

From Topological Defects to Quantum Entanglement

A Unified Perspective via Symbols, Qubits, and Geometric Phases

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AMS Eastern Sectional Meeting 03/28/26



Talk Roadmap

Setting the Stage The central question: are topology and entanglement fundamentally linked?

Symbol & Topological Defects in Graphene The Weyl symbol $H(k,r)$, defect dimension D , graphene vacancies, index theorem

Two Qubits Go Topological Cartan decomposition, Z_5 phases, Bell-state zero modes

Topological Sum Rule Geometric phases, SWAP vs 3-CNOT, noise measurement

The Unified Picture One framework connecting defects, qubits, and quantum gates

Two Apparently Unrelated Questions



Can a crystal defect be topological?

ACT I → Graphene vacancies and the Weyl symbol operator



Is quantum entanglement a topological property?

ACT II → Two-qubit Hilbert space and Cartan decomposition

Answer: YES to both – and they are connected by the same mathematics.

The Common Thread: Cartan's Tenfold Classification



Antiunitary symmetries

Time reversal Θ and particle-hole Π classify all Hamiltonians into 10 classes



Topological integers

Winding numbers and Chern numbers label disconnected phases



Symbol operator

Generalises Bloch Hamiltonians to non-periodic (defect) systems



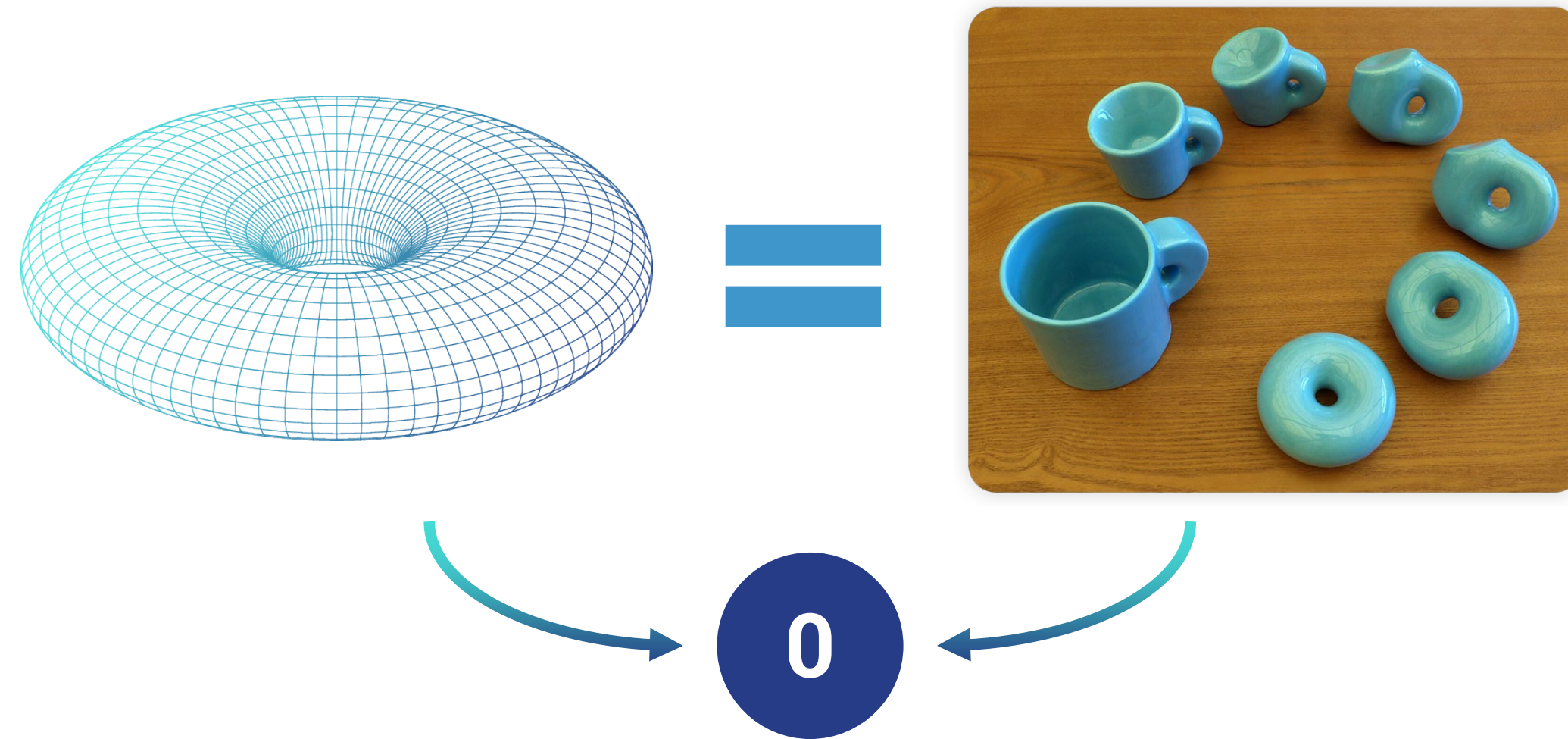
Finite-system topology

Five topological phases in a 4-dimensional two-qubit Hilbert space

Topological features : a general perspective

Speaking of topology requires a clear framework

Topology in a nutshell



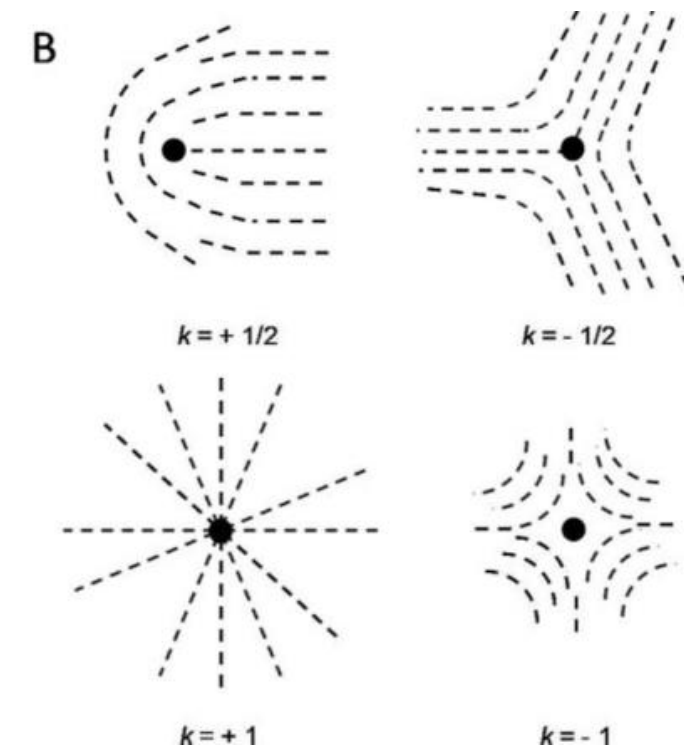
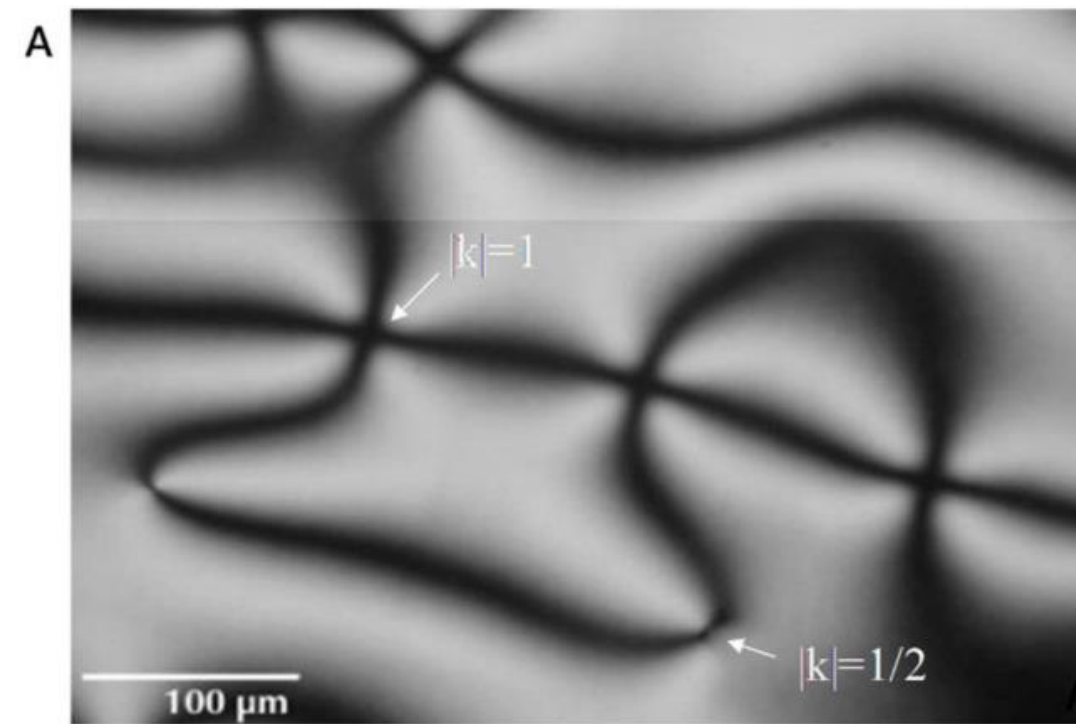
Classifies “objects” by assigning a single integer to each topological family

$$2(1-\#\text{holes})=\mathbb{Z} = \text{integer numbers}$$

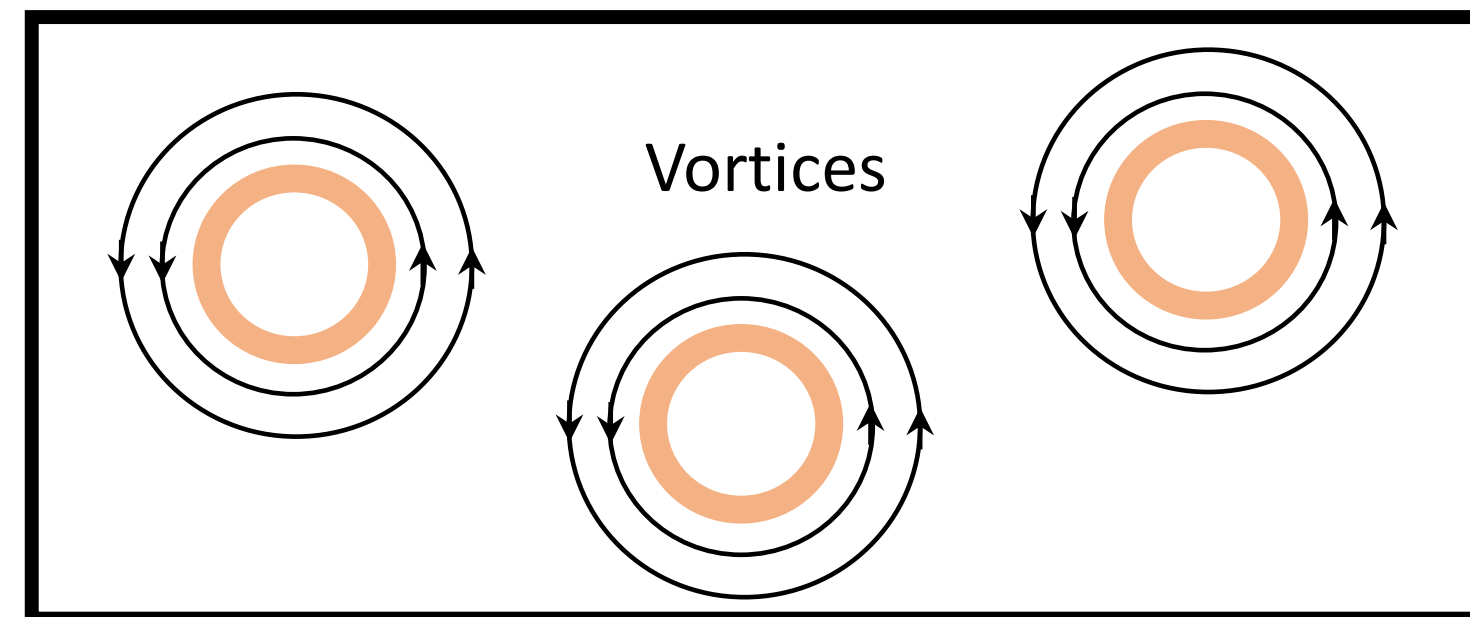
Topological defects in ordered phases - Toulouse-Kleman classification (1976)

