p)) \setminus \{0, 2\}) \cup \sigma(\mathbf{J}_{cs})$$

.

# Multi-parameter family of self-similar probabilistic Laplacians

- A key observation: centrosymmetric probabilistic Laplacians are invariant under the substitution operator in the sense that if  $\Delta_{\mathbf{p}_1}, \Delta_{\mathbf{p}_2} \in \mathcal{CSJ}$  with  $\Delta_{\mathbf{p}_1} \in \mathcal{L}_{G_p}$  then  $F_{G_p}(\Delta_{\mathbf{p}_1}, \Delta_{\mathbf{p}_2}) \in \mathcal{CSJ} \cap \mathcal{L}_G$ .

## Definition

Let  $\Delta_{\mathbf{p}}^{(1)} \in \mathcal{CSJ} \cap \mathcal{L}_{G_p}$  be given. We define the sequence  $\{\Delta_{\mathbf{p}}^{(\ell)}\}_{\ell \in \mathbb{N}} \subset \mathcal{CSJ}$  inductively by  $\Delta_{\mathbf{p}}^{(\ell+1)} := F_{G_p}(\Delta_{\mathbf{p}}^{(1)}, \Delta_{\mathbf{p}}^{(\ell)})$  for  $\ell \geq 1$ . We refer to  $\{\Delta_{\mathbf{p}}^{(\ell)}\}_{\ell \in \mathbb{N}}$  as a sequence of self-similar probabilistic Laplacians.

# Chebyshev polynomial

- We initialize  $\Delta_{\mathbf{p}}^{(1)}$  in the sequence  $\Delta_{\mathbf{p}}^{(\ell+1)} := F_{G_p}(\Delta_{\mathbf{p}}^{(1)}, \Delta_{\mathbf{p}}^{(\ell)})$

$$\Delta_{\mathbf{p}}^{(1)} := \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & \dots & 0 \\ 0 & -\frac{1}{2} & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & -\frac{1}{2} \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \in \mathbb{R}^{k_0 \times k_0}.$$

- The corresponding spectral decimation function can be derived from a Chebyshev polynomial, i.e.

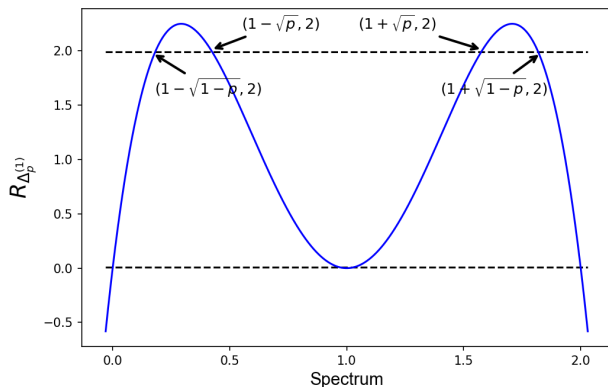
$$R_{\Delta_{\mathbf{p}}^{(1)}}(z) = 1 - (-1)^{k_0} \text{Cheb}_{k_0}(z - 1)$$

# One-parameter family of Laplacians

- We initialize  $\Delta_{\mathbf{p}}^{(1)}$  in the sequence  $\Delta_{\mathbf{p}}^{(\ell+1)} := F_{G_p}(\Delta_{\mathbf{p}}^{(1)}, \Delta_{\mathbf{p}}^{(\ell)})$

$$\Delta_{\mathbf{p}}^{(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ p-1 & 1 & -p & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -p & 1 & p-1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

## Spectrum



**Figure:** The spectral decimation function corresponding to the probabilistic Laplacians from the last slide. (It reduces to the Chebyshev polynomial case when  $p = \frac{1}{2}$ ).

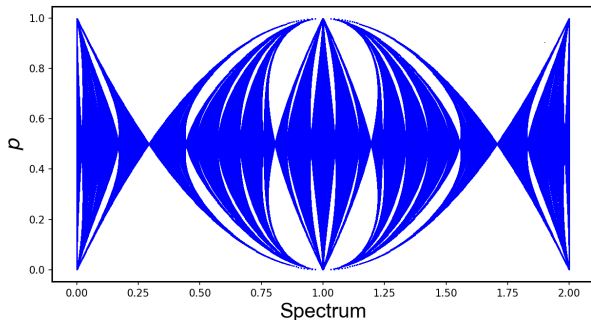
Spectrum for different  $p$ 

Figure: The spectrum of  $\Delta_p^{(\ell)}$  for  $\ell = 6$  (x-axis) is plotted for  $p \in (0, 1)$  (y-axis)..

