

# A Dynamical Approach to Studying the Lee-Yang Zeros for the Potts Model on the Cayley Tree

Diyath Pannipitiya  
Joint work with Roland Roeder

Indiana University - Indianapolis

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# Outline

## Ising Model

Basic Setup

The Hamiltonian and The Partition Function

Lee-Yang Theory

## Potts Model

Introduction

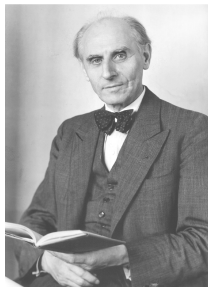
$q$ -State Potts Model on the Binary Cayley tree

## Some Results

## Dynamics

# Ising Model

- The Ising model is a theoretical model in statistical physics which was developed to describe ferromagnetism.
- It was invented in 1920 by Wilhelm Lenz, a German physicist and professor at the university of Hamburg. One of his students was Ernest Ising.



Wilhelm Lenz (1888-1957)



Ernest Ising (1900-1998)

- The classical two-dimensional Ising model is a system of magnetic particles, where one particle is at each lattice site.

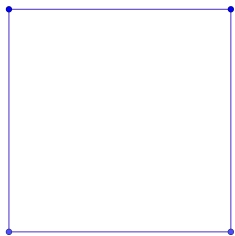
# Ising Model

## Basic Setup

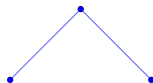
- Let  $(\Gamma_n)_{n=1}^{\infty}$  be a sequence of graphs and let  $(V_n)_{n=1}^{\infty}$  and  $(E_n)_{n=1}^{\infty}$  be the corresponding vertex set (sites) and edge set (bonds).
- One electron is at each vertex, and two neighboring electrons interact along the edges.
- The larger the  $n$ , the better the approximation to the magnetic material.
- The actual behavior of a magnet governed by the limit as the number of electrons goes to infinity.

# Ising Model

## Basic Setup



Actual magnetic material:  $\mathbb{Z}^2$

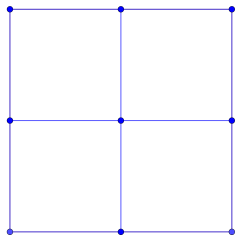


Model material

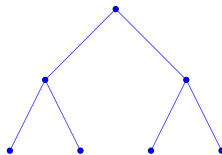
- One electron at each vertex, with pairs of electrons interacting along edges.
- Actual behavior of a magnet governed by the limit as the number of electrons goes to infinity.

# Ising Model

## Basic Setup



Actual magnetic material:  $\mathbb{Z}^2$

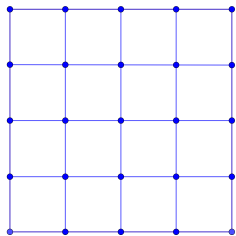


Model material

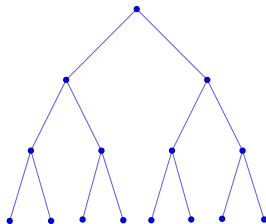
- One electron at each vertex, with pairs of electrons interacting along edges.
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# Ising Model

## Basic Setup



Actual magnetic material:  $\mathbb{Z}^2$



Model material

- One electron at each vertex, with pairs of electrons interacting along edges.
- Actual behavior of a magnet governed by the limit as the number of electrons goes to infinity.

# Ising Model

## The Hamiltonian and The Partition Function

- Each electron is assigned a magnetic moment, called *spin*, which is represented in the model by the discrete variable  $\sigma(i) \in \{-1, +1\}$ , assigning spin to the electron at the site  $i$ .
- For each spin configuration  $\sigma : V_n \rightarrow \{\pm 1\}$ , define the energy of interaction along edges  $I_n(\sigma)$  and the total magnetic moment  $M_n(\sigma)$  respectively by

$$I_n(\sigma) := \sum_{\langle i,j \rangle \in E_n} \sigma(i)\sigma(j) \quad \text{and} \quad M_n(\sigma) := \sum_{i \in V_n} \sigma(i).$$

- The Hamiltonian  $H_n$  of such a system (the energy function), in the presence of an external magnetic field  $h \in \mathbb{R}$ , is given by

$$H_n(\sigma) := -J \cdot I_n(\sigma) - h \cdot M_n(\sigma).$$

Here  $J > 0$  is the ferromagnetic coupling constant.