Example 1

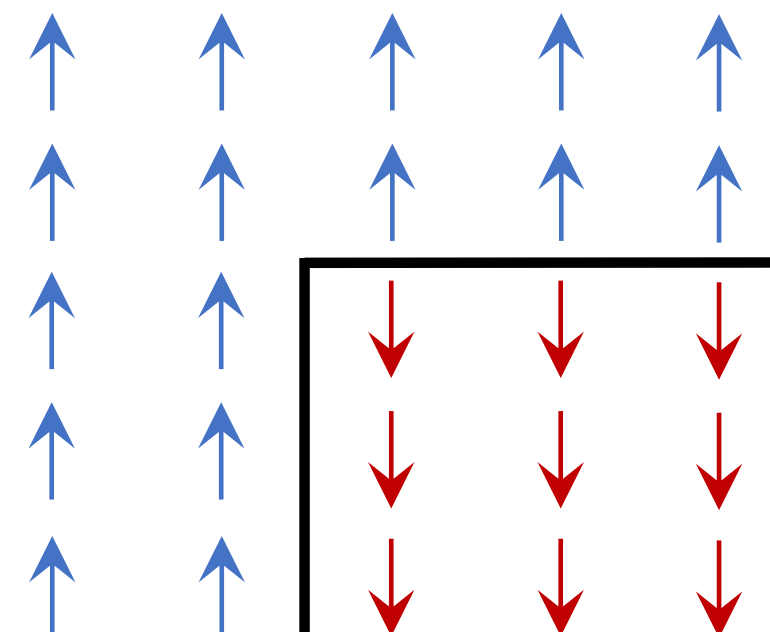
Liquid Crystals



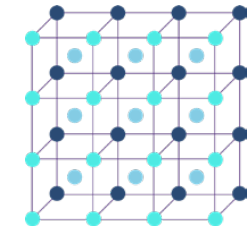
Vortices in Superconductors



Domain walls in Ising, Heisenberg models



Topology in Crystals

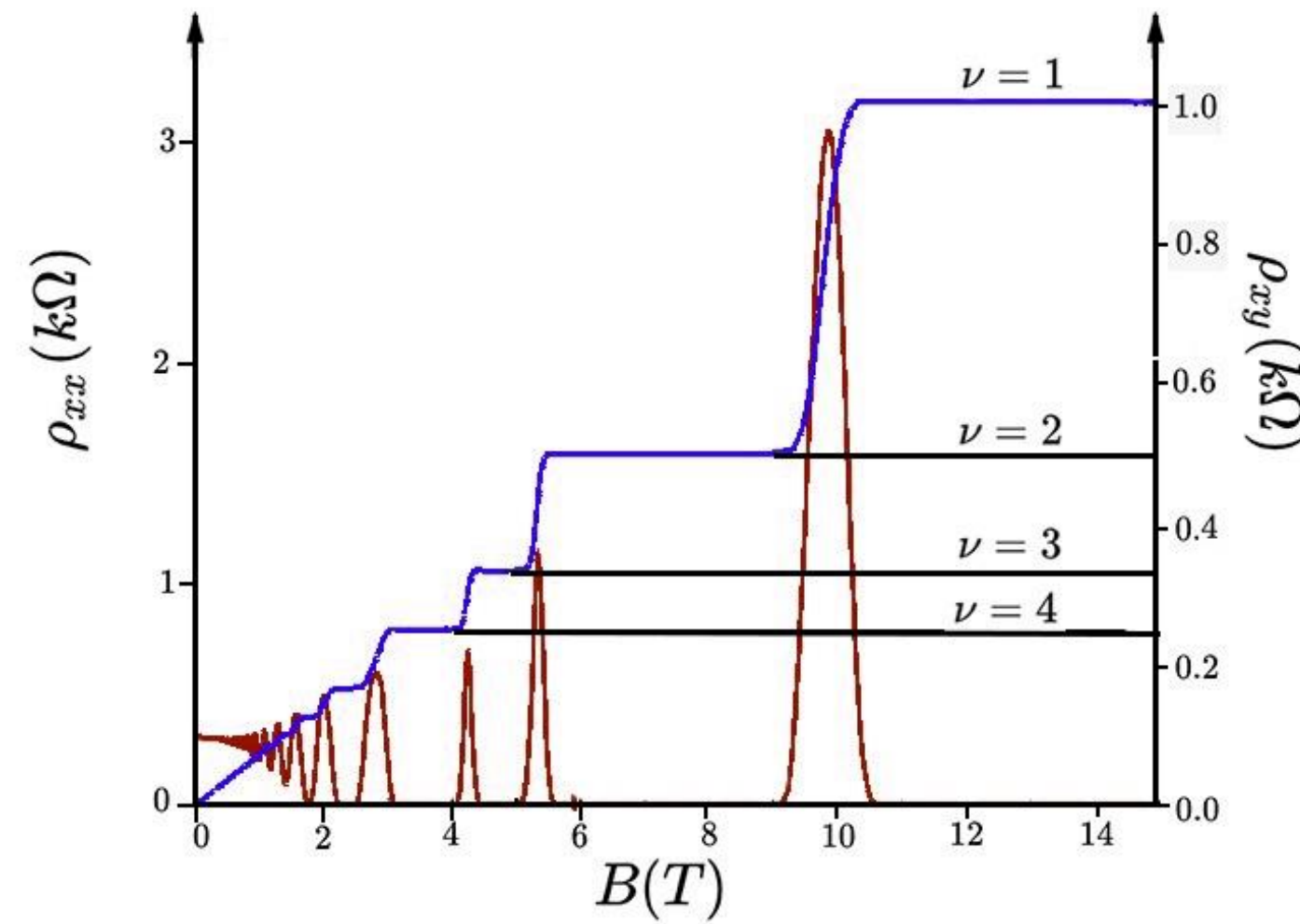


(2008)

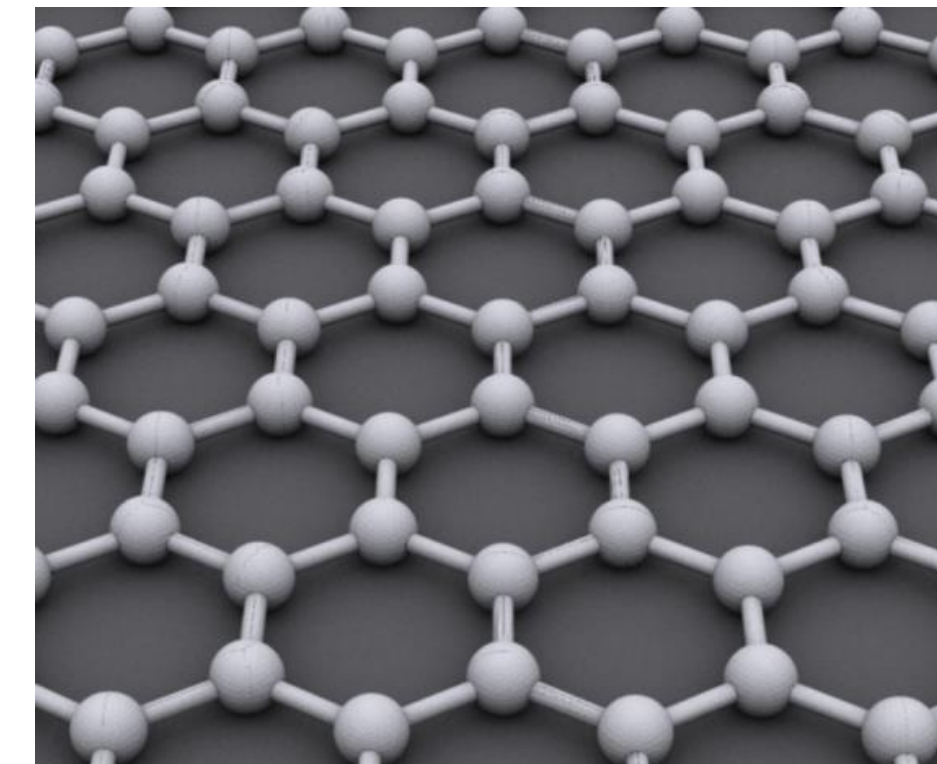
Example 2

Iconic examples

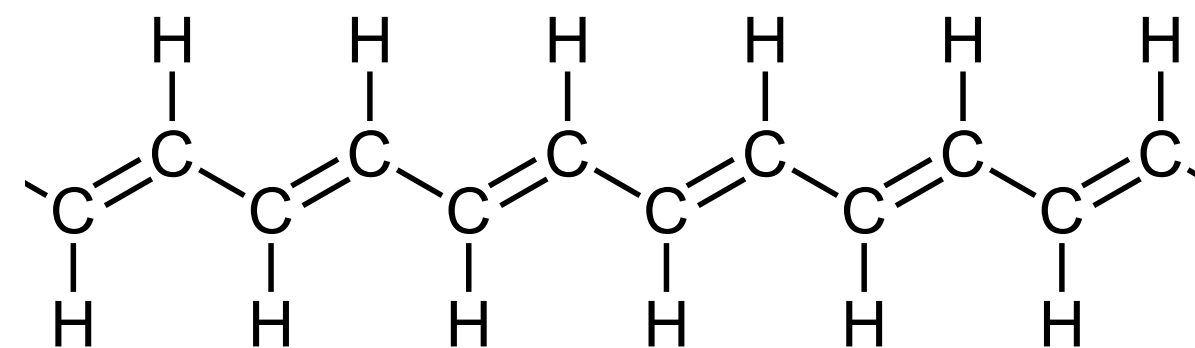
Quantum Hall effect



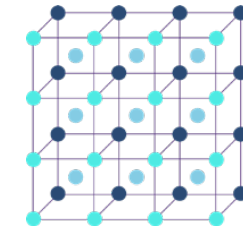
Graphene



Polyacetylene
(SSH model)



Topology in Crystals



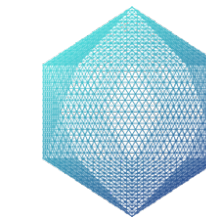
(2008)

Example 2

For translation invariance/crystals



Classified by Brillouin zones



Assigning integers to topological Bloch Hamiltonian

Quantum transport

IQHE

Berry curvature /
phase

Protected edge states

Majorana modes

Quantum computing

Tenfold classification

Wigner-Dyson classification (d=0).

Remaining symmetries of a random Hamiltonian once all **unitary** symmetries have been exhausted (Anderson localisation).

Name (Cartan)	T	
A (unitary)	0	← Magnetic Field/Aharonov-Bohm magnetic flux
AI (orthogonal)	+1	
AII (symplectic)	-1	← Time reversal in the presence of spin-orbit

Time reversal symmetry T

Altland & Zirnbauer (1997) : Extend the Wigner-Dyson classification (d=0).

Name (Cartan)	T	C	S= T C
A (unitary)	0	0	0
AI (orthogonal)	+1	0	0
AII (symplectic)	-1	0	0
AIII (chiral unitary)	0	0	1
BDI (chiral orth.)	+1	+1	1
CII (chiral sympl.)	-1	-1	1
D	0	+1	0
C	0	-1	0
DIII	-1	+1	1
CI	+1	-1	1

Time reversal symmetry (T)
 Particle-Hole symmetry (C)
 Their product $S = T C$ (chiral symmetry)

Anti-unitary symmetries !

Elegant mathematical structure inherited from Cartan classification.

The Tenfold classification in Crystals (2008)

Two anti-unitary symmetries:

"Time reversal" & "Particle-Hole"

d: space dimension

Tenfold: 10 classes

Class	s	T	P	C	d=0	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$

Cartan Classification of Lie groups

The Tenfold classification in Crystals (2008)

Reading the tenfold table

A crystal having both “**Time reversal**” and “**Particle-Hole**” in **d=1** is **topological**

Class	s	T	P	C	d=0	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$

The Symbol and Topological Defects

Goft, Abulafia, Orion, Schochet & Akkermans · Phys. Rev. B 108, 054101 (2023)

- The Weyl symbol $H(k,r)$: topology without translation symmetry
- Defect dimension D navigates the tenfold classification table
- Atiyah-Singer index theorem and protected zero modes

The Problem with Bloch Hamiltonians

Bloch's theorem works beautifully when translation symmetry holds:
 $H(k)$ encodes band topology in the Brillouin zone. But...

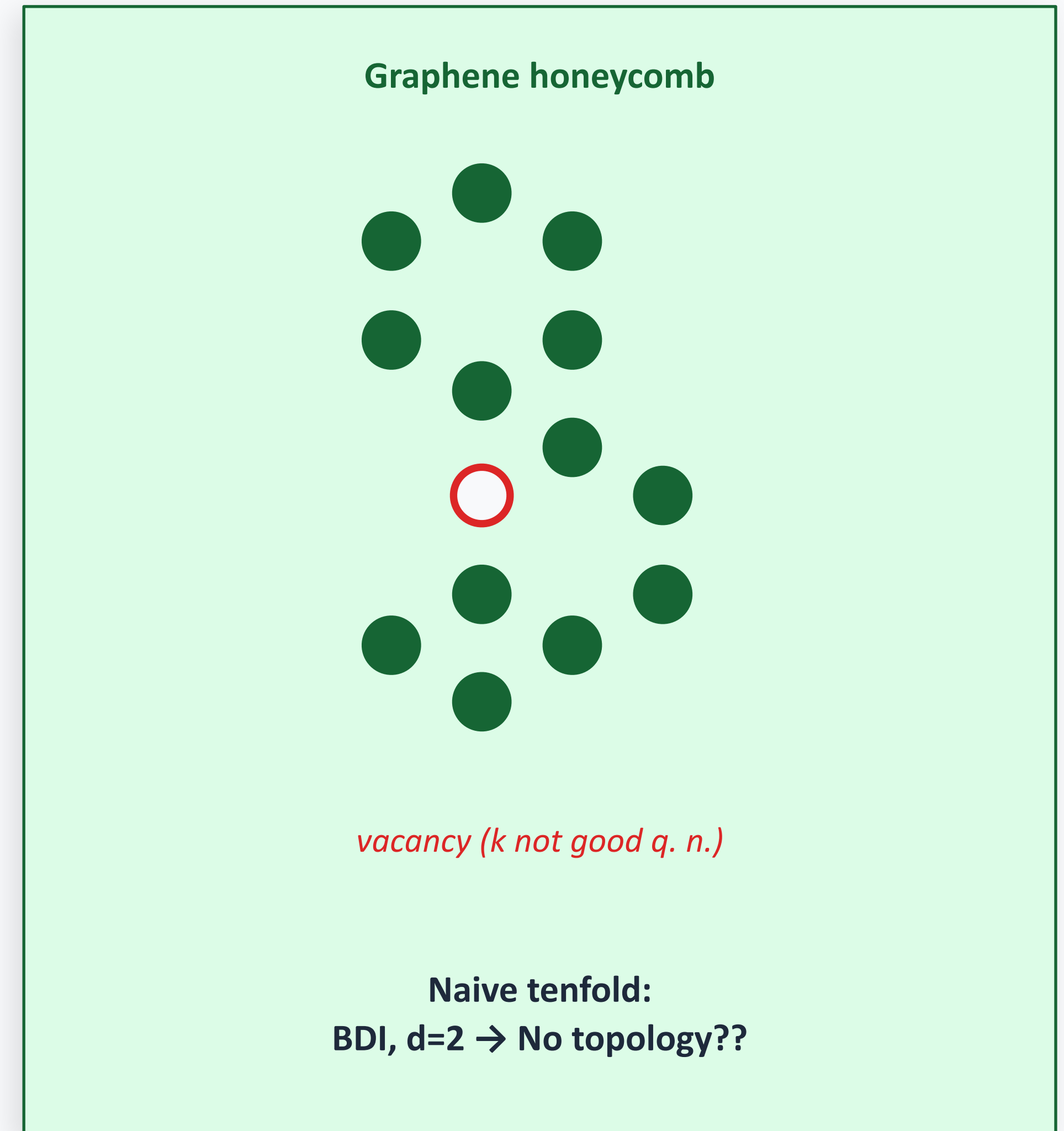
$H(k) \rightarrow$ band topology in the Brillouin zone

A defect — vacancy, adatom, Kekule distortion — breaks translation symmetry.

Naive classification: graphene \in BDI class, $d = 2 \rightarrow$ no Z-topology.

This conclusion is wrong. Understanding why is the entry point.

Defects ?