# Two-parameter family of Laplacians

- We initialize  $\Delta_{\mathbf{p}}^{(1)}$  in the sequence  $\Delta_{\mathbf{p}}^{(\ell+1)} := F_{G_p}(\Delta_{\mathbf{p}}^{(1)}, \Delta_{\mathbf{p}}^{(\ell)})$

$$\Delta_{\mathbf{p}}^{(1)} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ p_1 - 1 & 1 & -p_1 & 0 & 0 & 0 \\ 0 & p_2 - 1 & 1 & -p_2 & 0 & 0 \\ 0 & 0 & -p_2 & 1 & p_2 - 1 & 0 \\ 0 & 0 & 0 & -p_1 & 1 & p_1 - 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

Spectrum for different  $p_2$ , fix  $p_1 = 1/2$  (top),  $p_1 = 1/3$  (bottom)

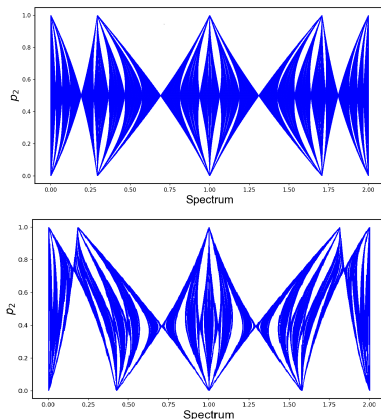


Figure: The spectrum of  $\Delta_{\mathbf{p}}^{(\ell)}$  for  $\ell = 5$  is plotted on the x-axis for  $p_2 \in (0, 1)$  (y-axis), while  $p_1$  is fixed

# Outline

- 1 Introduction
- 2 Orthogonal polynomials and centrosymmetric Jacobi matrices
- 3 Spectral decimation of piecewise centrosymmetric Jacobi matrices on graphs
- 4 The one-dimensional path graphs case
- 5 Applications to almost Mathieu-type operators

# Self-similar $p$ Laplacians on the half-integer lattice

- Let  $p \in (0, 1)$ , for each  $x \in \mathbb{Z}_+ \setminus \{0\}$ , we define  $m(x)$  to be the largest natural number  $m$  such that  $3^m$  divides  $x$ .
- For  $f \in \ell(\mathbb{Z}_+)$  we define a *self-similar Laplacian*  $\Delta_p$  by,

$$(\Delta_p f)(x) = \begin{cases} f(0) - f(1), & \text{if } x = 0 \\ f(x) - (1-p)f(x-1) - pf(x+1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod{3} \\ f(x) - pf(x-1) - (1-p)f(x+1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod{3} \end{cases} .$$

# Spectra of the self-similar $p$ Laplacians

- The spectrum  $\sigma(\Delta_p)$  is the Julia set  $\mathcal{J}(R_{\Delta_p})$  of a polynomial  $R_{\Delta_p}$ .
- For  $p = \frac{1}{2}$ , we have  $\mathcal{J}(R_{\Delta_{\frac{1}{2}}}) = [0, 2]$  and the spectrum is absolutely continuous.
- For  $p \neq \frac{1}{2}$ , the Julia set  $\mathcal{J}(R_{\Delta_p})$  is a Cantor set of Lebesgue measure zero and the spectrum is purely singularly continuous.

# Self-similar almost Mathieu operators

- Intuitively said: Substituting periodic potential in the self-similar  $p$ -Laplacian gives a self-similar almost Mathieu operator
- $p = \frac{1}{2}$  coincide up to a multiplicative constant with the standard one-dimensional almost Mathieu operators.
- Let  $f \in \ell(\mathbb{Z}_+)$ ,  $\alpha \in \mathbb{R}$ ,  $\theta \in [0, 2\pi)$  and  $\beta \in \mathbb{R}$ . We define

$$\mathbf{H}_{p,\beta,\alpha,\theta} f(x) = \begin{cases} \beta \cos(\theta) f(0) - f(1), & \text{if } x = 0, \\ \beta \cos(2\pi\alpha x + \theta) f(x) - p(x, x-1) f(x-1) - p(x, x+1) f(x+1), \\ \quad p(x, x-1) = 1-p, p(x, x+1) = p, & \text{if } 3^{-m(x)} x \equiv 1 \pmod{3} \\ \beta \cos(2\pi\alpha x + \theta) f(x) - p(x, x-1) f(x-1) - p(x, x+1) f(x+1), \\ \quad p(x, x-1) = p, p(x, x+1) = 1-p, & \text{if } 3^{-m(x)} x \equiv 2 \pmod{3}. \end{cases}$$