# Ising Model

## The Hamiltonian and The Partition Function

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$$H_n(\sigma) := -J \cdot I_n(\sigma) - h \cdot M_n(\sigma).$$

Here  $J > 0$  is the ferromagnetic coupling constant.

- For a fixed temperature  $T > 0$ , the “weight” of the spin configuration  $\sigma$  is given by

$$W_n(\sigma) := e^{-H_n(\sigma)/T}.$$

- Summing over all the possible spin configurations gives the **partition function**

$$Z_n(h, T) := \sum_{\sigma} W_n(\sigma).$$

# Ising Model

## The Hamiltonian and The Partition Function

- The probability  $P_n(\sigma)$  of the system in the state  $\sigma$  is proportional to  $W_n(\sigma)$ :

$$P_n(\sigma) = W_n(\sigma)/Z_n(h, T).$$

## Change of variables.

Let  $z = e^{-h/T}$  (field-like) and  $t = e^{-J/T}$  (temperature-like).

In these new variables,  $Z_n(z, t)$  becomes a polynomial.

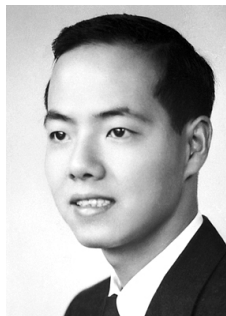
Have to multiply by  $z^{|V_n|}t^{|E_n|}$  to clear the denominator.

**Remark.** Because  $T > 0$  and  $h \in \mathbb{R}$ , we must have  $t \in (0, 1)$  and  $z \in (0, \infty)$  for ‘physical’ values of  $T, h$ .

# Ising Model

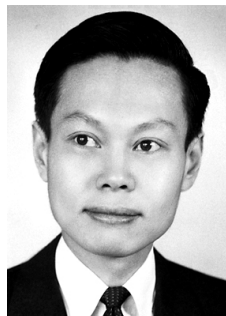
## Lee-Yang Theory

In 1952,



Tsung-Dao Lee (1926-2024)

and



Chen-Ning Yang (1922-2025)

published two important papers in statistical mechanics and proved a series of theorems. Today they are called Lee-Yang theorems.

# Ising Model

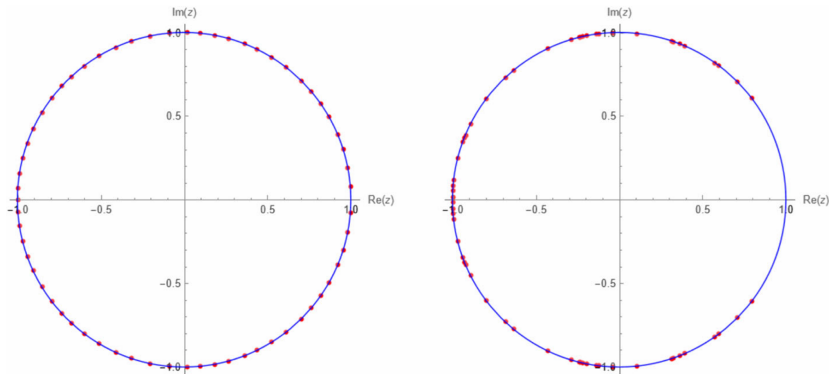
## Theorem (Lee-Yang Circle Theorem, 1952)

*For  $t \in [0, 1]$ , the complex zeros in  $z$  of the partition function  $Z_n(z, t)$ , for any  $n \in \mathbb{N}$ , of the Ising model on any graph lie on the unit circle  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ .*

### Remarks.

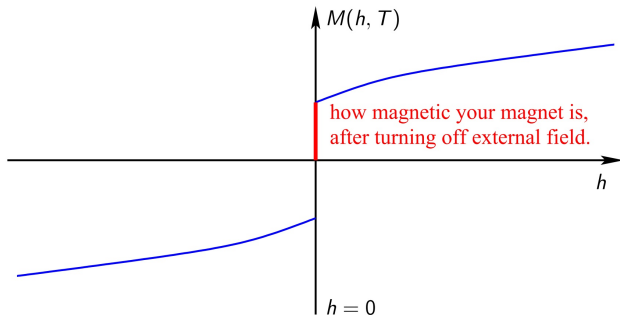
- Though  $Z_n(z, t) \neq 0$  for any  $n \in \mathbb{N}$  and  $z > 0$ , the zeros of  $Z_n(z, t)$  can accumulate to the positive real axis as  $n \rightarrow \infty$  leading to phase transitions in the thermodynamic limit.
- The Lee-Yang theorem says that for fixed  $t \in [0, 1]$ , the only possible parameter at which we can observe (for physical values of  $h \in \mathbb{R}$ ) a phase transition is when  $z = 1$  (equivalently when  $h = 0$ ).
- There exists a critical temperature  $t_c \in (0, 1)$  such that when  $t < t_c$  the Lee-Yang zeros accumulate to 1 and when  $t > t_c$  the Lee-Yang zeros stay away from the positive real axis (in the complex  $z$ -plane).

