Spectral Theory on Self-Similar Groups, Graphs, and Fractals

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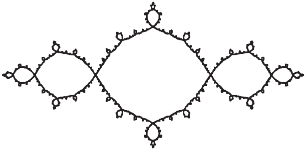


Rochester * 2025

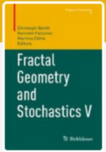
outline of the talk:

- Introduction and motivation.
- **Algebraic applications: spectrum of the Laplacian on the Basilica Julia set** (with Rogers, Brzoska, George, Jarvis arXiv:1908.10505).
- selected technical details (if time permits)

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by **carefully building approximations by graphs or manifolds**.

abstract of the talk

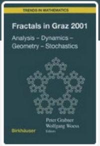
This talk explores how spectral theory, graph geometry, and dynamical systems are applied to study the random walk generator on finitely ramified self-similar graphs and fractals. Such structures often exhibit pure point or singular continuous spectra, as seen in examples like the Sierpinski triangle, the Vicsek tree, and the Schreier graphs of the Hanoi group studied by Bartholdi, Grigorchuk, Lyubich, Nagnibeda, Sunic, Zuk et al. A more intricate instance involves the Basilica Julia set of the polynomial $z^2 - 1$ and its Iterated Monodromy Group, introduced by Nekrashevych. The spectrum of the Basilica Julia set was analyzed numerically by Strichartz et al. and analytically in collaboration with Luke Rogers and students at UConn, as well as independently by Dang, Grigorchuk, and Lyubich in the paper "Self-similar groups and holomorphic dynamics: renormalization, integrability, and spectrum." We will discuss the background and new results on the spectral analysis of self-similar graphs and their fractal limits, highlighting spectral dimensions, self-similar random walks, diffusion limits, and the role of symmetries and finite ramification in explicit spectral computations.



Fractal Geometry and Stochastics V pp 175–207 | Cite as

From Self-Similar Groups to Self-Similar Sets and Spectra

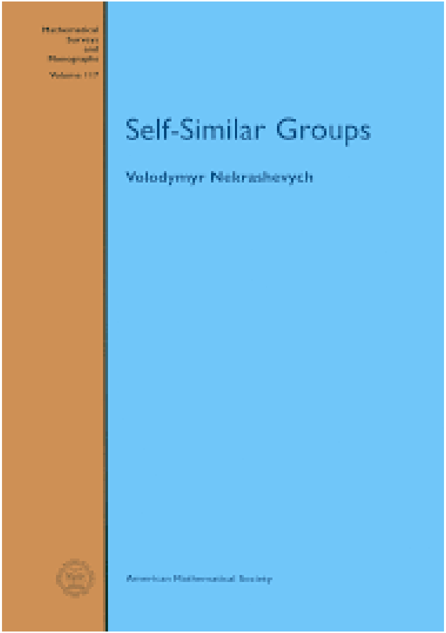
Rostislav Grigorchuk, Volodymyr Nekrashevych & Zoran Šunić



Fractals in Graz 2001 pp 145–183 | Cite as

Random Walks on Sierpiński Graphs: Hyperbolicity and Stochastic Homogenization

Vadim A. Kaimanovich


Self-Similar Groups
Volodymyr Nekrashevych
American Mathematical Society

[Submitted on 1 Oct 2020 (v1), last revised 13 Jan 2021 (this version, v2)]

Self-similar groups and holomorphic dynamics: Renormalization, integrability, and spectrum

Nguyen-Bac Dang, Rostislav Grigorchuk, Mikhail Lyubich

In this paper, we explore the spectral measures of the Laplacian on Schreier graphs for several self-similar groups (the Grigorchuk, Lamplighter, and Hanoi groups) from the dynamical and algebro-geometric viewpoints. For these graphs, classical Schur renormalization transformations act on appropriate spectral parameters as rational maps in two variables. We show that the spectra in question

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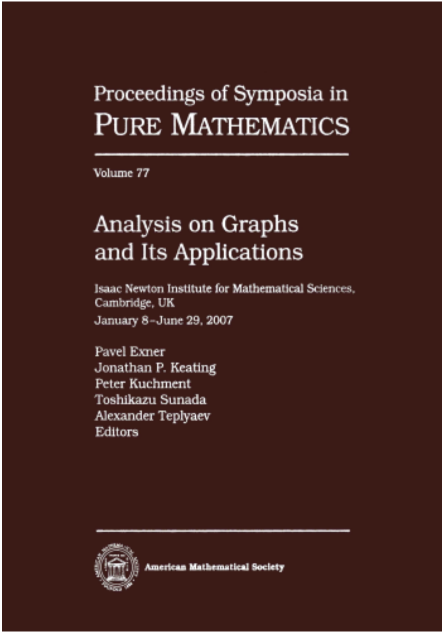
Journal of Fractal Geometry