# Self-similar almost Mathieu operators as substitution operators

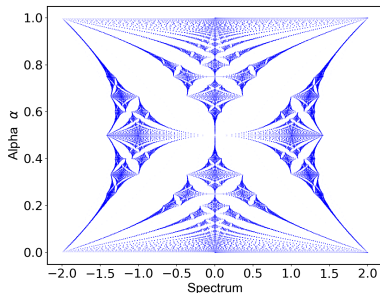
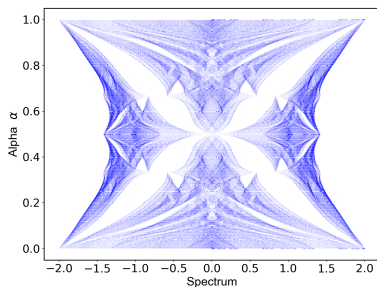
## Theorem

Let  $p \in (0, 1)$ ,  $\beta \in \mathbb{R}$  and  $n \geq 1$  be fixed. We set  $\theta = 0$ ,  $\alpha = \frac{k}{3^n}$ , for  $k \in \{1, \dots, 3^n - 1\}$ . There exists a polynomial  $R_{p, \beta, \frac{k}{3^n}, 0}$  of order  $3^n$  such that,

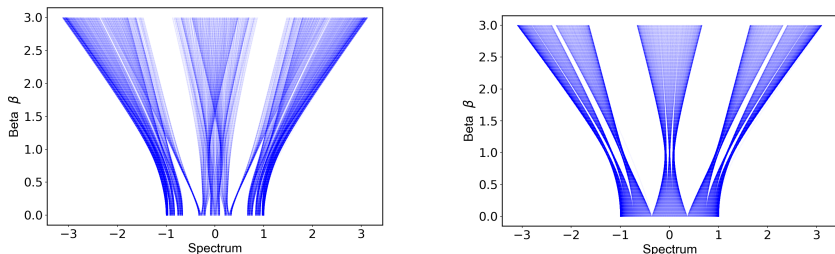
$$\sigma\left(\mathbf{H}_{p, \beta, \frac{k}{3^n}, 0}\right) = R_{p, \beta, \frac{k}{3^n}, 0}^{-1}\left(\sigma(\Delta_p)\right).$$

Moreover,  $\mathbf{H}_{p, \beta, \frac{k}{3^n}, 0}$  has purely singularly continuous spectrum if  $p \neq \frac{1}{2}$ .

# Hofstadter butterfly



**Figure:** The left panel of the figure shows a Hofstadter butterfly for a self-similar almost Mathieu operator corresponding to  $\frac{1}{3}$ -Laplacian whose spectrum is a Cantor set. For comparison, the (classical) Hofstadter butterfly corresponding to the standard AMO is shown in right panel.

Numerical results for irrational  $\alpha$ 

**Figure:** The spectrum of  $\mathbf{H}_{p,\beta,\alpha,\theta}$  is plotted ( $x$ -axis) for the fixed parameters  $\alpha = \frac{\sqrt{5}-1}{2}$  and  $\theta = 0$  while varying  $\beta \in [0, 3]$  ( $y$ -axis). The parameter  $p$  is equal to  $\frac{1}{3}$  for the left- and equal to  $\frac{1}{2}$  for the right-panel. Both panels depict examples corresponding to an irrational  $\alpha$ .

## Spectral gaps

- For  $p \neq \frac{1}{2}$ , the spectrum  $\sigma\left(\mathbf{H}_{p,\beta,\frac{k}{3^n},0}\right)$  has infinitely many spectral gaps.
- We can generate the spectral gaps iteratively using the spectral decimation function  $R_{p,\beta,\frac{k}{3^n},0}$ .
- We illustrate these ideas with the example  $p = \frac{1}{3}$ ,  $\beta = 1$ ,  $\alpha = \frac{1}{3}$ ,  $\theta = 0$ .
- Recall the general formula  $R_{J_{cs}}(z) = 1 - \frac{P_{n_0}(z)}{\prod_{i=1}^{n_0} a(i)}$ , we use it and compute the spectral decimation function for the example,

$$R_{\frac{1}{3},1,\frac{1}{3},0}(z) = \frac{9z^3}{2} - \frac{55z}{8} - \frac{9}{8}. \quad (11)$$