# Ising Model



**Figure:** The Lee–Yang zeros for the Ising model on the  $5^{\text{th}}$  rooted binary Cayley Tree. Left:  $t = 0.0625 < t_c = \frac{1}{3}$ . Right:  $t = 0.5 > t_c = \frac{1}{3}$ .

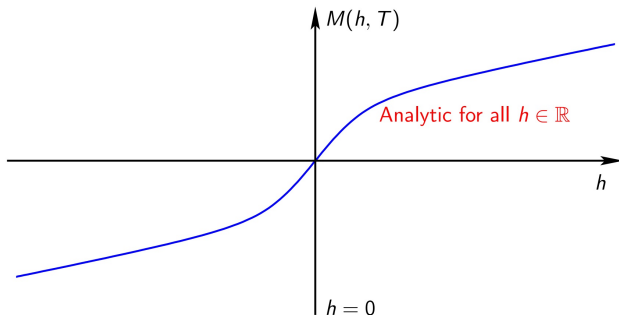
# Ising Model



When  $T < T_c$

- $h$  is the external magnetic field applied to your piece of iron.
- $T$  is the temperature.
- $M(h, T)$  is how magnetic your piece of iron is (magnetization).

# Ising Model

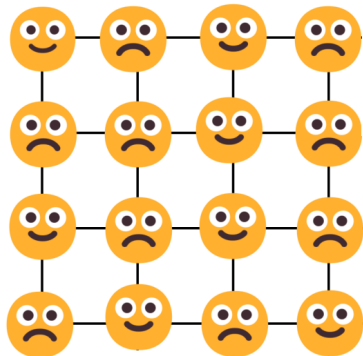


When  $T > T_c$

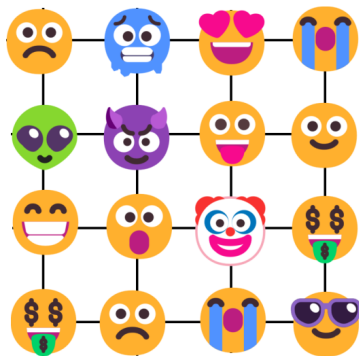
- $h$  is the external magnetic field applied to your piece of iron.
- $T$  is the temperature.
- $M(h, T)$  is how magnetic your piece of iron is (magnetization).

# $q$ -state Potts Model

Difference between the models



**Ising Model**  
Only two states



**Potts Model**  
 $q \geq 3$  number of states

# Potts Model

## Introduction

- In the Ising model, the spin configuration is  $\sigma : V_n \rightarrow \{\pm 1\}$ .
- In the  $q$ -state Potts model, it is  $\sigma : V_n \rightarrow \{0, \dots, q-1\}$ , where  $q \geq 3$ .
- The **Hamiltonian** of the configuration  $\sigma$  is given by

$$H_n(\sigma) = -J \sum_{(i,j) \in E_n} \left(1 - \delta(\sigma(i), \sigma(j))\right) - h \sum_{i \in V_n} \delta(\sigma(i), 0).$$

Here  $J > 0$  is the coupling constant and  $h \in \mathbb{R}$  is a parameter related to the external magnetic field.

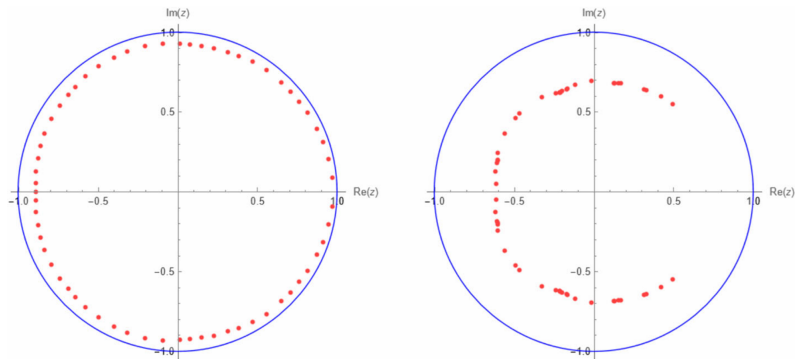
Also,  $\delta(a, b) = 0$  if  $a \neq b$  and  $\delta(a, b) = 1$  if  $a = b$ .