Ends of Schreier graphs and cut-points of limit spaces of self-similar groups

Ievgen Bondarenko,[1] Daniele D'Angeli,[2] and Tatiana Nagnibeda[3]



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Groups and analysis on fractals

V. Nekrashevych, A. Teplyaev • Published 2005 • Mathematics

We describe relation between analysis on fractals and the theory of self-similar groups. In particular, we focus on the construction of the Laplacian on limit sets of such groups in several concrete examples, and in the general p.c.f. case. We pose a number of open questions.

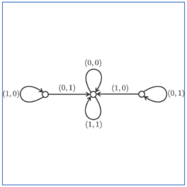


Figure 1

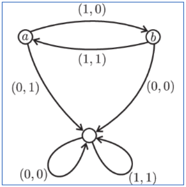


Figure 2

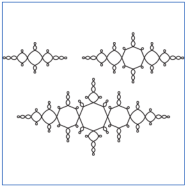


Figure 3

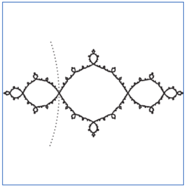


Figure 4

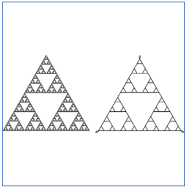


Figure 6

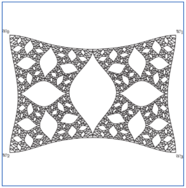


Figure 7

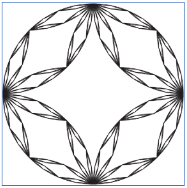


Figure 8

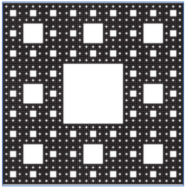


Figure 9

What are Hausdorff and spectral dimensions of a self-similar set?

For the circle, $d_2 = 1$
For Riemannian d-manifolds, $d_S = d_H = d$

In general, d_S can be defined using the asymptotics of eigenvalues or, equivalently, asymptotics of the heat kernel.

If d_s is well defined, then

$$\textit{recurrence of the diffusion} \quad \Longleftrightarrow \quad d_S < 2$$

in which case we sometimes can prove Kigami's formula

$$d_S = 2\frac{d_{H,R}}{d_{H,R}+1}$$

where $d_{H,R}$ is the **effective resistance Hausdorff dimension**.

On the Sierpinski gasket (S.Goldstein 1984)

$$d_{topo} = 1 < d_S = \frac{\log 9}{\log 5} < d_H = \frac{\log 3}{\log 2}$$

On the basilica Julia set we formally computed (Rogers-T, 2010)

$$d_S = \frac{4}{3}$$

On the Sierpinski carpet $\exists! d_S$ (Barlow, Bass, Kumagai, T. 1989-2010)

$$d_{topo} = 1 < d_{H,topo} = 1 + \frac{\log 2}{\log 3} < d_S < d_H = \frac{\log 8}{\log 3}$$

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

$$\textit{distance} \sim \sqrt{\textit{time}}$$

"Einstein space–time relation for Brownian motion"

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

- **De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;**
- **Li-Yau (1986) type estimates** on a geodesically complete Riemannian manifold with *Ricci* $\geqslant 0$:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

$$distance \sim \sqrt{time}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here d_w is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