## Spectral gaps

- Implications of  $\sigma\left(\mathbf{H}_{p,\beta,\frac{k}{3n},0}\right) = R_{p,\beta,\frac{k}{3n},0}^{-1}\left(\sigma(\Delta_p)\right)$
- Note that  $\sigma(\Delta_{\frac{1}{3}}) \subset [0, 2]$ .
- Hence,

$$z \in \sigma\left(\mathbf{H}_{\frac{1}{3},1,\frac{1}{3},0}\right) \Rightarrow R_{\frac{1}{3},1,\frac{1}{3},0}(z) \in [0, 2],$$

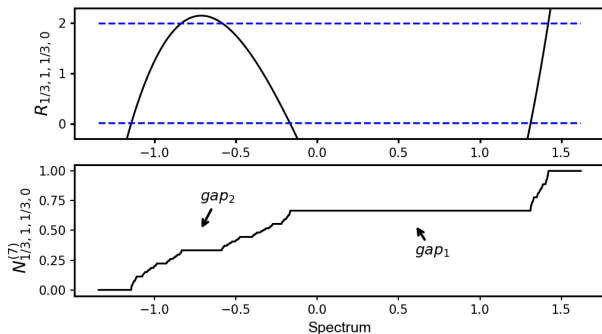
or equivalently

$$R_{\frac{1}{3},1,\frac{1}{3},0}(z) \notin [0, 2] \Rightarrow z \in \rho\left(\mathbf{H}_{\frac{1}{3},1,\frac{1}{3},0}\right).$$

where  $\rho(H)$  is the resolvent set of  $H$ .

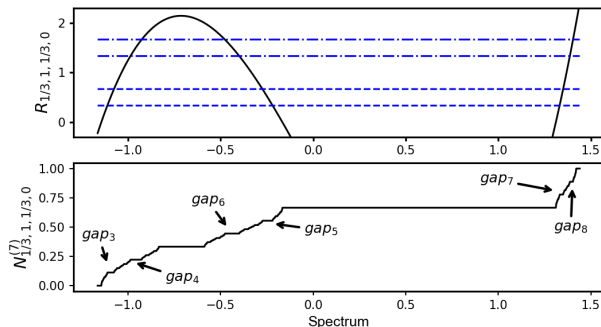
- Iterating the argument, see next two slides.

## Spectral gaps



**Figure:** (Top) The spectral decimation function  $R_{\frac{1}{3}, 1, \frac{1}{3}, 0}$ . (Bottom) The integrated density of states  $N_{\frac{1}{3}, 1, \frac{1}{3}, 0}^{(l)}$ . The dashed cutoff lines and spectral decimation function are used to locate the spectral gaps, which coincide with the indicated plateaus of the integrated density of states.

## Spectral gaps



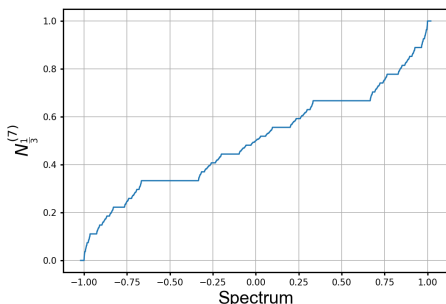
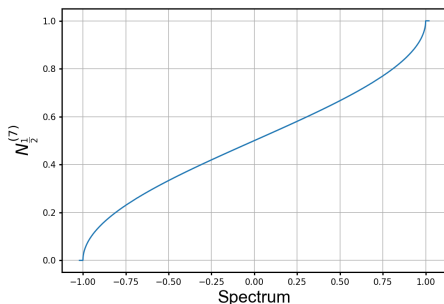
**Figure:** (Top) The spectral decimation function  $R_{\frac{1}{3}, 1, \frac{1}{3}, 0}$  is plotted. (Bottom) The integrated density of states  $N_{\frac{1}{3}, 1, \frac{1}{3}, 0}^{(l)}$ . The cutoff lines and spectral decimation function are used to locate the spectral gaps, which coincide with the indicated plateaus of the integrated density of states.

# Integrated density of states

- $\Delta_p$  is approximated by finite graph probabilistic Laplacians  $\{\Delta_p^{(l)}\}_l$ .
- The *density of states* of  $\Delta_p^{(l)}$  is given by (sum of Dirac measures)

$$\nu_{l,p}(\{x\}) = \frac{1}{3^l + 1} \sum_{\lambda \in \sigma(\Delta_p^{(l)})} \delta_\lambda(x) \quad (12)$$

- The *normalized eigenvalue counting* function of  $\Delta_p^{(l)}$  is then given by  $N_p^{(l)}(x) := \nu_{l,p}((-\infty, x])$ .