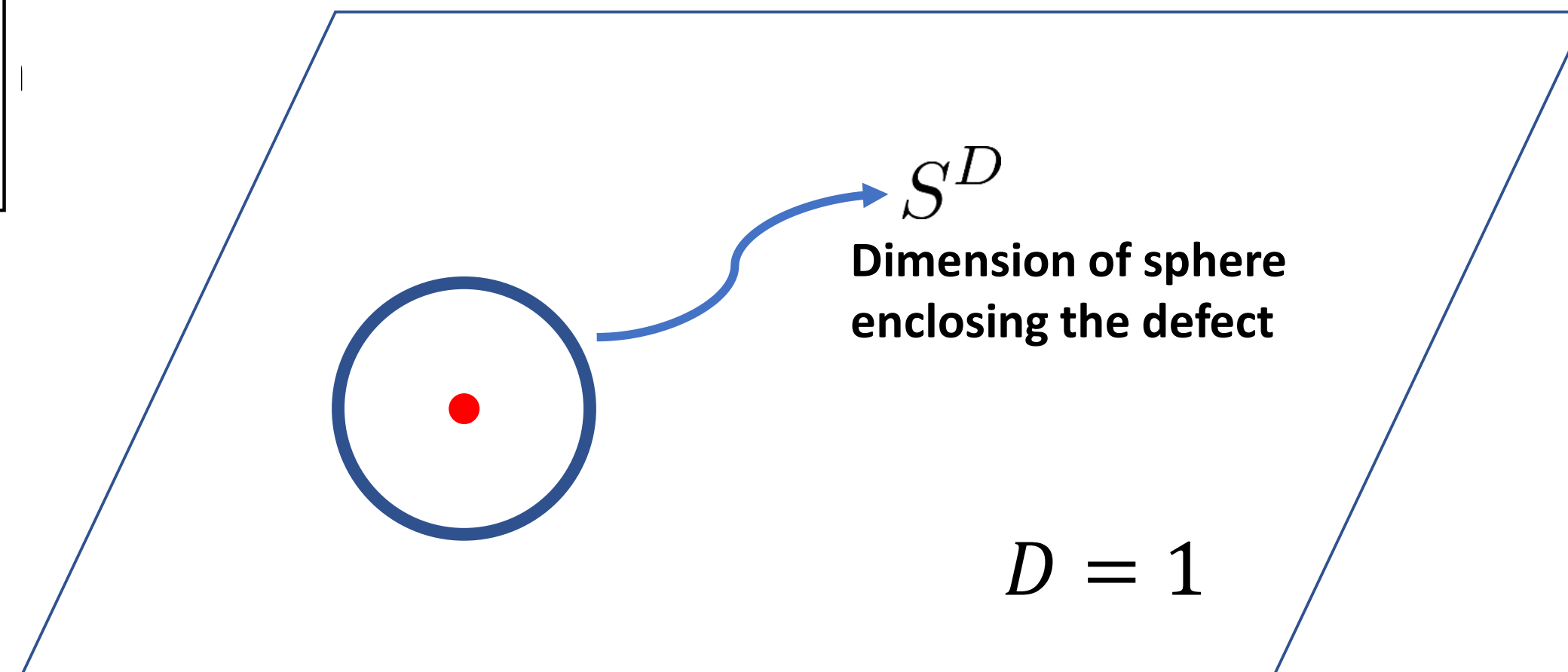


Defects in topological crystals

	d=1	d=2	d=3
D=0			
D=1			
D=2			

Codimension $\delta = d - D$

Spatial dimension d Defect dimension D



Defects in topological crystals

Two anti-unitary symmetries:
 "Time reversal" & "Particle-Hole"

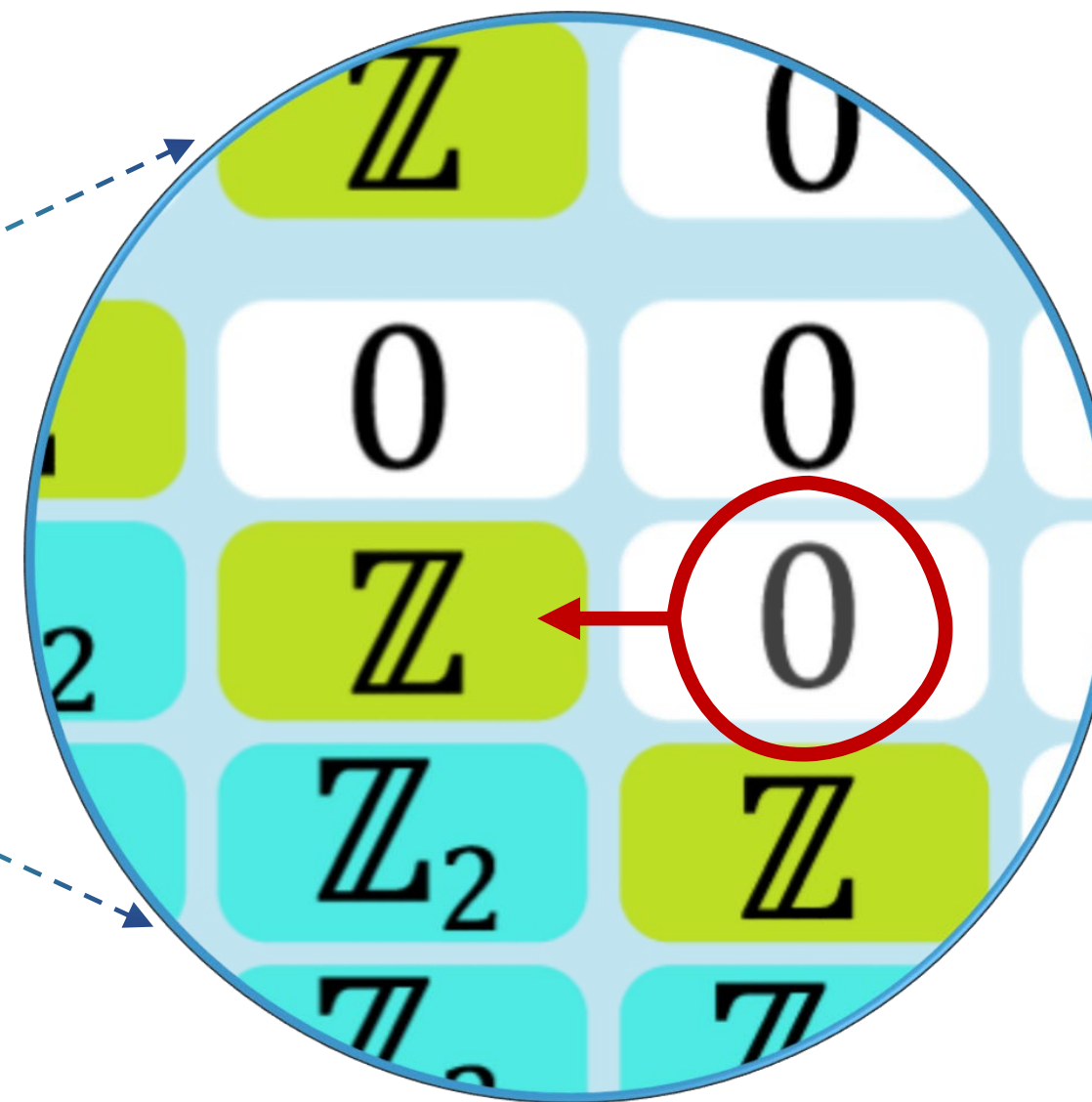
δ
 d : space dimension

Tenfold: 10 classes

Class	s	T	P	C	d=0	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$

Defects in topological crystals: Creating Quantum Matter

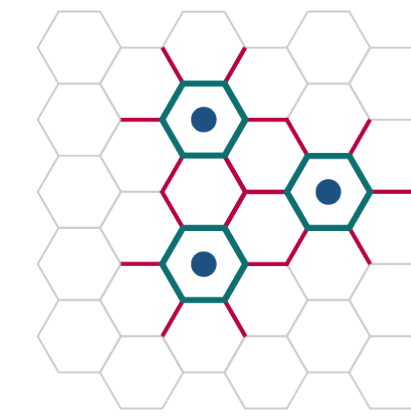
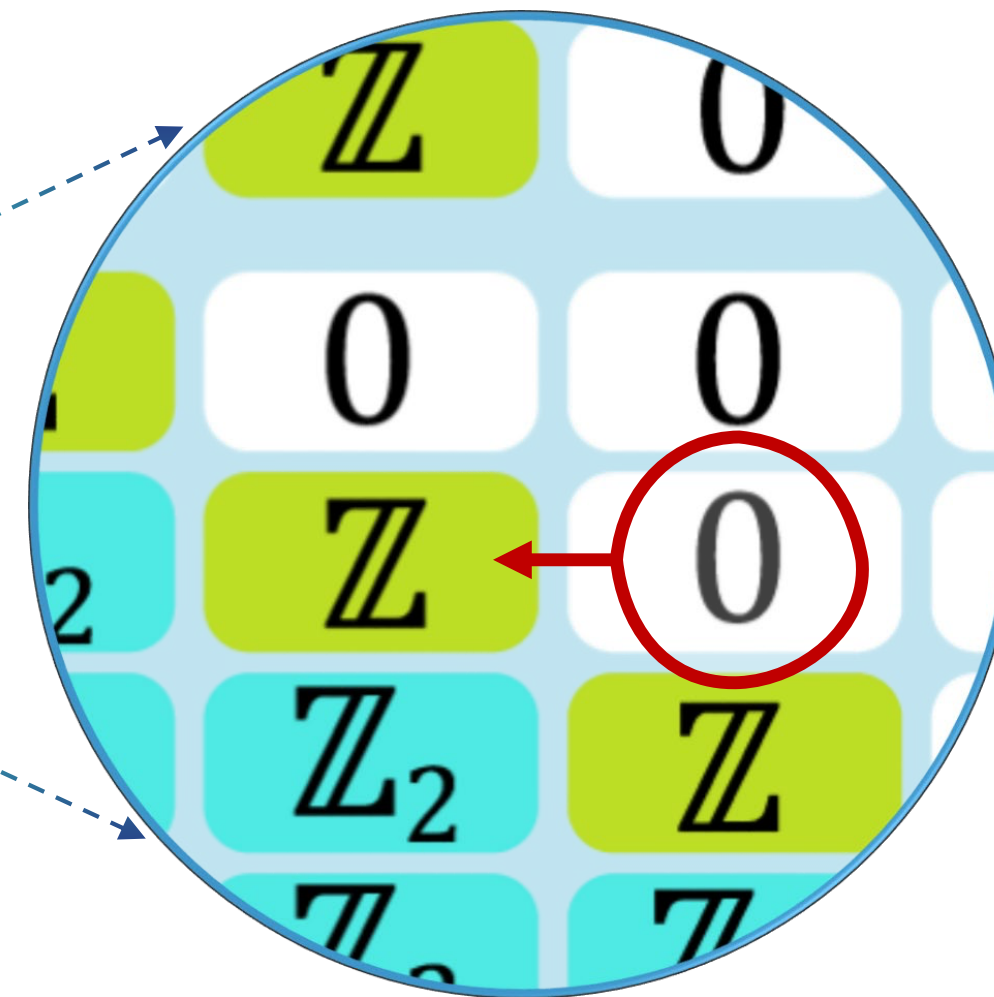
Class	S	T	P	C	$d=0$	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$



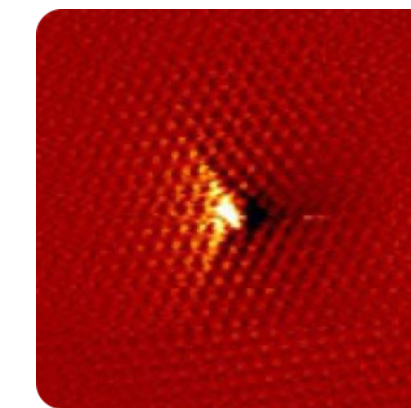
To navigate across the tenfold classification

Defects in topological crystals: Creating Quantum Matter

Class	S	T	P	C	d=0	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$



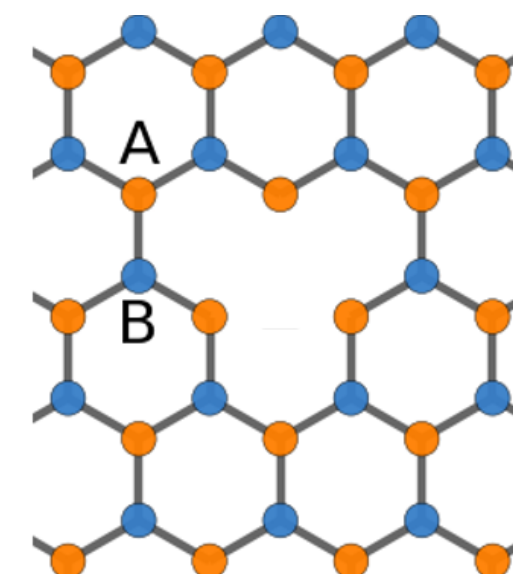
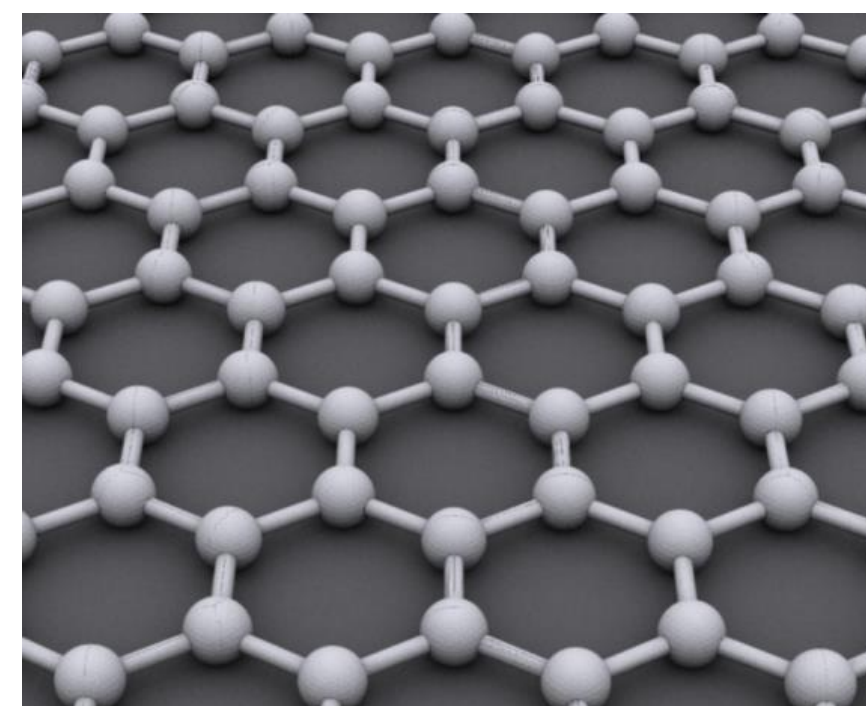
Kekule textures



Lacunar Graphene

Dig a hole in graphene makes it topological!

Graphene



Topological defects in ordered phases -

Topological defects and phase transitions

Landau Theory

Spontaneous
symmetry breaking



Order parameter $\phi(r)$
 n dimensional

What types of topological defects?