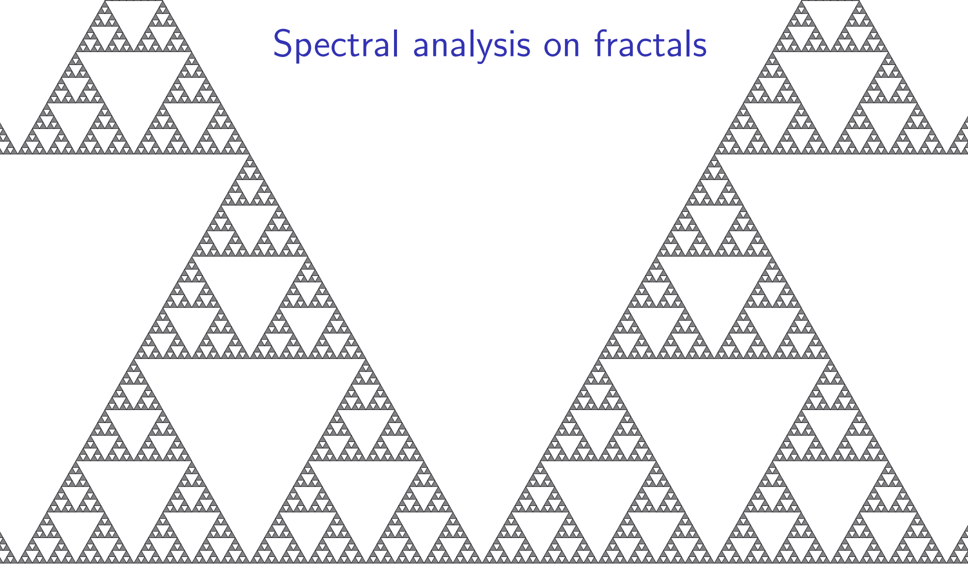
$$\begin{aligned} d_H &= \text{Hausdorff dimension} \\ \frac{1}{\gamma} = d_w &= \text{“walk dimension” } (\gamma\text{=diffusion index}) \\ \frac{2d_H}{d_w} = d_S &= \text{“spectral dimension” (diffusion dimension)} \end{aligned}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

$$p_t(x,x) \sim \frac{1}{t^{d_S/2}}$$

Note: $t \to \infty$ for random walks but $t \to 0$ for diffusions.

Spectral analysis on fractals



A part of an infinite Sierpiński gasket.

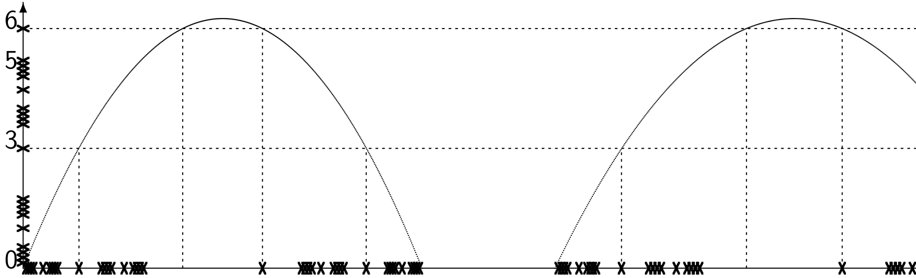
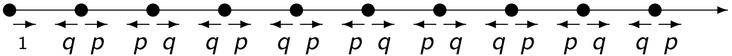


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

Theorem (Rammal, Toulouse 1983, Béllissard 1988, Fukushima, Shima 1991, T. 1998, Quint 2009)

On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues** $\mathfrak{R}^{-1}(\Sigma_0)$ **of infinite multiplicity** *and a* **singularly continuous component of spectral multiplicity one supported on** $\mathfrak{R}^{-1}(\mathcal{J}_R)$.

Half-line example



Transition probabilities in the pq random walk. Here $p \in (0,1)$ and $q = 1 - p$.

$$(\Delta_p f)(x) = \begin{cases} f(0) - f(1), & \text{if } x = 0 \\ f(x) - qf(x-1) - pf(x+1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod 3 \\ f(x) - pf(x-1) - qf(x+1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod 3 \end{cases}$$

Theorem (J.P.Chen, T., 2016)

If $p \neq \frac{1}{2}$, the Laplacian Δ_p on $\ell^2(\mathbb{Z}_+)$ has **purely singularly continuous spectrum**. *The spectrum is the Julia set, a* **topological Cantor set of Lebesgue measure zero**, *of the polynomial*

$$R(z) = \frac{z(z^2 - 3z + (2 + pq))}{pq}$$

This is a simple, possibly the simplest, quasi-periodic example related to the recent results of A.Avila, D.Damanik, A.Gorodetski, S.Jitomirskaya, Y.Last, B.Simon et al.