- Just like in the Ising model, summing over all the possible spin configurations gives the **partition function**

$$Z_n(h, T) := \sum_{\sigma} W_n(\sigma) = \sum_{\sigma} e^{-H_n(\sigma)/T}.$$

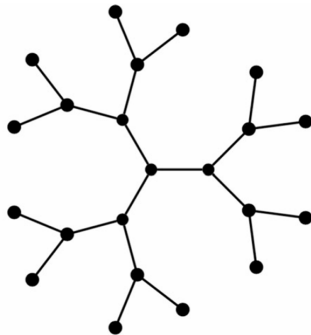
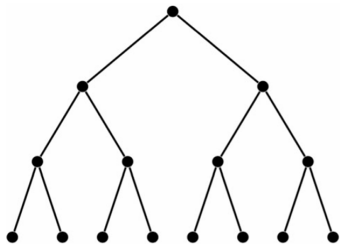
- **Lee-Yang theorem does not hold for  $q \geq 3$ .**

# Potts Model



**Figure:** The Lee–Yang zeros for the 3-state Potts model on the 5<sup>th</sup> rooted binary Cayley Tree. Left:  $t = 0.0625 < t_c = \frac{1+\sqrt{73}}{36} \approx 0.265$ . Right:  $t = 0.5 > t_c = \frac{1+\sqrt{73}}{36} \approx 0.265$ .

# Potts Model on the Cayley tree with branching number two



**Figure:** Left: rooted binary Cayley Tree of depth 3. Right: unrooted binary Cayley tree of depth 3.

# Some Results

Theorem (Roeder, P. 2023)

For  $q \in \mathbb{N}_{\geq 2}$ ,  $t \in \mathbb{R}$ , and  $z, w \in \mathbb{C}$  define  $R_{z,t,q}(w)$  and  $\hat{R}_{z,t,q}(w)$  by

$$R_{z,t,q}(w) := z \left[ \frac{t + w + (q-2)tw}{1 + (q-1)tw} \right]^2, \text{ and}$$

$$\hat{R}_{z,t,q}(w) := z \left[ \frac{t + w + (q-2)tw}{1 + (q-1)tw} \right]^3.$$

Then, the Lee-Yang zeros of the  $q$ -state Potts model on the  $n^{\text{th}}$  level rooted and unrooted binary Cayley trees are the solutions  $z$  to:

$$R_{z,t,q}^n(z) = \frac{1}{1-q} \quad \text{and} \quad \hat{R}_{z,t,q} \circ R_{z,t,q}^{n-1}(z) = \frac{1}{1-q}$$

respectively.

## Some Results

- Here,  $R_{z,t,q}^n(z)$  means that one first iterates  $R_{z,t,q}(w)$   $n$ -times with respect to  $w$  and then substitute  $w = z$ .
- This renormalization procedure allows us to use methods from dynamical systems to study the Lee-Yang zeros for the Potts model on the Cayley tree.
- The analogous problems would be way harder for the classical  $\mathbb{Z}^2$  or  $\mathbb{Z}^3$  lattices.

# Some Results

Theorem (Roeder, P. 2023)

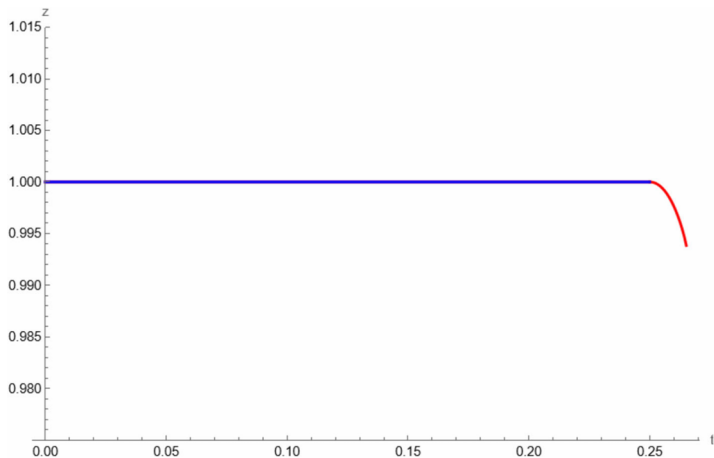
*(Ferromagnetic Case)* For any  $t \in [0, 1]$ , as  $n \rightarrow \infty$  the Lee-Yang zeros for the  $q \geq 2$  state Potts Model on the (rooted or unrooted) binary Cayley Tree accumulate to  $z \in (0, \infty)$  if and only if  $t \in [0, t_2(q)]$ . They do so at a single point:

$$z_c(t, q) = \begin{cases} 1 & \text{if } 0 \leq t \leq t_1(q), \\ \mathcal{Z}_q(t) & \text{if } t_1(q) < t \leq t_2(q), \end{cases}$$

with  $t_1(q) = \frac{1}{1+q}$ ,  $t_2(q) = \frac{q-2 + \sqrt{q^2 + 32q - 32}}{18(q-1)}$ , and

$$\mathcal{Z}_q(t) := \frac{-27(q-1)^2 t^4 + 18(q^2 - 3q + 2)t^3 + (q^2 + 14q - 14)t^2 + 2(q-2)t + 1 - \sqrt{(t-1)((q-1)t+1)(9(q-1)t^2 - (q-2)t - 1)^3}}{8t((q-2)t+1)^3}.$$

# Some Results



**Figure:** Graph of  $z = z_c(t, 3)$ . For  $0 \leq t \leq t_1(3) = \frac{1}{4}$  we have  $z_c(t, 3) = 1$  (blue) and for  $\frac{1}{4} < t \leq t_2(q) \approx 0.2651$  we have  $z_c(t, 3) = \mathcal{Z}_q(t) < 1$  (red).

# Some Results

Theorem (Roeder, P. 2024)

*(Antiferromagnetic Case) For any  $t > 1$ , as  $n \rightarrow \infty$  the Lee-Yang zeros for the  $q \geq 2$  state Potts Model on the (rooted or unrooted) binary Cayley tree accumulate to  $z \in (0, \infty)$  if and only if  $t \geq t_3(q)$  and  $z = z_c^\pm(t, q)$  where*

$$t_3(q) = \frac{3 \left( 3q - 6 + \sqrt{9q^2 - 32q + 32} \right)}{2(q - 1)}, \text{ and}$$

$$z_c^\pm(t, q) := \frac{(-3 - 6(-2 + q)t - 3(2 + (-2 + q)q)t^2 - 6(-2 + q)(-1 + q)t^3 + (-1 + q)^2 t^4) \pm \sqrt{(-1 + t)^3(1 - t + qt)^3(-9 + 18t - 9qt - t^2 + qt^2)}}{8t(1 - 2t + qt)^3}.$$

## Some Results

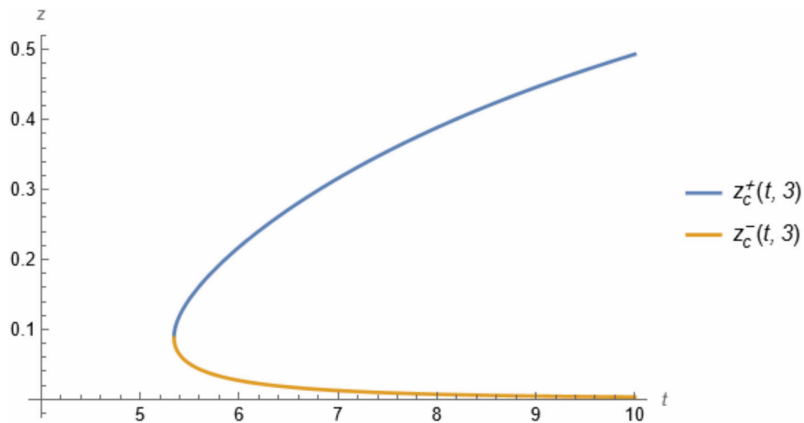


Figure: Graph of  $z_c^-(t, 3)$  and  $z_c^+(t, 3)$ .

# Dynamics

- Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a family of rational maps from the Riemann sphere to itself such that  $f_\lambda$  varies holomorphically on the parameter  $\lambda$ .
- Let  $a(\lambda)$  be a choice of initial conditions for the iterates of  $f_\lambda$  which also depends holomorphically on  $\lambda$ . It is called a **Marked Point** for  $f_\lambda$ .
- A marked point  $a(\lambda)$  is called **passive** for  $f_\lambda$  at  $\lambda_0$  if the sequence  $(g_n)_{n=1}^\infty$  of functions defined by  $g_n(\lambda) := f_\lambda^n(a(\lambda))$  forms a normal family in some neighborhood of  $\lambda_0$ .
- Else, we say  $a(\lambda)$  is **active** for  $f_\lambda$  at the parameter  $\lambda_0$ .