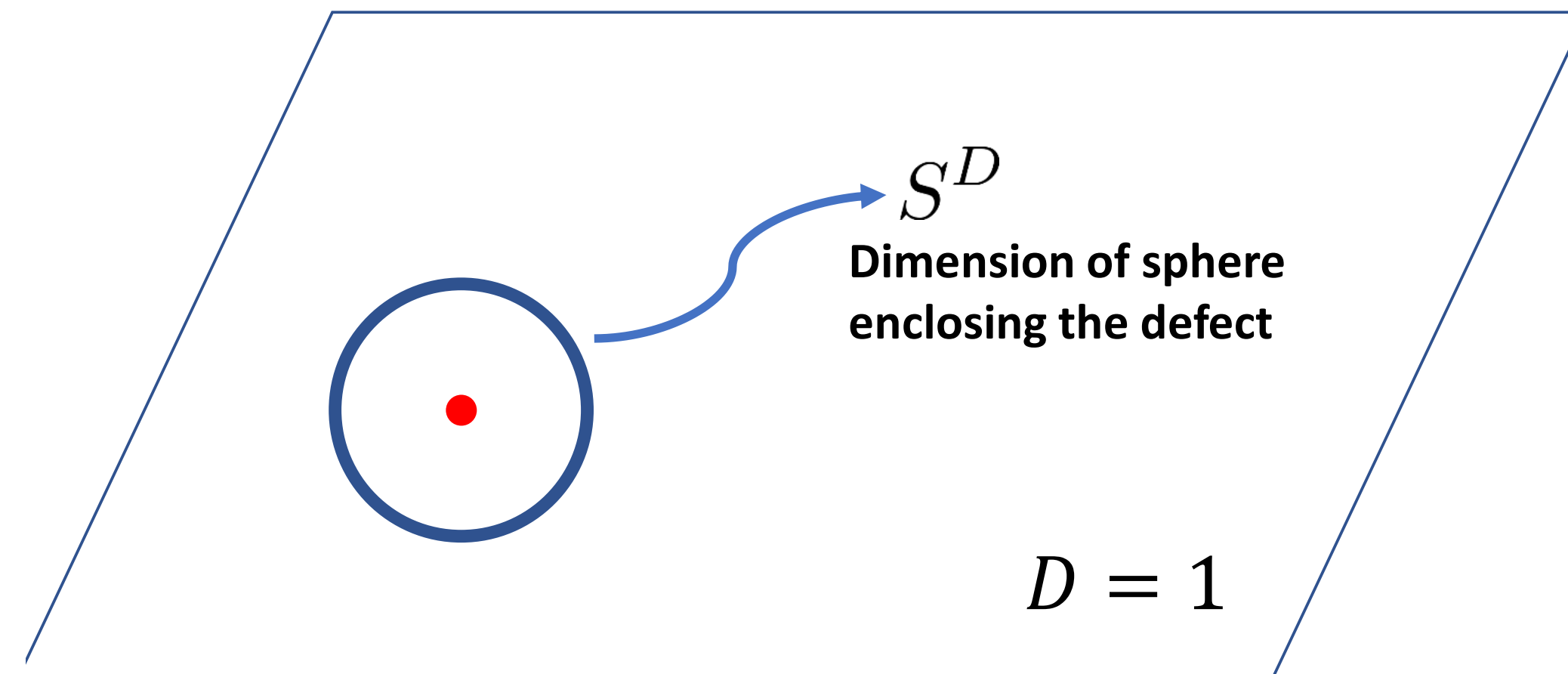
Topological defects in ordered phases -

Homotopy group
of spheres

$$\pi_D (S^{n-1}) = \begin{cases} 0 & D < n - 1 \\ \mathbb{Z} & D = n - 1 \end{cases}$$

n = dimension of order
parameter $\phi(r)$

D = dimension of sphere
enclosing the defect



Topological defects in ordered phases -

Homotopy group
of spheres

$$\pi_D (S^{n-1}) = \begin{cases} 0 & D < n - 1 \\ \mathbb{Z} & D = n - 1 \end{cases}$$

n = dimension of order
parameter $\phi(r)$

D = dimension of sphere
enclosing the defect

How many ways can a
 D -dimensional sphere
wrap (wind) around an
 $n - 1$ - dimensional sphere



Topological defects in ordered phases -

Homotopy Group $\pi_1(S^1)$

How many ways can

S^1

1-dimensional sphere

**Wrap
around**

S^1

1-dimensional sphere

Topological defects in ordered phases -

Homotopy Group $\pi_1 (S^1)$

How many ways can

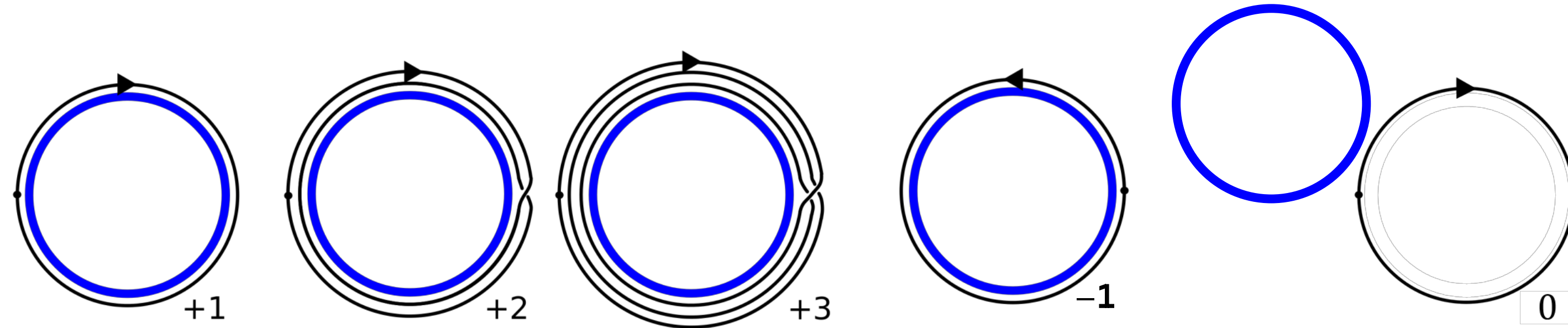
S^1

**Wrap
around**

S^1

1-dimensional sphere

1-dimensional sphere



$$\pi_1 (S^1) = \mathbb{Z}$$

Topological defects in ordered phases -

Homotopy Group $\pi_1 (S^2)$

How many ways can

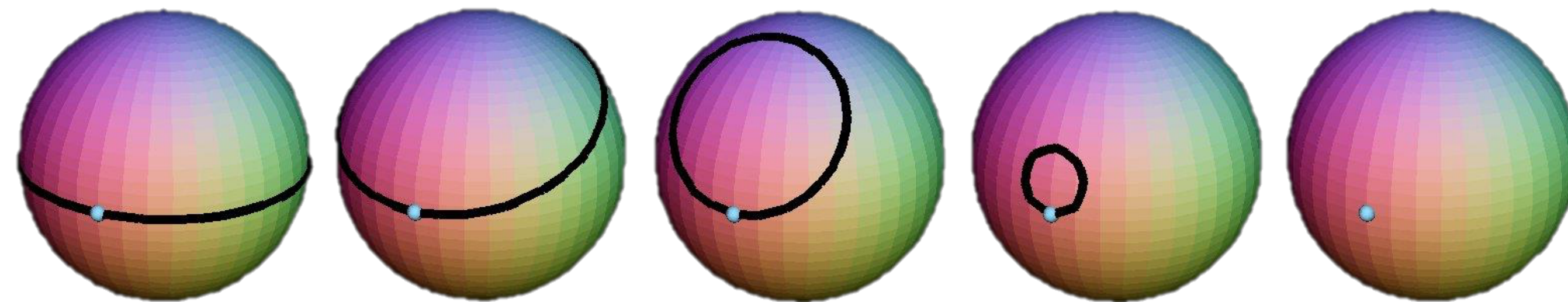
S^1

1-dimensional sphere

**Wrap
around**

S^2

2-dimensional sphere



$$\pi_1 (S^2) = 0$$

Topological defects in ordered phases -

Homotopy group of spheres

$$\pi_D (S^{n-1}) = \begin{cases} 0 & D < n - 1 \\ \mathbb{Z} & D = n - 1 \end{cases}$$

n = dimension of order parameter $\phi(r)$
 D = dimension of sphere enclosing the defect

Winding of order parameter around a defect

$$\nu_D \propto \int_{S^D} d^D r J_n(\phi)$$

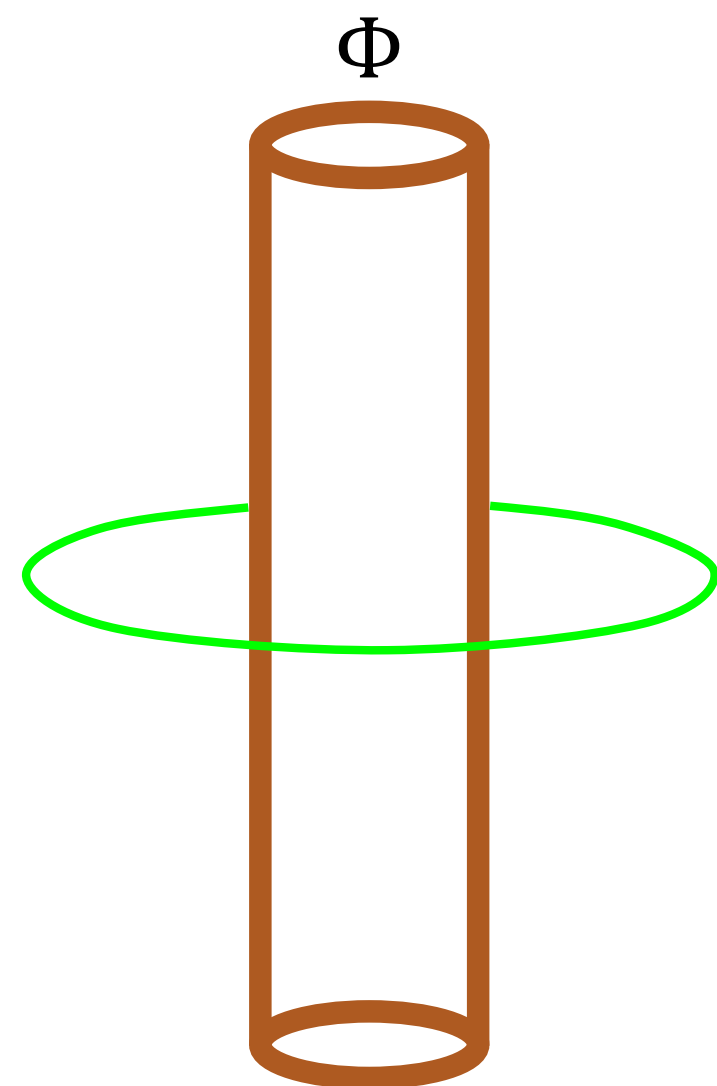
$$S^D \rightarrow S^{n-1}$$

$$J_n(\phi) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \partial_1 \phi_1 & \partial_1 \phi_2 & \cdots & \partial_1 \phi_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial_D \phi_1 & \partial_D \phi_2 & \cdots & \partial_D \phi_n \end{vmatrix}$$

Topological defects in ordered phases -

Topological Defects in Superconductors

Order parameter $\phi = |\phi|e^{i\chi} = \phi_1 + i\phi_2 \rightarrow n = 2 \rightarrow \pi_D(S^{n-1}) = \pi_1(S^1)$

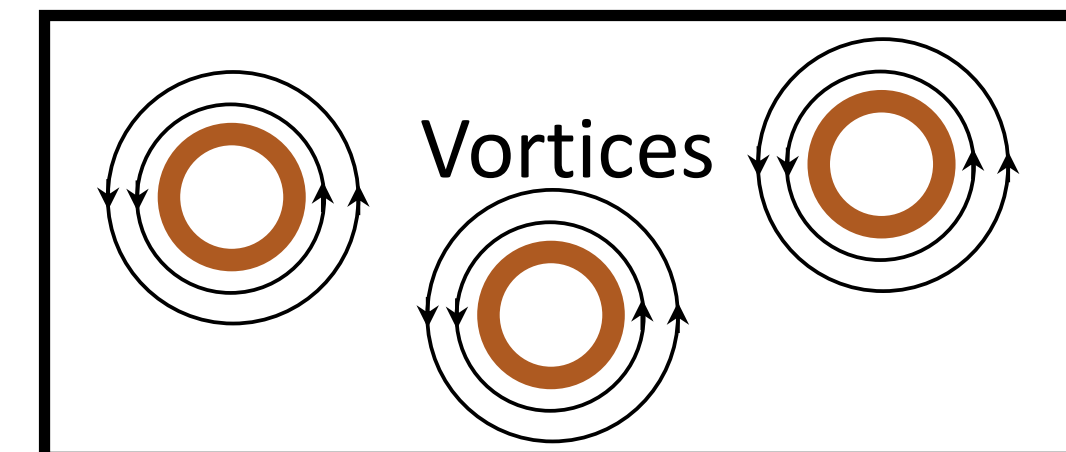


Possible topological defects

$D = 1$

In 3d \rightarrow line defect

Vortices



The Common Thread: Cartan's Tenfold Classification



Antiunitary symmetries

Time reversal Θ and particle-hole Π classify all Hamiltonians into 10 classes



Topological integers

Winding numbers and Chern numbers label disconnected phases



Symbol operator

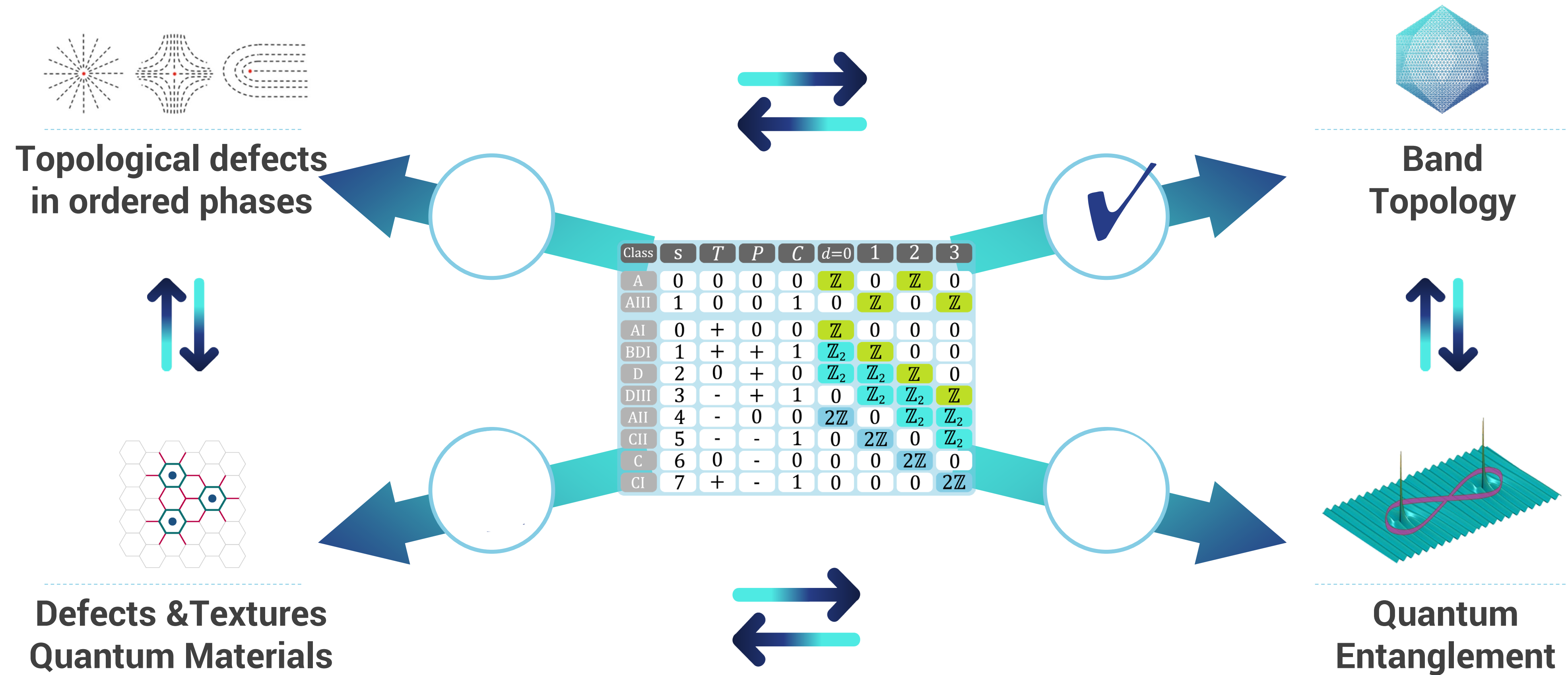
Generalises Bloch Hamiltonians to non-periodic (defect) systems