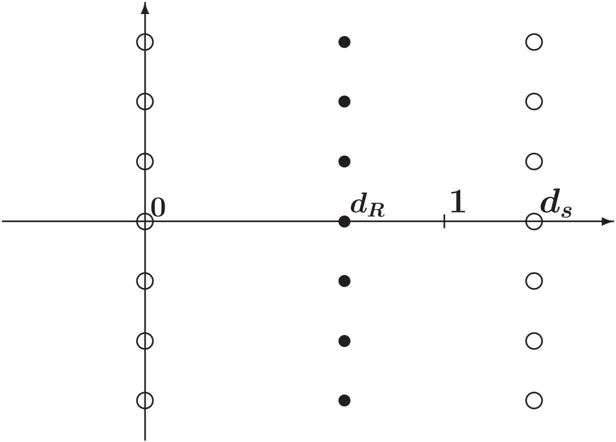
Spectral zeta function

Theorem. (Derfel-Grabner-Vogl, Steinhurst-T., Chen-T.-Tsougkas, Kajino (2007–2017)) For a large class of **finitely ramified symmetric fractals** the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $Re(s) > d_S$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here λ_j are the eigenvalues if the unique symmetric Laplacian.

- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

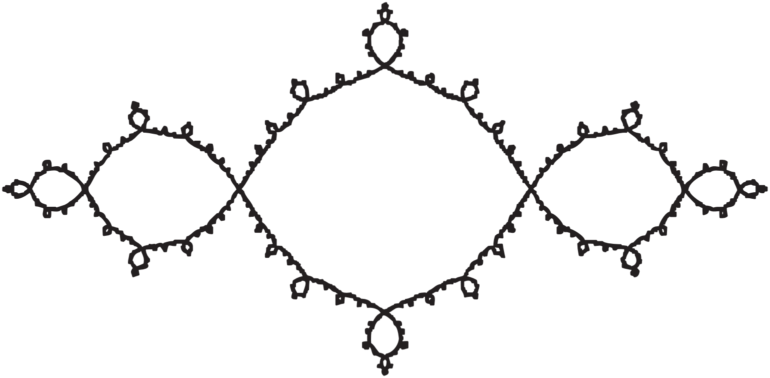
Poles (white circles) of the spectral zeta function of the Sierpiński gasket.

Spectral Analysis of the Basilica Graphs (with Luke Rogers, Toni Brzoska, Courtney George, Samantha Jarvis)

The question of existence of groups with **intermediate growth, i.e. subexponential but not polynomial**, was asked by **Milnor in 1968** and answered in the positive by **Grigorchuk in 1984**. There are still open questions in this area, and a complete picture of which orders of growth are possible, and which are not, is missing.

The Basilica group is a group generated by a finite automation acting on the binary tree in a self-similar fashion, introduced by **R. Grigorchuk and A. Zuk in 2002**, does not belong to the closure of the set of groups of subexponential growth under the operations of group extension and direct limit.

In 2005 L. Bartholdi and B. Virag further showed it to be amenable, making the Basilica group **the 1st example of an amenable but not subexponentially amenable** group (also **"Münchhausen trick" and amenability of self-similar groups by V.A. Kaimanovich**).



The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).

In 2005, V. Nekrashevych described the Basilica as the iterated monodromy group, and there exists a natural way to associate it to the Basilica fractal (**Nekrashevych+T., 2008**).

In **Schreier graphs of the Basilica group (2010), Nagnibeda et al.** classified up to isomorphism all possible limits of finite Schreier graphs of the Basilica group.

In **Laplacians on the Basilica Julia set (2010), L. Rogers+T.** constructed Dirichlet forms and the corresponding Laplacians on the Basilica fractal in two different ways: by imposing a self-similar harmonic structure and a graph-directed self-simliar structure on the fractal.

In **2012-2015, Dong, Flock, Molitor, Ott, Spicer, Totari and Strichartz** provided numerical techniques to approximate eigenvalues and eigenfunctions on families of Laplacians on the Julia sets of $z^2 + c$.

Theorem (Rogers-T., 2010)

The simple random walks in the basilica Schreier graphs are **re-normalizable** and converge, with a change of time, to a diffusion process on the basilica Julia set. These random walks and the diffusion process have **spectral dimension**

$$d_S = \frac{4}{3}$$

This spectral dimension appears in the **weak Weyl's law** for the Laplacian on the basilica Julia set.

Note: this is an informal re-statement of the main result.

Theorem (Rogers-T. et al., 2017)

The graph Laplacian on a **generic infinite basilica Schreier graph has pure point spectrum** and a complete set of eigenfunctions with finite support.