# Dynamics

## Definition

Let  $z_{rep}(\lambda_0)$  be a repelling fixed point of  $f_{\lambda_0}$ . Because  $f_\lambda$  varies holomorphically with  $\lambda$ , we can find a holomorphic map  $z_{rep}(\lambda)$  defined in a neighborhood  $U$  of  $\lambda_0$  such that  $z_{rep}(\lambda)$  is a repelling fixed point of  $f_\lambda$  for all  $\lambda \in U$ . Let  $n_0 \in \mathbb{N}$ . We say  $f_{\lambda_0}^{n_0}$  maps  $a(\lambda_0)$  **non-persistently** onto the repelling fixed point  $z_{rep}(\lambda_0)$  if

$$f_{\lambda_0}^{n_0}(a(\lambda_0)) = z_{rep}(\lambda_0) \text{ and } f_\lambda^{n_0}(a(\lambda)) \neq z_{rep}(\lambda) \text{ on } U.$$

# Dynamics

## Lemma

*Suppose  $f_{\lambda_0}^{n_0}$  maps  $a(\lambda_0)$  non-persistently onto the repelling fixed point  $z_{rep}(\lambda_0)$ . Then  $\lambda_0$  is an active parameter for the marked point  $a(\lambda)$  under  $f_\lambda$ . The same statement holds if  $f_{\lambda_0}^{n_0}$  maps  $a(\lambda_0)$  nonpersistently onto a point of a repelling periodic cycle.*

*If  $f_{\lambda_0}^{n_0}$  maps  $a(\lambda_0)$  into the basin of attraction of an attracting fixed point  $z_{att}(\lambda_0)$ , then  $\lambda_0$  is a passive parameter for the marked point  $a(\lambda)$  under  $f_\lambda$ .*

- A number  $b \in \hat{\mathbb{C}}$  is called **exceptional** for the rational map  $f : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$  if  $|\{z \in \hat{\mathbb{C}} : f^m(z) = b \text{ for some } m \in \mathbb{N}_0\}| \leq 2$ .
- A marked point  $b(\lambda)$  is **persistently exceptional** for a holomorphic family of rational maps  $f_\lambda$  if for every parameter  $\lambda$ , the point  $b(\lambda)$  is exceptional for  $f_\lambda$ .

# Dynamics

## Lemma (Lyubich, 1984)

Let  $\{f_\lambda\}_{\lambda \in \Lambda}$  be a family of rational maps which depends holomorphically on the parameter  $\lambda$ . Let  $a(\lambda)$  and  $b(\lambda)$  be marked points for  $f_\lambda$  with  $b(\lambda)$  being not persistently exceptional for  $f_\lambda$ . If  $\lambda_0$  is an active parameter for  $a(\lambda)$  under  $f_\lambda$ , then  $\lambda_0 \in \overline{\{\lambda \in \Lambda : f_\lambda^m(a(\lambda)) = b(\lambda) \text{ for some } m \in \mathbb{N}\}}$ .

## Lemma (Roeder, P. 2023)

For fixed  $s < 0$ ,  $t \in \mathbb{R}$ , and  $q \in \mathbb{N}_{\geq 2}$ , define

$$\mathcal{B}(t, q) := \overline{\{z \in \mathbb{C} \setminus \{0\} : R_{z,t,q}^m(z) = s \text{ for some } m \in \mathbb{N}\}}.$$

If  $z_\bullet \in [0, \infty)$  is a passive parameter for the marked point  $a(z) = z$  under the map  $R_{z,t,q}$ , then  $z_\bullet \notin \mathcal{B}(t, q)$ .

# Dynamics

Lemma (Roeder, P. 2023)

*For fixed  $t \in (0, 1) \cup (1, \infty)$  and  $q \in \mathbb{N}_{\geq 2}$ , let  $\mathcal{A}(t, q)$  be the active locus of the marked point  $a(z) = z$  under the holomorphic family  $R_{z,t,q}(w)$ . Then,  $\mathcal{A}(t, q) \subset \mathcal{B}(t, q)$  and*

$$\mathcal{B}(t, q) \cap (0, \infty) = \mathcal{A}(t, q) \cap (0, \infty).$$

# Dynamics

With above two lemmas, the proofs of the two theorems reduce to the proofs of the following two theorems:

## Theorem (Roeder, P. 2023)

Suppose  $0 < t < 1$  and let  $\mathcal{A}(t, q)$  be the active locus for the marked point  $a(z) = z$  under  $R_{z,t,q}(w)$ . Then

$$\mathcal{A}(t, q) \cap (0, \infty) = \begin{cases} \{1\} & \text{if } 0 < t \leq t_1(q), \\ \{\mathcal{Z}_q(t)\} & \text{if } t_1(q) < t \leq t_2(q), \\ \emptyset & \text{if } t_2(q) < t < 1. \end{cases}$$

Here,  $t_1(q) = \frac{1}{1+q}$ ,  $t_2(q) = \frac{q-2 + \sqrt{q^2 + 32q - 32}}{18(q-1)}$ , and

$$\mathcal{Z}_q(t) := \frac{-27(q-1)^2 t^4 + 18(q^2 - 3q + 2)t^3 + (q^2 + 14q - 14)t^2 + 2(q-2)t + 1 - \sqrt{(t-1)((q-1)t+1)(9(q-1)t^2 - (q-2)t - 1)^3}}{8t((q-2)t+1)^3}.$$

# Dynamics

and

Theorem (Roeder, P. 2024)

Suppose  $t > 1$ ,  $q \geq 3$  and let  $\mathcal{A}(R_{z,t,q})$  be the active locus for the marked point  $a(z) = z$  under  $R_{z,t,q}(w)$ . Then

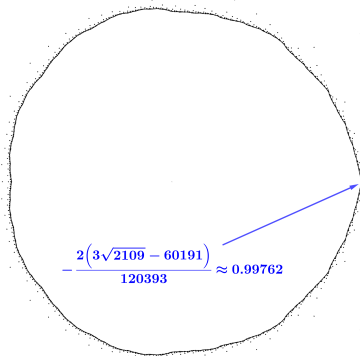
$$\mathcal{A}(R_{z,t,q}) \cap (0, \infty) = \begin{cases} \emptyset & \text{if } 1 < t < t_3(q), \\ \{z_c^\pm(t, q)\} & \text{if } t_3(q) \leq t. \end{cases}$$

Where

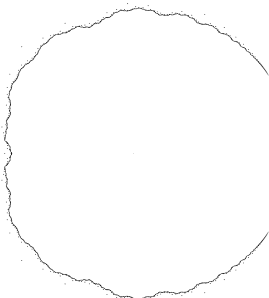
$$t_3(q) = \frac{3 \left( 3q - 6 + \sqrt{9q^2 - 32q + 32} \right)}{2(q - 1)}, \text{ and}$$

$$z_c^\pm(t, q) := \frac{(-3 - 6(-2 + q)t - 3(2 + (-2 + q)q)t^2 - 6(-2 + q)(-1 + q)t^3 + (-1 + q)^2 t^4) \pm \sqrt{(-1 + t)^3(1 - t + qt)^3(-9 + 18t - 9qt - t^2 + qt^2)}}{8t(1 - 2t + qt)^3}.$$

# Dynamics



$$t = 0.26 < t_c \approx 0.2651$$

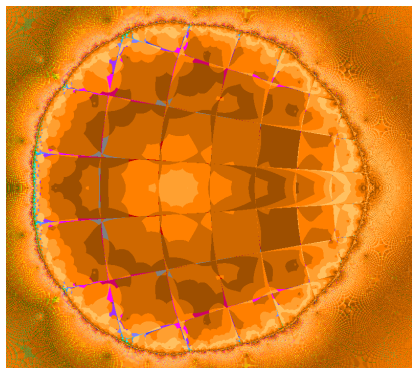


$$t = 0.5 > t_c \approx 0.2651$$

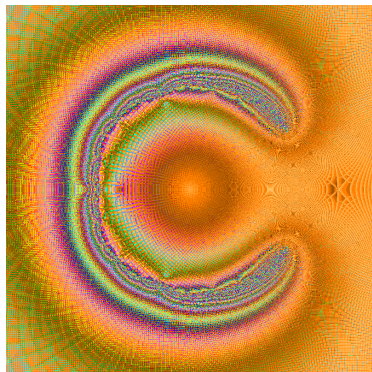
The active locus for the marked point  $a(z) = z$  under  $R_{z,t,3}(w)$ . Points in the active locus are black, and the points in the passive locus are white.