Finite-system topology

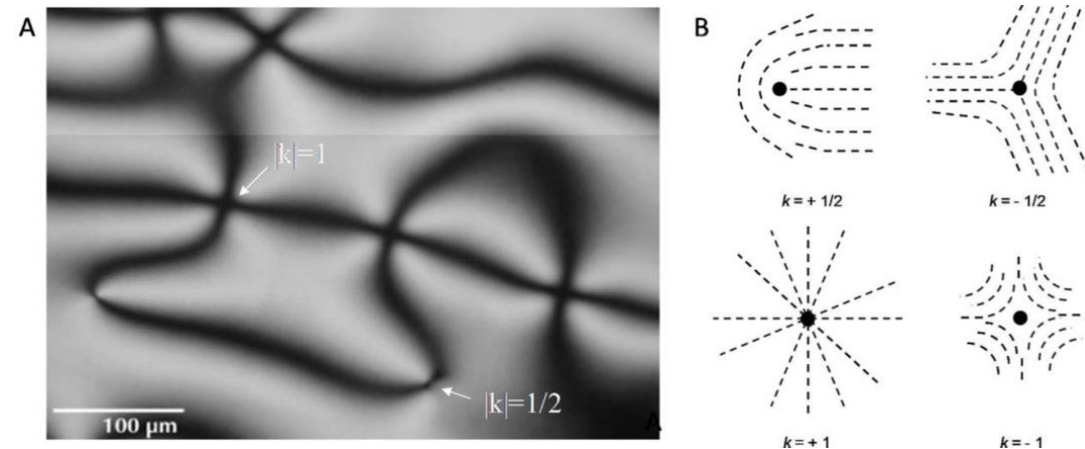
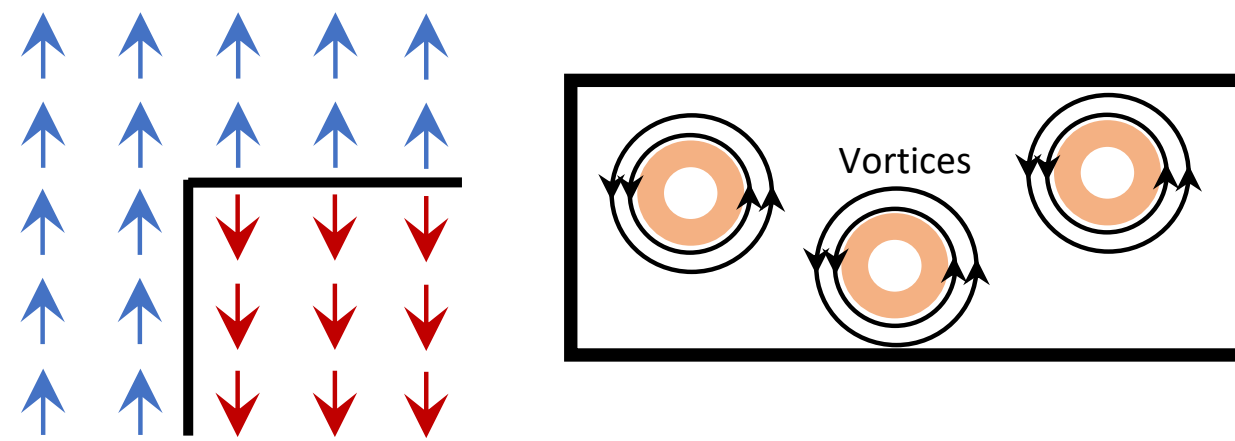
Five topological phases in a 4-dimensional two-qubit Hilbert space

A single topological classification to account for apparently “unrelated” topics



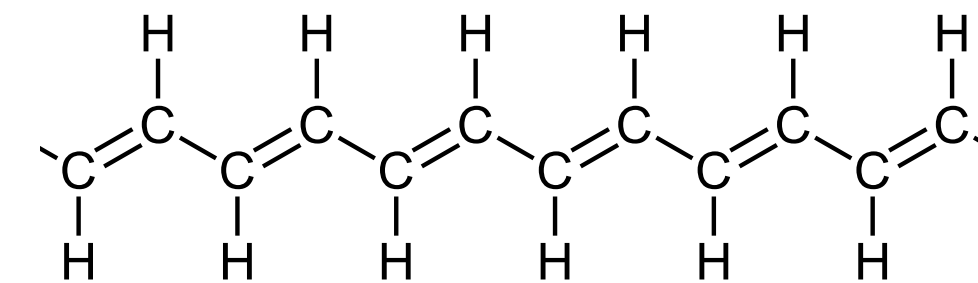
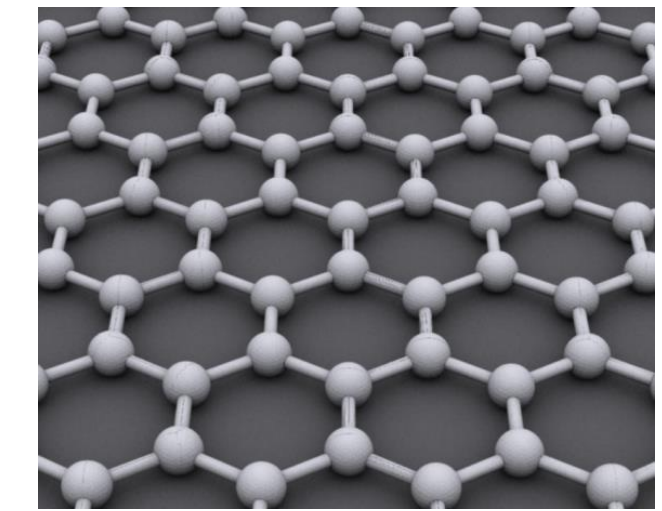
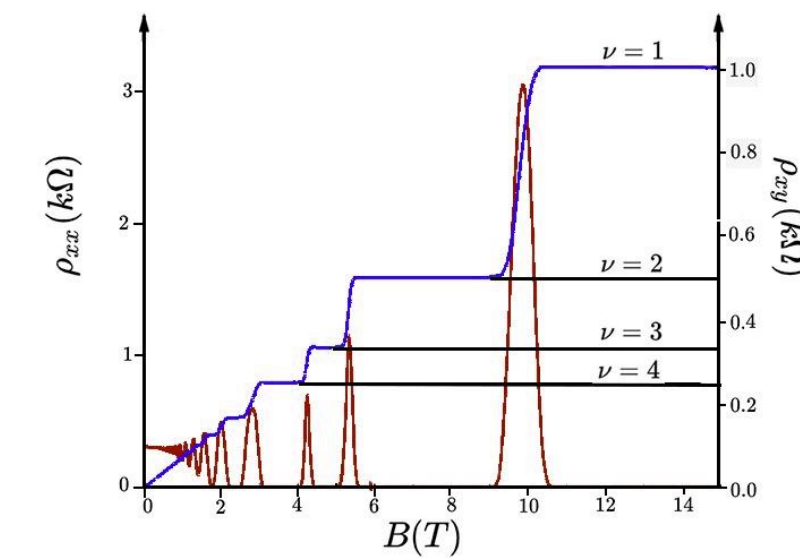
A single topological classification to account for apparently “unrelated” topics

Topological defects of ordered phases




Cartan classification

Topological materials



Computing invariant numbers associated to defects in topological materials

Identifying homotopy groups $\pi_?(S^?)$

Class	S	T	P	C	$d=0$	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$

Mapping between spheres

Wigner Symbol

$$\mathcal{H}(\mathbf{k}, \mathbf{r}) = \int_{-\infty}^{\infty} d\mathbf{r}' e^{-i\mathbf{k} \cdot \mathbf{r}'} \left\langle \mathbf{r} + \frac{\mathbf{r}'}{2} \left| H \right| \mathbf{r} - \frac{\mathbf{r}'}{2} \right\rangle$$

Weyl transform

Hamiltonian \rightarrow Symbol

$$H(\partial_{\mathbf{r}}, \hat{\mathbf{r}}) \rightarrow \mathcal{H}(\mathbf{k}, \mathbf{r})$$

$$\partial_{\mathbf{r}} \rightarrow \mathbf{k}$$

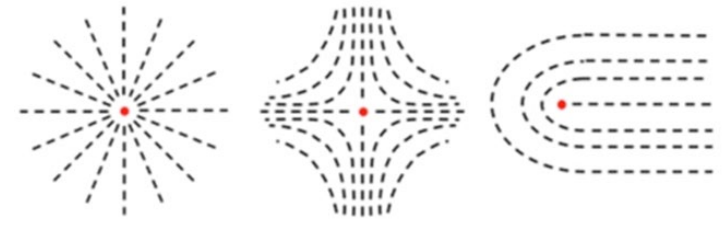
Operators \rightarrow Parameters

The Symbol $H(k,r)$: The Right Object

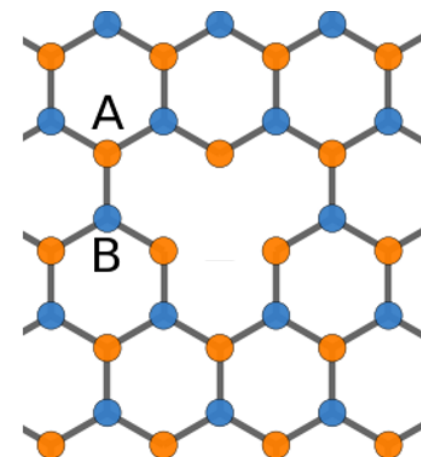
The Wigner/Weyl symbol generalises the Bloch Hamiltonian to non-periodic systems.

Wigner Symbol

Toulouse - Kleman classification



Topological defects
in ordered phases



Defects & Textures
Quantum Materials

ψ

Order
Parameter

$F(\psi)$

Landau thermodynamic
theory of ordered phases



ϕ



Wigner Symbol

$$\mathcal{H}(\mathbf{k}, \mathbf{r}) = \mathbf{k} \cdot \boldsymbol{\gamma}_a + \boldsymbol{\phi}(\mathbf{r}) \cdot \boldsymbol{\gamma}_s$$

Topology Lives in the Symbol, Not in the Hamiltonian

Hamiltonian

$$H = \sum_{i,j} t_{ij} c_i^\dagger c_j + \Delta c_i^\dagger c_j^\dagger + \Delta^\dagger c_i c_j$$

Naive: cannot apply
tenfold classification

✗ No spectral gap



Symbol

$$\mathcal{H}(\mathbf{k}, \mathbf{r}) = \mathbf{h}_a \cdot \boldsymbol{\gamma}_a + \mathbf{h}_s \cdot \boldsymbol{\gamma}_s \equiv \mathbf{h}(\mathbf{k}, \mathbf{r}) \cdot \boldsymbol{\gamma}$$

p components q+1 components

↑
Anti commuting
Dirac matrices

✓ Spectral gap exists

Topology is encoded in the symbol, not the Hamiltonian. This distinction is invisible when translation symmetry holds.

Wigner Symbol

- 1** No translation invariance \longrightarrow Use the Symbol
- 2** Hamiltonian spectrum \neq Symbol spectrum
- 3** What about topology?
In math: Atiyah-Singer index theorem
In physics: Bulk-Edge correspondence

Symbol in Tenfold Classification **with Defects**

$$H = \sum_{i,j} t_{ij} c_i^\dagger c_j + \Delta c_i^\dagger c_j^\dagger + \Delta^\dagger c_i c_j$$

Bogoliubov-de Gennes Hamiltonians

$$\mathcal{H}(\mathbf{k}, \mathbf{r}) = \mathbf{h}_a \cdot \boldsymbol{\gamma}_a + \mathbf{h}_s \cdot \boldsymbol{\gamma}_s \equiv \mathbf{h}(\mathbf{k}, \mathbf{r}) \cdot \boldsymbol{\gamma}$$

p components q+1 components

↑
Anti commuting
Dirac matrices

$$\begin{aligned} \mathbf{k} &\in S^d \\ \mathbf{r} &\in S^D \\ \mathbf{h} &\in S^{p+q} \end{aligned}$$

Symmetry class $s = p - q$
Codimension $\delta = d - D$

$$\pi_{d+D} (S^{p+q})$$

?

Integer Topological Invariants

Non chiral – Chern

$$\mathcal{C}_{2m} \propto \int_{S^{d+D}} \text{Tr} \left[\hat{\mathcal{H}} (d\hat{\mathcal{H}})^{2m} \right]$$

Chiral – Winding

$$\nu_{2m-1} \propto \int_{S^{d+D}} \text{Tr} \left[(q^\dagger dq)^{2m-1} \right]$$

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & q \\ q^\dagger & 0 \end{pmatrix}$$

Class	S	T	P	C	d=0	1	2	3
A	0	0	0	0	\mathbb{Z}	0	\mathbb{Z}	0
AIII	1	0	0	1	0	\mathbb{Z}	0	\mathbb{Z}
AI	0	+	0	0	\mathbb{Z}	0	0	0
BDI	1	+	+	1	\mathbb{Z}_2	\mathbb{Z}	0	0
D	2	0	+	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}	0
DIII	3	-	+	1	0	\mathbb{Z}_2	\mathbb{Z}_2	\mathbb{Z}
AII	4	-	0	0	$2\mathbb{Z}$	0	\mathbb{Z}_2	\mathbb{Z}_2
CII	5	-	-	1	0	$2\mathbb{Z}$	0	\mathbb{Z}_2
C	6	0	-	0	0	0	$2\mathbb{Z}$	0
CI	7	+	-	1	0	0	0	$2\mathbb{Z}$

Integer Topological Invariants

Non chiral – Chern #

$$\mathcal{C}_{2m} \propto \int_{S^{d+D}} \text{Tr} \left[\hat{\mathcal{H}} \left(d\hat{\mathcal{H}} \right)^{2m} \right]$$

$$\mathcal{H}(\mathbf{k}, \mathbf{r}) = \mathbf{h}_a \cdot \boldsymbol{\gamma}_a + \mathbf{h}_s \cdot \boldsymbol{\gamma}_s \equiv \mathbf{h}(\mathbf{k}, \mathbf{r}) \cdot \boldsymbol{\gamma}$$

p components q+1 components



$$\nu_{d+D} \propto \int_{S^{d+D}} d^d k d^D r J_{p+q+1}(\mathbf{h})$$

Chiral – Winding #

$$\nu_{2m-1} \propto \int_{S^{d+D}} \text{Tr} \left[(q^\dagger dq)^{2m-1} \right]$$

$$\hat{\mathcal{H}} = \begin{pmatrix} 0 & q \\ q^\dagger & 0 \end{pmatrix}$$

$$J_{p+q+1}(\mathbf{h}) = \begin{vmatrix} h_1 & h_2 & \cdots & h_{p+q+1} \\ \partial_1 h_1 & \partial_1 h_2 & \cdots & \partial_1 h_{p+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{d+D} h_1 & \partial_{d+D} h_2 & \cdots & \partial_{d+D} h_{p+q+1} \end{vmatrix}$$

$(\mathbf{k}, \mathbf{r}) \rightarrow \mathbf{h}$

$$\pi_{d+D} \left(S^{p+q} \right)$$

!