General Geometric Pure Point Spectrum Theorem (1995 — ... work in progress: T. Nagnibeda, L. Rogers et al.)

$$G_\infty = \bigcup_{n \geqslant 0} G_n$$

is a strictly increasing union of finite graphs and each point $x \in \partial G_n$ on the boundary of G_n has a symmetry

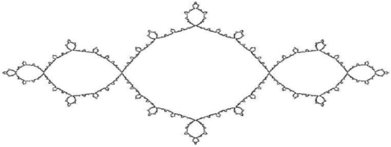
$$g_{x,n} : G_n \to G_n$$

which fixes x, $g_{x,n}(x) = x$.

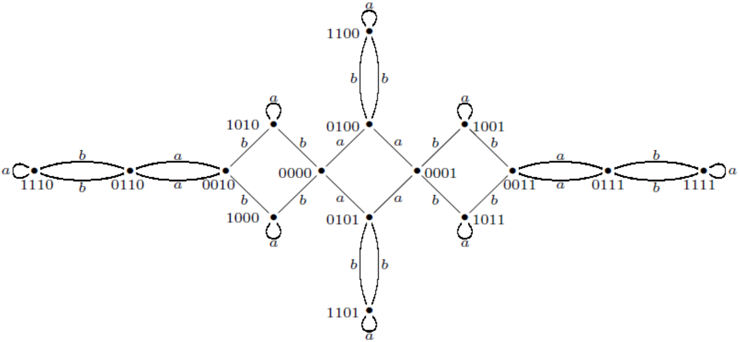
Let $\mathcal{G}_n$ be the sub-group of symmetries of G_n generated by $g_{x,n}$

Theorem (informal). If $\sup_{n \geqslant 0} |\mathcal{G}_n| < \infty$ and each $\mathcal{G}_{n+1}$ acts "sufficiently transitively" on the orbit of G_n, then the spectrum on G_∞ is pure point with a complete set of localized eigenfunctions.

$$g_{x,n+1}(G_n) \cap G_n = \varnothing$$

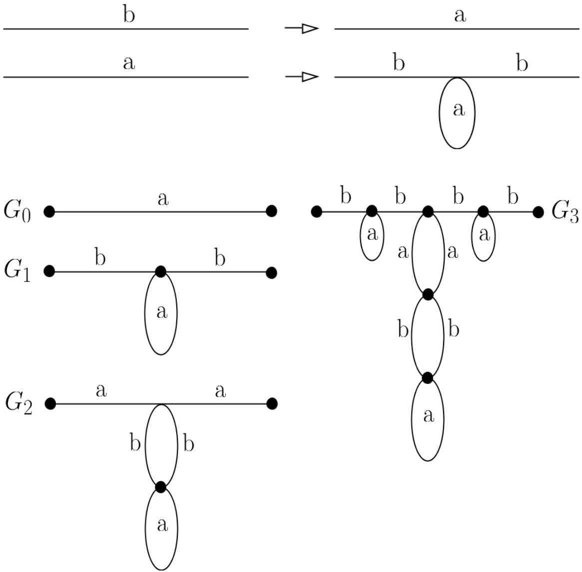


Γ_4

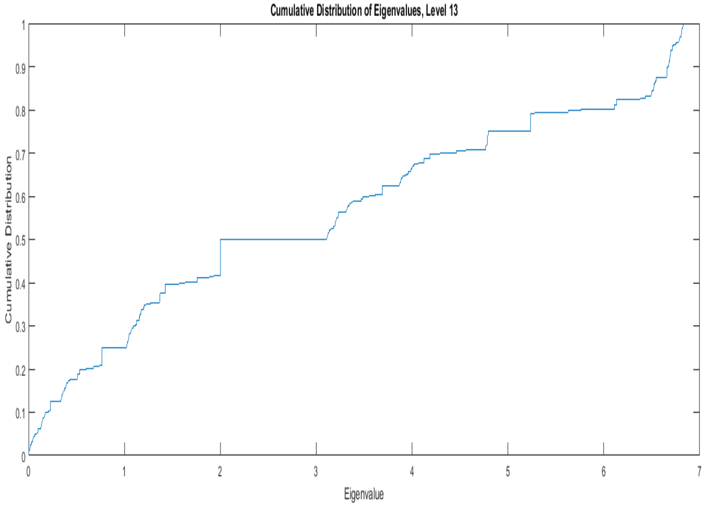


pictures taken from paper by Nagnibeda et. al.