*Courtesy: Arnaud Chéritat - University of Toulouse.*

# Dynamics



$$t = 0.1 < t_c \approx 0.2651$$

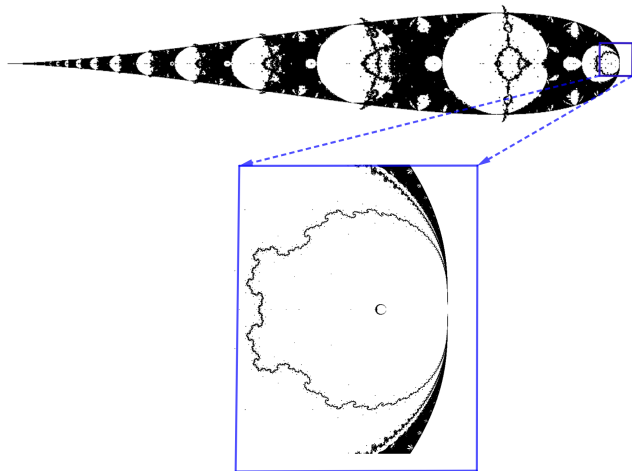


$$t = 0.5 > t_c \approx 0.2651$$

Lee-Yang zeros accumulate at places where the color changes - it is the active locus for the marked point  $a(z) = z$  under  $R_{z,t,3}(w)$ .

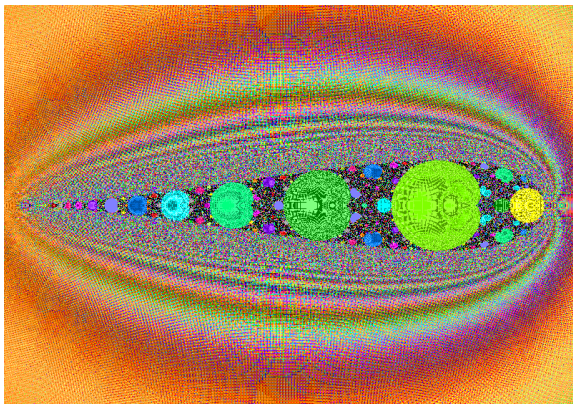
*Courtesy: Suzanne Boyd - University of Wisconsin Milwaukee and Brian Boyd.*

# Dynamics



**Figure:** The active locus for the marked point  $a(z) = z$  for  $R_{z,t,3}(w)$  at  $t = 6 > t_c \approx 5.3423$ . Points in the active locus are black, and the points in the passive locus are white.

# Dynamics

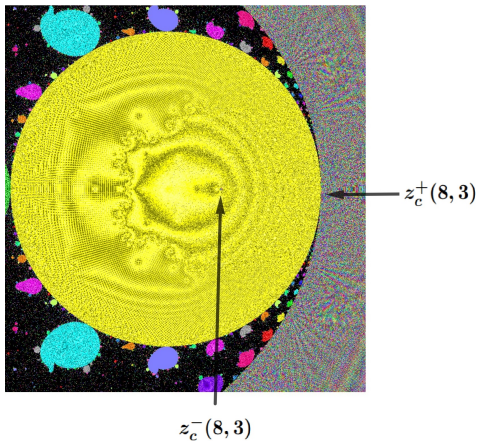


$$t = 8 > t_c \approx 5.3423$$

$$z_c^+(8, 3) = \frac{119\sqrt{5593}+9229}{46656} \approx 0.388558$$

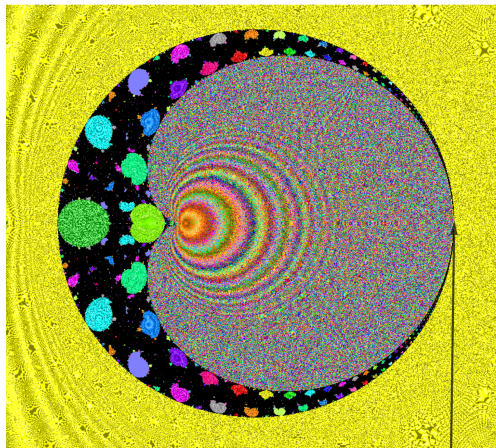
$$z_c^-(8, 3) = \frac{9229-119\sqrt{5593}}{46656} \approx 0.007060$$

# Dynamics



$$z_c^+(8, 3) = \frac{119\sqrt{5593}+9229}{46656} \approx 0.388558$$
$$z_c^-(8, 3) = \frac{9229-119\sqrt{5593}}{46656} \approx 0.007060$$

# Dynamics



$z_c^-(8, 3)$

**Thank You!**