Symbol in Tenfold Classification **with Defects**

Thermodynamics

Order parameter $\phi(\mathbf{r})$

Winding number

$$\nu_D \propto \int_{S^D} d^D r J_n(\phi)$$

$$J_n(\phi) = \begin{vmatrix} \phi_1 & \phi_2 & \cdots & \phi_n \\ \partial_1 \phi_1 & \partial_1 \phi_2 & \cdots & \partial_1 \phi_n \\ \vdots & \vdots & \ddots & \vdots \\ \partial_D \phi_1 & \partial_D \phi_2 & \cdots & \partial_D \phi_n \end{vmatrix}$$

$$\pi_D(S^{n-1})$$

Tenfold classification

Dirac Symbol $\mathcal{H}(\mathbf{k}, \mathbf{r}) = \mathbf{h}_a \cdot \gamma_a + \mathbf{h}_s \cdot \gamma_s$

Winding number

$$\nu_{d+D} \propto \int_{S^{d+D}} d^d k d^D r J_{p+q+1}(\mathbf{h})$$

$$J_{p+q+1}(\mathbf{h}) = \begin{vmatrix} h_1 & h_2 & \cdots & h_{p+q+1} \\ \partial_1 h_1 & \partial_1 h_2 & \cdots & \partial_1 h_{p+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{d+D} h_1 & \partial_{d+D} h_2 & \cdots & \partial_{d+D} h_{p+q+1} \end{vmatrix}$$

$$\pi_{d+D}(S^{p+q})$$

Two Qubits Go Topological

Orion & Akkermans · Phys. Rev. B 111, 245408 (2025)

- The Wootters concurrence = antiunitary time-reversal operator Θ
- Cartan decomposition splits operators into entangling and non-entangling
- Five topological phases Z_5 in a four-dimensional Hilbert space
- Bell-state zero modes at topological phase boundaries

Wootters Concurrence and Time-Reversal Symmetry

Can the tenfold classification — built for large systems — say anything about a 4-dimensional Hilbert space?

Wootters concurrence: $C\psi = |\langle \psi | \Theta | \psi \rangle|$ with $\Theta = (\sigma_y \otimes \sigma_y)K$

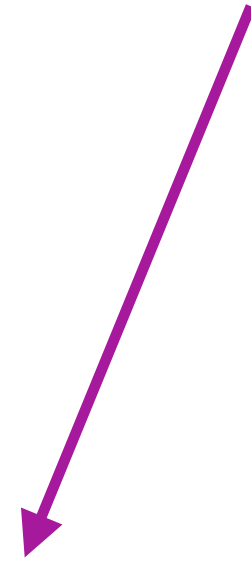
$\Theta = (\sigma_y \otimes \sigma_y)K$: Time-reversal operator for two spin-1/2 particles. K = complex conjugation in computational basis.

$\Theta^2 = +1_4$: Squares to +1 \rightarrow class AI symmetry. This is the same operator that classifies the symmetry class.

$C\psi = 0$: Separable state. $C\psi = 1$: maximally entangled (Bell) state. Same Θ defines both entanglement and topology.

Not a coincidence: The operator that measures entanglement identifies the symmetry class. This is the heart of the connection.

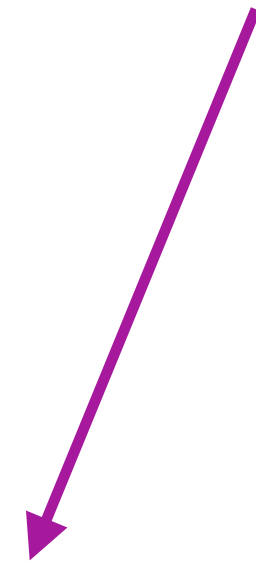
Cartan decomposition using anti-unitaries



Tenfold topological classification

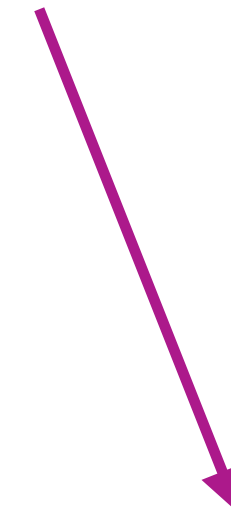
Stone et al (2010)

Cartan decomposition using anti-unitaries



Tenfold topological
classification

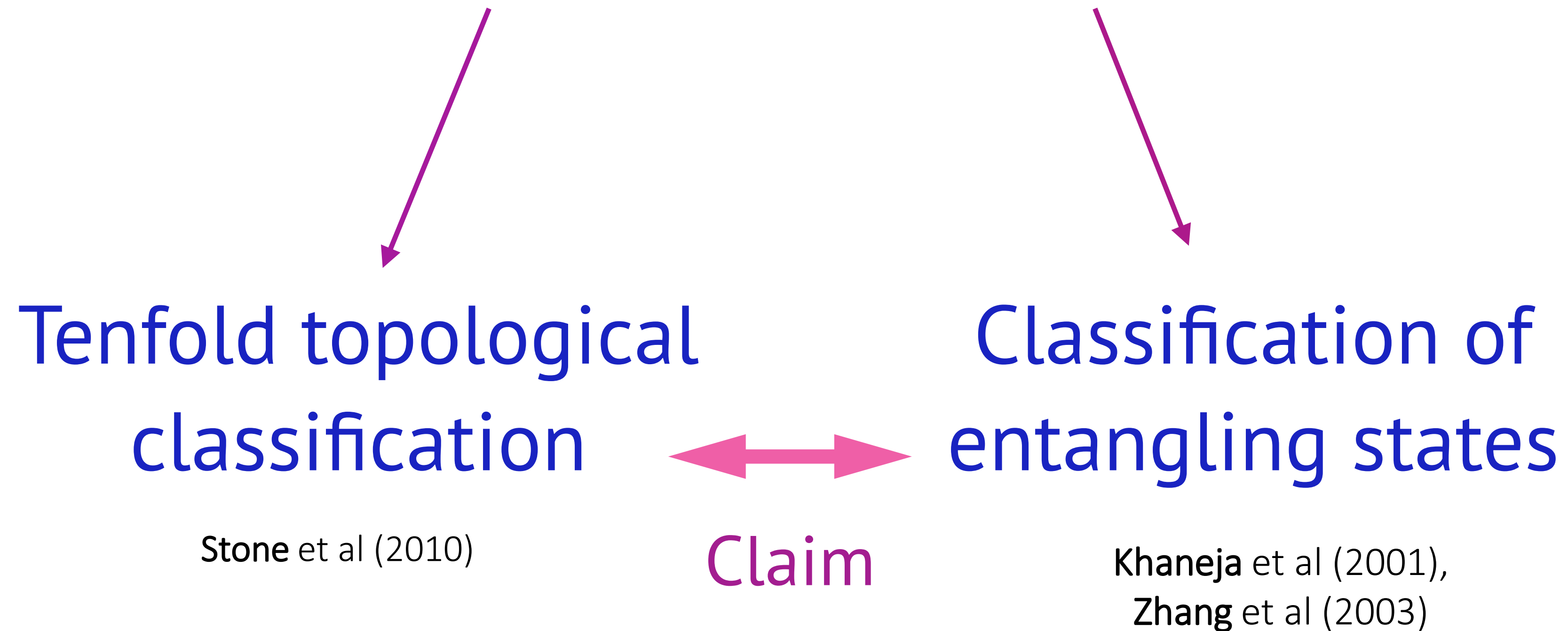
Stone et al (2010)



Classification of
entangling states

Khaneja et al (2001),
Zhang et al (2003)

Cartan decomposition using anti-unitaries



Topology characterises entanglement

Classification of entangling states

Cartan decomposition

Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I, I \otimes \sigma_j, \sigma_i \otimes \sigma_j$

Cartan

Decomposition:

$$g = h \oplus m$$

$$\Theta = (\sigma_y \otimes \sigma_y)K$$

Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I, I \otimes \sigma_j, \sigma_i \otimes \sigma_j$

Cartan

Decomposition:

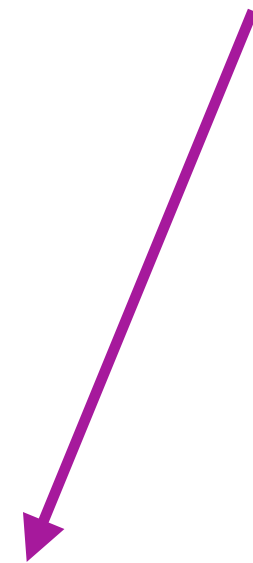
$$g = h \oplus m$$

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$$C_\psi \equiv |\langle \psi | \Theta | \psi \rangle|$$

Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I, I \otimes \sigma_j, \sigma_i \otimes \sigma_j$



Cartan
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$$g = h \oplus m$$

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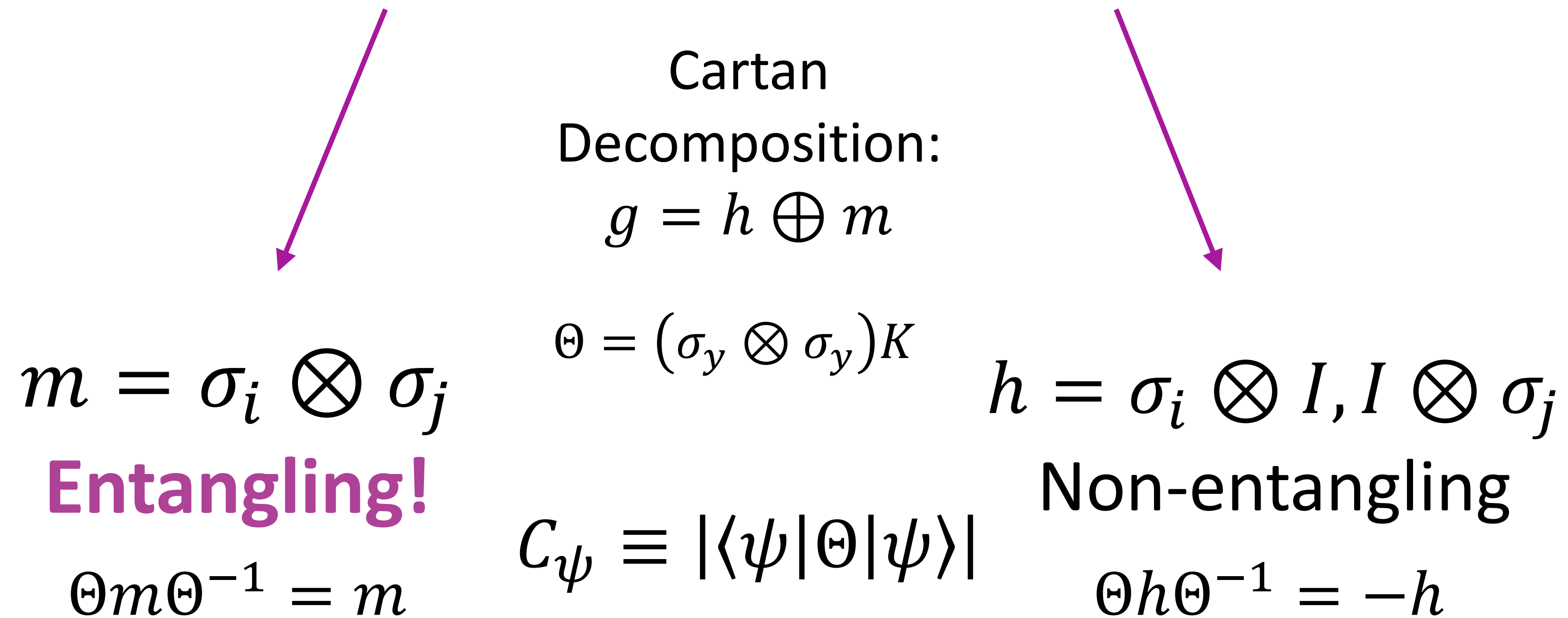
Entangling!

$$\Theta m \Theta^{-1} = m$$

$$C_\psi \equiv |\langle \psi | \Theta | \psi \rangle|$$

Cartan decomposition

Two qubit Hamiltonians: $g = \sigma_i \otimes I, I \otimes \sigma_j, \sigma_i \otimes \sigma_j$



Cartan decomposition

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$$m = \sigma_i \otimes \sigma_j$$

Entangling!

$$\Theta m \Theta^{-1} = m$$

Class AI

$$C_\psi \equiv |\langle \psi | \Theta | \psi \rangle|$$

$$h = \sigma_i \otimes I, I \otimes \sigma_j$$

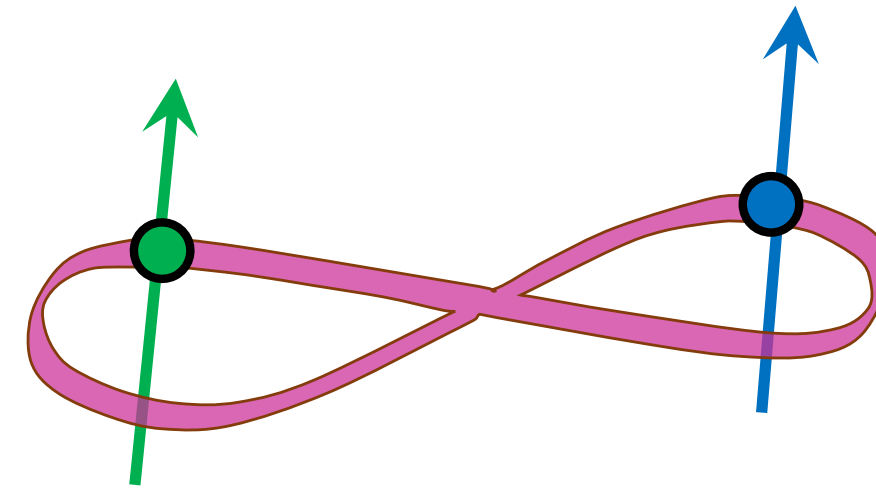
Non-entangling

$$\Theta h \Theta^{-1} = -h$$

Class A

Two qubits in the tenfold classification

Class	Θ	Π	S	$d = 0$
A	0	0	0	\mathbb{Z}
AIII	0	0	1	0
AI	+	0	0	\mathbb{Z}
BDI	+	+	1	\mathbb{Z}_2
D	0	+	0	\mathbb{Z}_2
DIII	-	+	1	0
AII	-	0	0	$2\mathbb{Z}$
CII	-	-	1	0
C	0	-	0	0
CI	+	-	1	0