Replacement Rule and the Graphs G_n



Distribution of Eigenvalues, Level 13



One can define a Dirichlet to Neumann map for the two boundary points of the graphs G_n. One can construct a dynamical system to determine these maps (which are two by two matrices). The dynamical system allows us to prove the following.

Theorem

In the Hausdorff metric, $\limsup_{n\to\infty} \sigma(L^{(n)})$ has a gap that contains the interval $(2.5, 2.8)$.

Theorem (arXiv:1908.10505)

In the Hausdorff metric, $\limsup_{n\to\infty} \sigma(L^{(n)})$ has infinitely many gaps.

Infinite Blow-ups of G_n

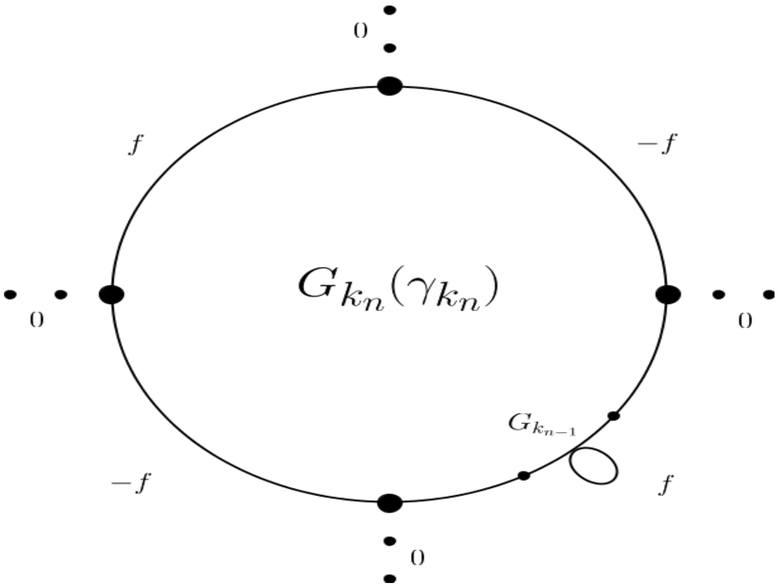
Definition

Let $\{k_n\}_{n\in\mathbb{N}}$ be a strictly increasing subsequence of the natural numbers. For each n, embed G_{k_n} in some isomorphic subgraph of $G_{k_{n+1}}$. The corresponding infinite blow-up is $G_\infty := \cup_{n\geq 0} G_{k_n}$.

Assumption

The infinite blow-up G_∞ satisfies:

- For $n \geq 1$, the long path of $G_{k_{n-1}}$ is embedded in a loop γ_n of G_{k_n}.
- Apart from $l_{k_{n-1}}$ and $r_{k_{n-1}}$, no vertex of the long path can be the $3, 6, 9$ or 12 o'clock vertex of γ_n.
- The only vertices of G_{k_n} that connect to vertices outside the graph are the boundary vertices of G_{k_n}.


0
f
$-f$
$G_{k_n}(\gamma_{k_n})$
0
0
$G_{k_{n-1}}$
$-f$
f
0

Theorem

(1) $\sigma(L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}) = \sigma(L_0^{(j_n)})$.

(2) The spectrum of $L^{(\infty)}$ is pure point. The set of eigenvalues of $L^{(\infty)}$ is

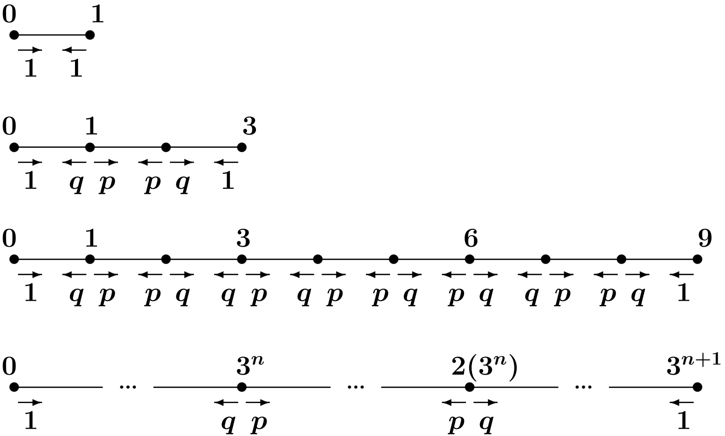
$$\bigcup_{n\geq 0} \sigma(L_0^{(j_n)}) = \bigcup_{n\geq 0} c_{j_n}^{-1}\{0\},$$

where the polynomials c_n are the characteristic polynomials of $L_0^{(n)}$, as defined in the previous proposition.