Integrated density of states  $N_p^{(l)}(x)$ 

**Figure:** (Left)  $p = \frac{1}{2}$ , i.e., standard probabilistic graph Laplacian with  $\sigma(\Delta_{\frac{1}{2}}) = [0, 2]$ . (Right)  $p = \frac{1}{3}$ , i.e., a self-similar graph Laplacian where  $\sigma(\Delta_{\frac{1}{3}})$  is a cantor set.

## Integrated density of states

- The spectrum  $\sigma(\Delta_p)$  is the Julia set  $\mathcal{J}(R_{\Delta_p})$  of a polynomial  $R_{\Delta_p}$ .
- For  $p = \frac{1}{2}$ , we have  $\mathcal{J}(R_{\Delta_{\frac{1}{2}}}) = [0, 2]$  and the spectrum is absolutely continuous.
- For  $p \neq \frac{1}{2}$ , the Julia set  $\mathcal{J}(R_{\Delta_p})$  is a Cantor set of Lebesgue measure zero and the spectrum is purely singularly continuous.

### Proposition

*The sequence of density of states  $\{\nu_{l,p}\}_{l \in \mathbb{N}}$  converges weakly to the balanced invariant measure  $\nu_p$  of the Julia set  $\mathcal{J}(R_{\Delta_p})$ .*

## Integrated density of states

- Recall  $R_{p,\beta,\frac{k}{3^n},0}$  is a polynomial of order  $3^n$ .
- Let  $S_1, S_2, \dots, S_{3^n}$  be the  $3^n$  branches of the inverse  $R_{p,\beta,\frac{k}{3^n},0}^{-1}$ .
- We define the following measure

$$\nu_{p,\beta,\frac{k}{3^n},0}(E) := \frac{1}{3^n} \sum_{i=1}^{3^n} \int_{\sigma(\Delta_p)} \chi_E(S_i(x)) \nu_p(dx) \quad (13)$$

where  $\chi_E$  is the characteristic function of the set  $E$ , i.e.

$$\chi_E(x) = \begin{cases} 0 & \text{if } x \notin E \\ 1 & \text{if } x \in E \end{cases} \quad (14)$$

and  $E \subset \mathbb{R}$  ( $\nu_p$ -measurable).

# Integrated density of states

## Theorem

Let  $\text{supp}(\nu_{p,\beta,\frac{k}{3^n},0})$  denotes the support of  $\nu_{p,\beta,\frac{k}{3^n},0}$ . Then  $\text{supp}(\nu_{p,\beta,\frac{k}{3^n},0}) = \sigma(\mathbf{H}_{p,\beta,\frac{k}{3^n},0})$ . The sequence of density of states  $\{\nu_{p,\beta,\frac{k}{3^n},0}^{(l)}\}_{l \in \mathbb{N}}$  converges weakly to  $\nu_{p,\beta,\frac{k}{3^n},0}$ . Moreover, the following identity holds

$$\int_{\sigma(\mathbf{H}_{p,\beta,\frac{k}{3^n},0})} f(x) \nu_{p,\beta,\frac{k}{3^n},0}(dx) = \frac{1}{3^n} \sum_{i=1}^{3^n} \int_{\sigma(\Delta_p)} f(S_i(x)) \nu_p(dx) \quad (15)$$

for  $f \in C_b(\mathbb{C})$ , i.e.  $f$  is a continuous bounded function on  $\mathbb{C}$ .

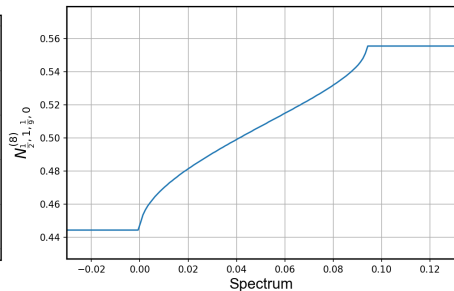
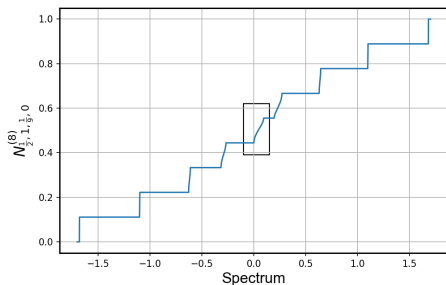
# Integrated density of states

## Corollary

Let  $E \subset \sigma(\Delta_p)$ , then  $\nu_{p,\beta,\frac{k}{3^n},0}(S_j(E)) = \frac{1}{3^n} \nu_p(E)$ .