$d = 0$

$$\Theta = (\sigma_y \otimes \sigma_y)K$$

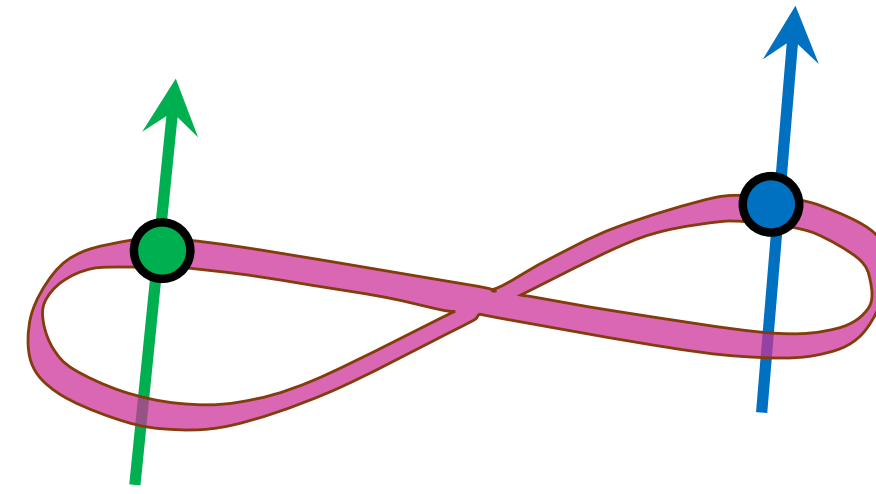
$$\Theta^2 = +$$

No Π

$$\Pi^2 = "0"$$

Two qubits in the tenfold classification

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CI	+	-	1	0



$d = 0$

$$\Theta = (\sigma_y \otimes \sigma_y)K$$

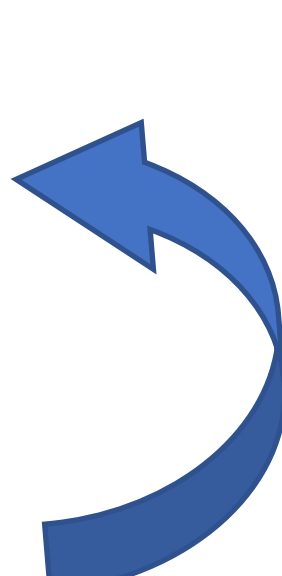
No Π

$$\Theta^2 = +$$

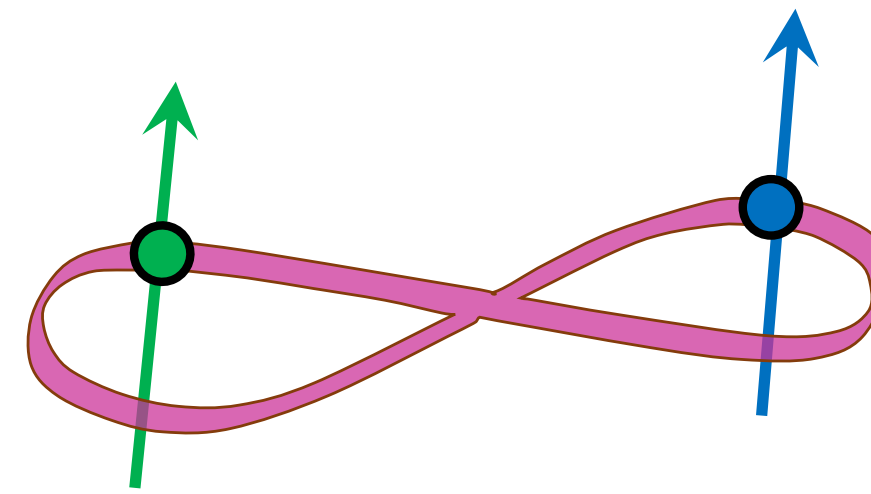
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CI	+	-	1	0



Breaks time reversal



$d = 0$

Add non-entangling terms:

$$\Theta h \Theta^{-1} = -h$$

No Θ

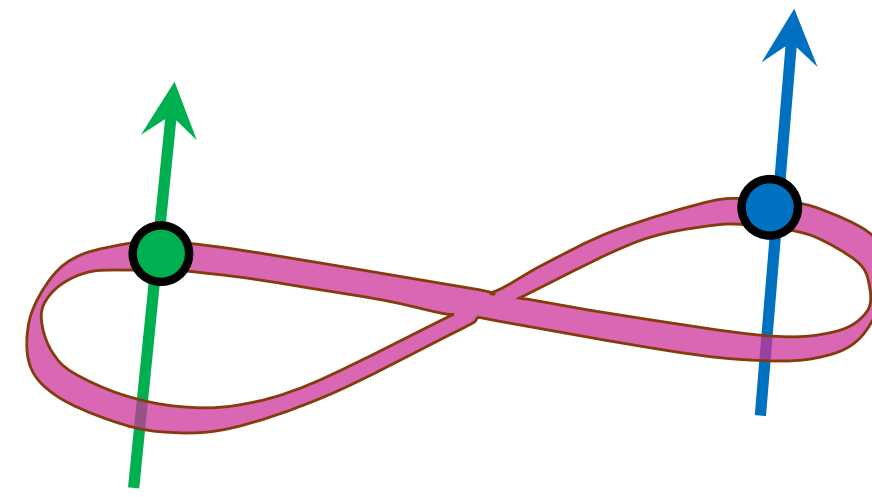
$$\Theta^2 = "0"$$

No Π

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CI	+	-	1	0



$d = 0$

What is the integer?

$$\Theta = (\sigma_y \otimes \sigma_y)K$$

$$\Theta^2 = +$$

No Π

$$\Pi^2 = "0"$$

Topological integers

- Understanding and measuring them

Geometric phase - Concurrence - Quantum gates

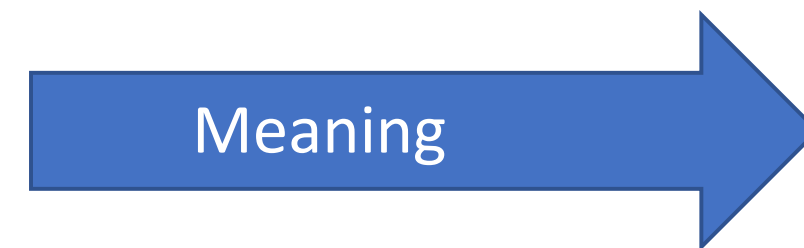
- Computing topological integers

Quantum graphs - Effective tight binding model -
topological phase transitions

Understanding topological integers

$\gamma(C)$ Berry Phase is \mathbb{Z}

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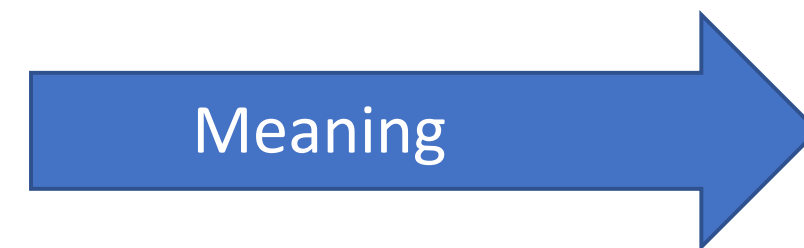


Describes Hamiltonians via topological invariants

Understanding topological integers

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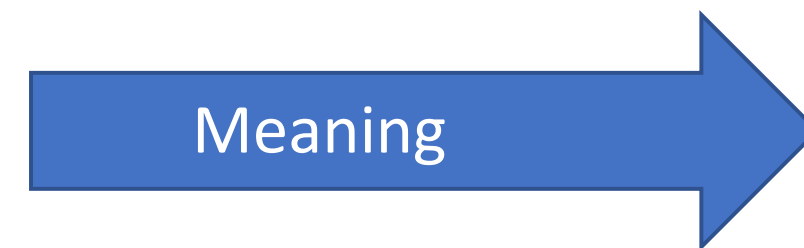
Describes Hamiltonians via topological invariants

Mathematically: $\pi_0(H)$ Homotopy Group

Understanding topological integers

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Describes Hamiltonians via topological invariants

Mathematically: $\pi_0(H)$ Homotopy Group

$$\pi_0(H) = \pi_1(e^{-iHt})$$

Understanding topological integers

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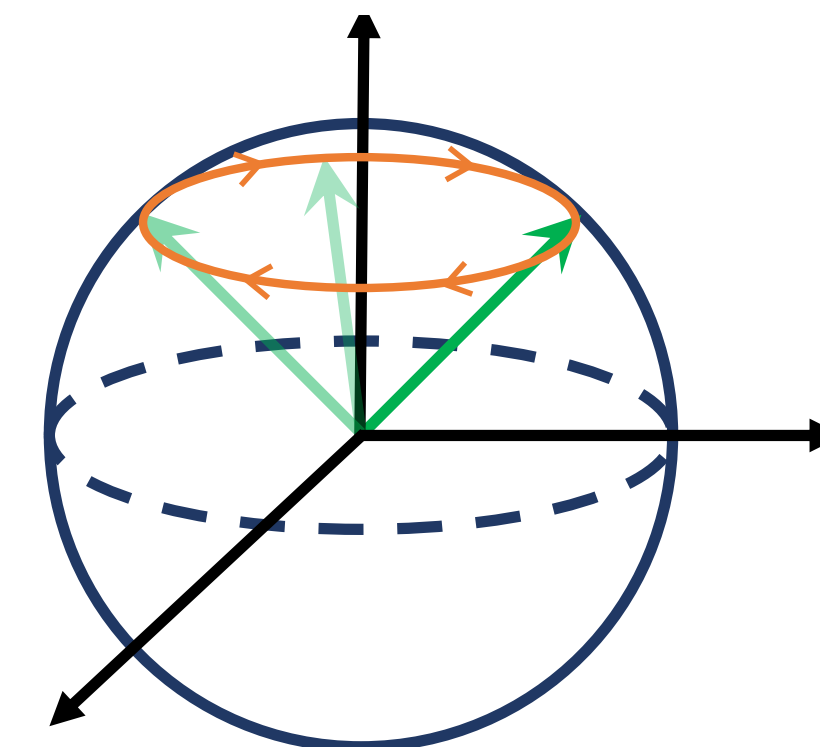


Describes Hamiltonians via topological invariants

Mathematically: $\pi_0(H)$ Homotopy Group

$$\pi_0(H) = \pi_1(e^{-iHt})$$

$\pi_1(e^{-iHt}) =$ **Closed paths of time evolution**



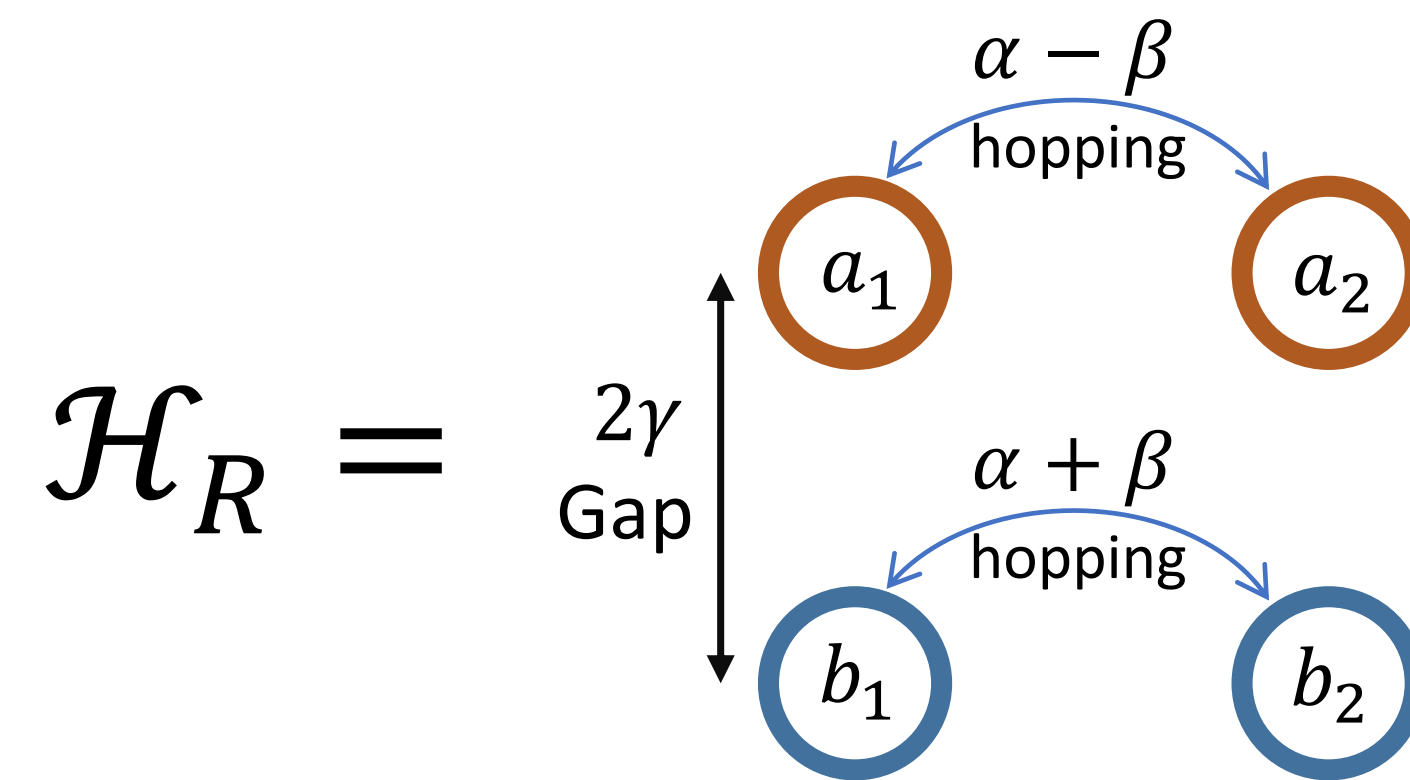
Computing topological integers

Quantum graphs - Effective tight binding model -
topological phase transitions

$$H_R = \alpha \sigma_x \otimes \sigma_x + \beta \sigma_y \otimes \sigma_y + \gamma \sigma_z \otimes \sigma_z$$

Interpret H_R as hopping in a lattice:

$|00\rangle \rightarrow$ site a_1 $|11\rangle \rightarrow$ site a_2 $|01\rangle \rightarrow$ site b_1 $|10\rangle \rightarrow$ site b_2

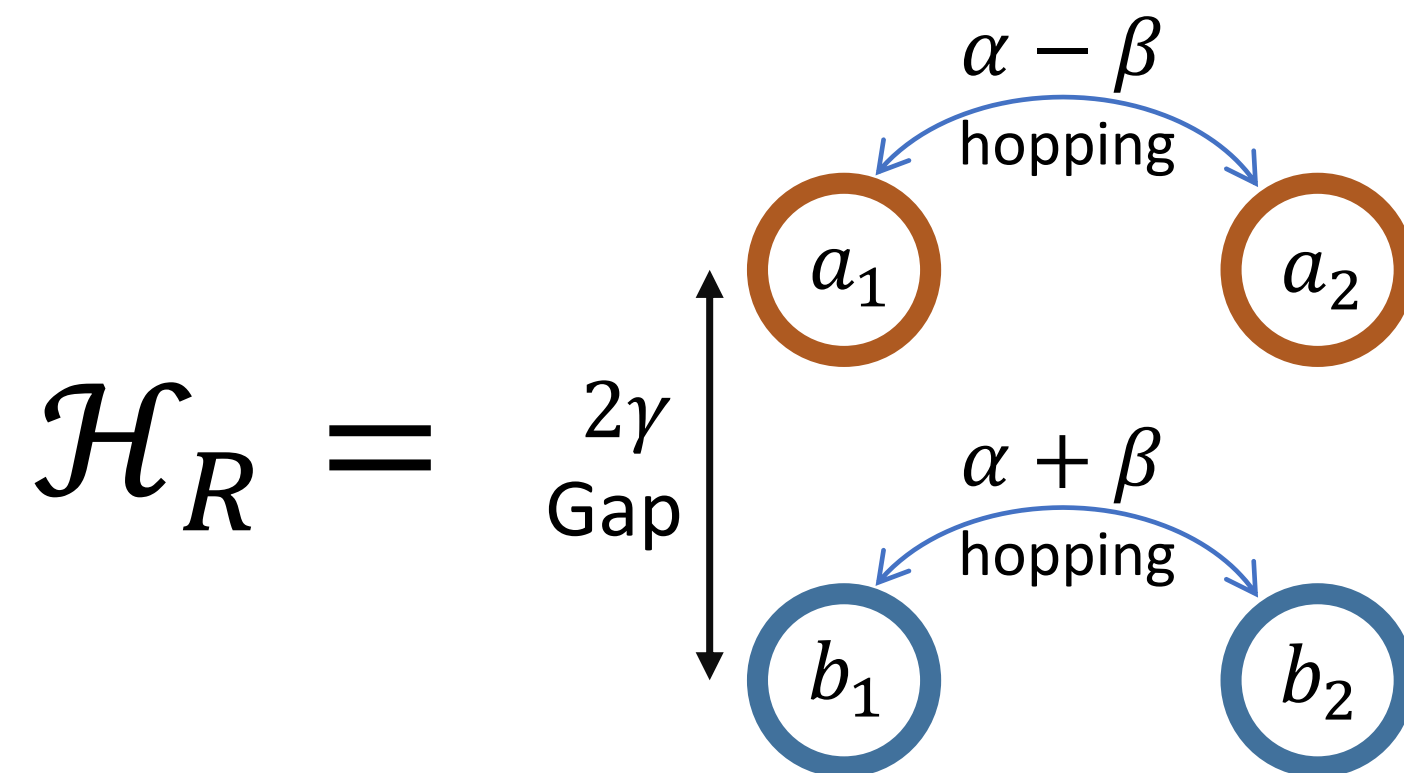


Computing topological integers

$$\text{Topological number:}$$
$$\nu = \nu_a + \nu_b \in \{0, \pm 1, \pm 2\}$$

$$\nu_a = \begin{cases} 1 & \gamma > \alpha - \beta \\ -1 & \gamma < -(\alpha - \beta) \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_b = \begin{cases} 1 & \gamma > \alpha + \beta \\ -1 & \gamma < -(\alpha + \beta) \\ 0 & \text{otherwise} \end{cases}$$



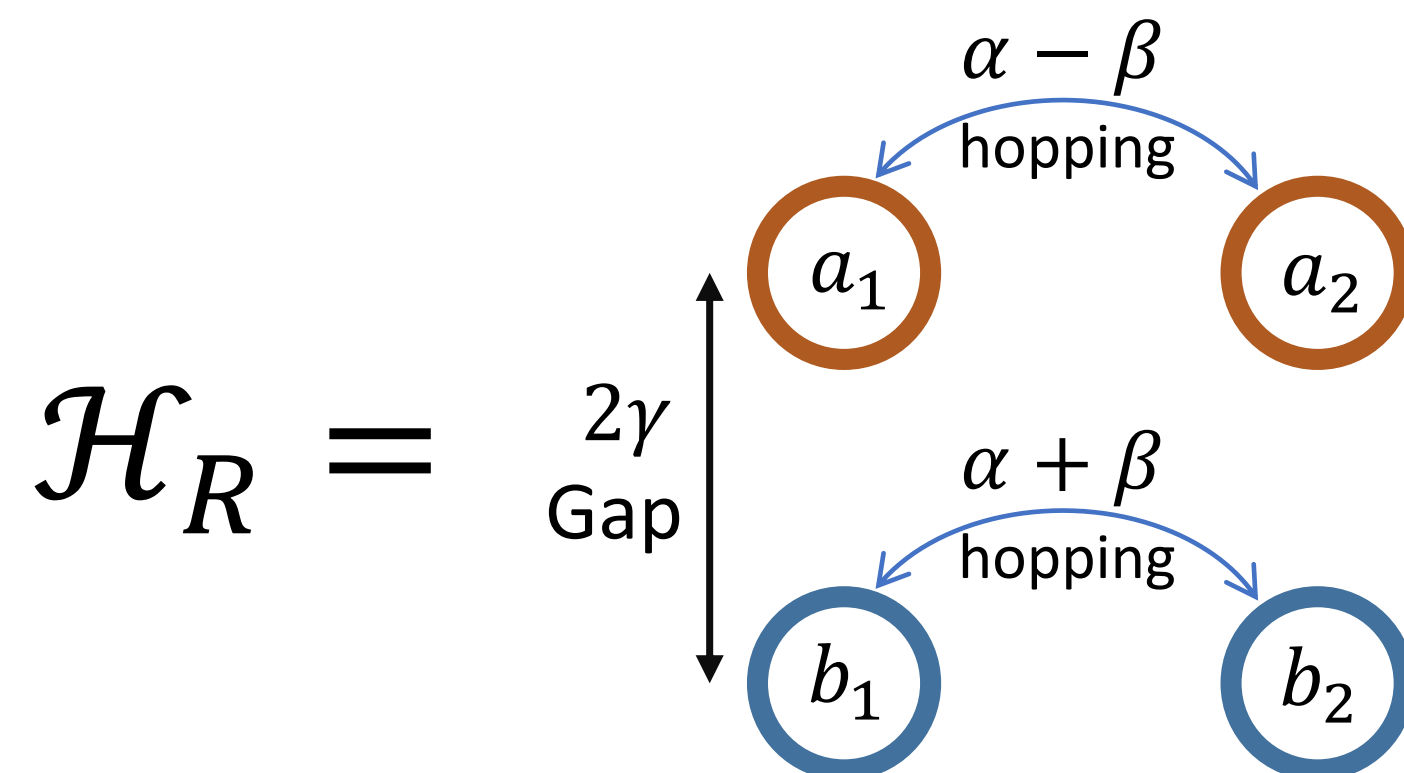
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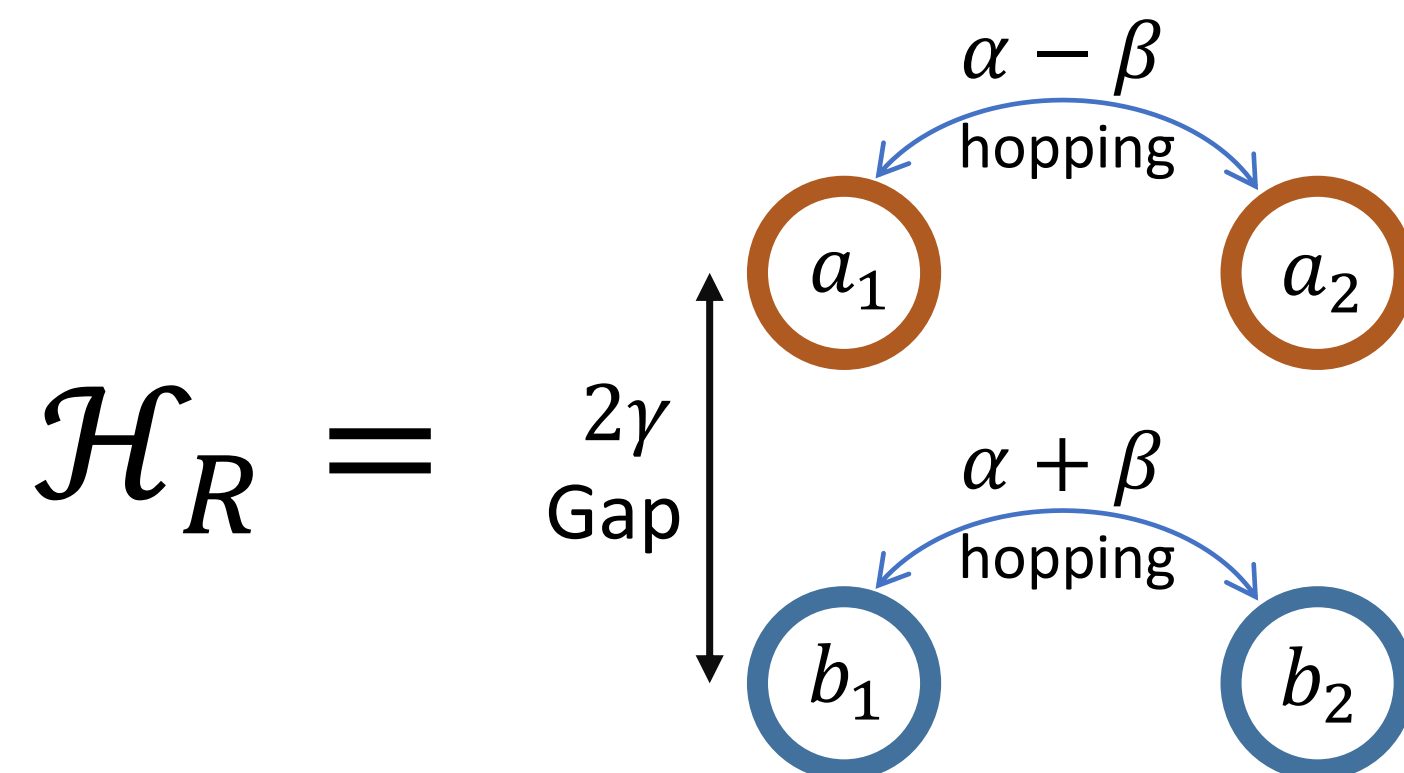
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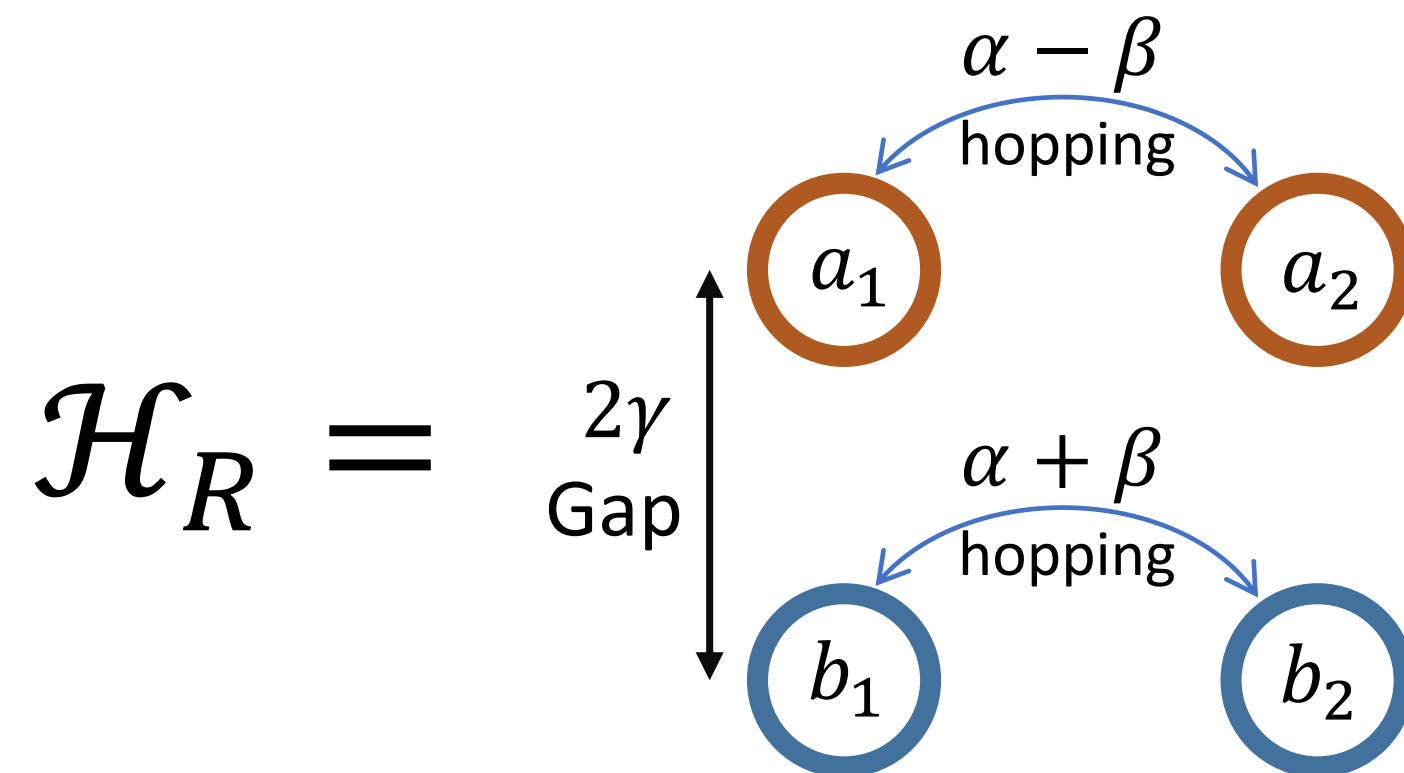
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topological phase transitions

CLOSING — The Unified Picture

	Graphene + defects	Two qubits	Quantum gates
Key object	● Symbol $H(k,r)$	● Cartan decomposition	● Winding number νH
Topology	● Defect dimension D	● Z_5 phases, class AI	● Disconnected H classes
Observable	● Zero modes (edge states)	● Concurrence / Bell states	● Geometric phases
Tool	● Index theorem	● Homotopy groups	● Sum rule $\sum \gamma_n = 2\pi\nu U$

Conclusions and Open Directions

Topology is a necessary condition for entanglement.

The Weyl symbol $H(k,r)$ is the right object for topological classification — not the Hamiltonian.

Bell states are zero-dimensional topological zero modes — maximally entangled states at phase boundaries.

Open directions

- Extension to $N > 2$ qubits via Uhlmann concurrences and multi-qubit Cartan decompositions
- Topological modes in condensed matter (Hubbard model, quantum dot arrays) as protected distinguishable qubits
- Pre-calibration diagnostics: geometric phases identify which topological class is most noise-robust

Thanks to my research group



**Yuval
Abulafia**



**Alon
Strugatsky**



**Boris
Rotstein**



**Vishesh
Makwana**



**Nadav
Orion**



**Anna
Hasine**



**Sean
Faktor**



**Amit
Goft**

Thank you for your attention.

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