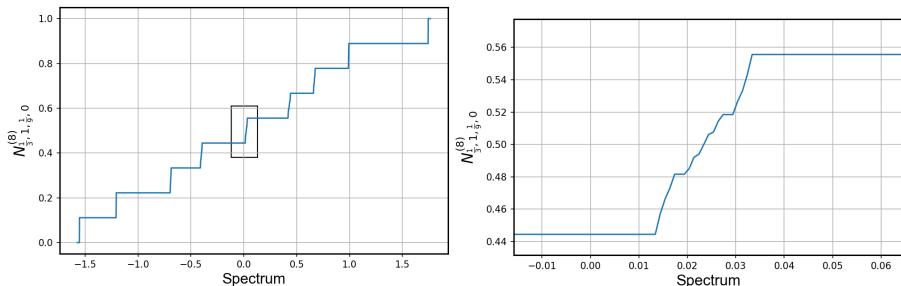
- On each range of the  $3^n$  branches  $S_i(\Delta_p)$ , we have a copy of the density of states  $\nu_p$  rescaled by  $\frac{1}{3^n}$ .
- See next two slides for  $n = 2$ , i.e. 9 copies for  $p = \frac{1}{2}$  and  $p = \frac{1}{3}$ .

# Integrated density of states



**Figure:** (Left) Numerical computation of the integrated density of states for  $\mathbf{H}_{p,\beta,\alpha,\theta}^{(l)}$ . The computations are done for level  $l = 8, p = \frac{1}{2}, \beta = 1, \alpha = \frac{1}{9}, \theta = 0$ . (Right) A resized version of the box in the left-hand side figure is displayed. It shows a copy of p-Laplacian IDS rescaled by  $\frac{1}{9}$ .

# Integrated density of states



**Figure:** (Left) Numerical computation of the integrated density of states for  $\mathbf{H}_{p,\beta,\alpha,\theta}^{(l)}$ . The computations are done for level  $l = 8, p = \frac{1}{3}, \beta = 1, \alpha = \frac{1}{9}, \theta = 0$ . (Right) A resized version of the box in the left-hand side figure is displayed. It shows a copy of p-Laplacian IDS rescaled  $\frac{1}{9}$ .

# Gap labeling

- Let  $\rho(\Delta_p)$  be the resolvent set of  $\Delta_p$ .
- We define the set of spectral gap labels of  $\Delta_p$  by

$$\mathcal{GL}(\Delta_p) = \{N_p(x) \mid x \in \rho(\Delta_p) \cap \mathbb{R}\}.$$

- It is not difficult to see that  $\mathcal{GL}(\Delta_{\frac{1}{2}}) = \{0, 1\}$ .
- For  $p \neq \frac{1}{2}$ , we have

$$\mathcal{GL}(\Delta_p) = \left\{ \frac{j}{3^i} \mid i \in \mathbb{N}, j \in \{0, 1, \dots, 3^i\} \right\}.$$

# Gap labeling

We define the set of spectral gap labels of  $\mathbf{H}_{p,\beta,\frac{k}{3^n},0}$  similarly and denote it by  $\mathcal{GL}(\mathbf{H}_{p,\beta,\frac{k}{3^n},0})$ .

## Corollary (Gap labeling)

*The set of spectral gap labels of  $\mathbf{H}_{p,\beta,\frac{k}{3^n},0}$  is given by*

$$\mathcal{GL}(\mathbf{H}_{p,\beta,\frac{k}{3^n},0}) \subset \left\{ \frac{j}{3^n} + \frac{1}{3^n} \mathcal{GL}(\Delta_p) \mid j \in \{0, 1, \dots, 3^n - 1\} \right\}.$$

# References



G. Mograby, R. Balu, K. Okoudjou, and A. Teplyaev. *Spectral decimation of a self-similar version of almost Mathieu-type operators*. arXiv:2105.09896, 2021.



G. Mograby, R. Balu, K. Okoudjou, and A. Teplyaev. *Spectral decimation of piecewise centrosymmetric Jacobi operators on graphs*. arXiv:2201.05693, 2022.

# Thank you!