(3) Moreover, the set of eigenfunctions of $L^{(\infty)}$ with finite support is complete in ℓ^2.

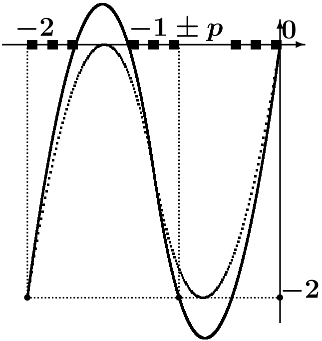
TECHNICAL DETAILS

Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians Δ_n on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:



and let $\Delta = \lim_{n \to \infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$.

If $z \neq -1 \pm p$ and $R(z)=z(z^2+3z+2+pq)/pq$, then
$R(z) \in \sigma(\Delta_n) \Longleftrightarrow z \in \sigma(\Delta_{n+1})$



Theorem (Joe P. Chen and T., JMP 2016). $\sigma(\Delta) = \mathcal{J}_R$, the Julia set of $R(z)$.

If $p=q$, then $\sigma(\Delta)=[-2,0]$, spectrum is a.c.

If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.

U.Andrews, J.P.Chen, G.Bonik, R.W.Martin, A.Teplyaev, **Wave equation on one-dimensional fractals with spectral decimation**. J. Fourier Anal. Appl. 23 (2017) 994–1027. `http://teplyaev.math.uconn.edu/fractalwave/`

Bellissard, Geronimo, Volberg, Yuditskii, **Are they limit periodic?** Complex analysis and dynamical systems II, 43–53, Contemp. Math., 382, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2005.
(Reviewed by Maxim S. Derevyagin)

Bellissard, **Renormalization group analysis and quasicrystals**. Ideas and methods in quantum and statistical physics (Oslo, 1988), 118–148, Cambridge Univ. Press, Cambridge, 1992.

Barnsley, Geronimo and Harrington, **Almost periodic Jacobi matrices associated with Julia sets for polynomials**. Comm. Math. Phys. 99 (1985), 303–317.

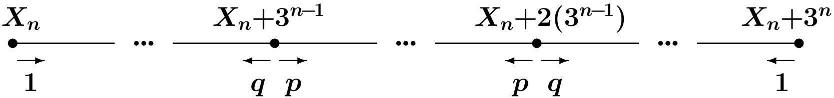
Bellissard, Bessis and Moussa, **Chaotic states of almost periodic Schrodinger operators**. Phys. Rev. Lett. 49 (1982), 701–704.

Bellissard and Simon, **Cantor spectrum for the almost Mathieu equation** J. Funct. Anal. 48 (1982), 408–419.

There are uncountably many “random” self-similar Laplacians $\boldsymbol{\Delta}$ on $\mathbb{Z}$:
For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0, 1, 2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

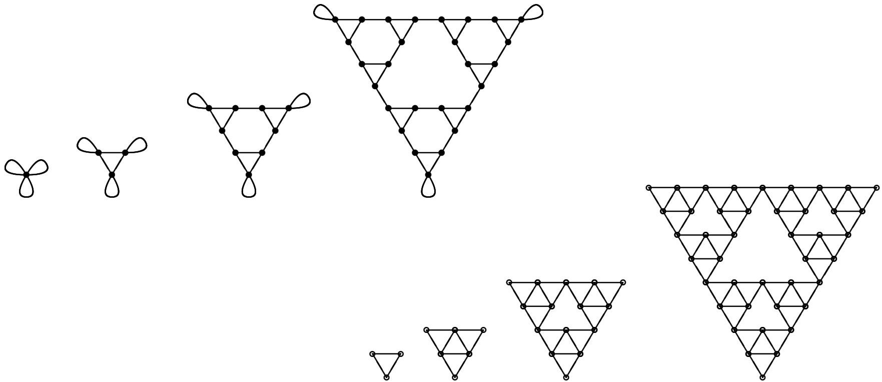
and $\boldsymbol{\Delta}_n$ is a probabilistic Laplacian on $[X_n, X_n{+}3^n]$:

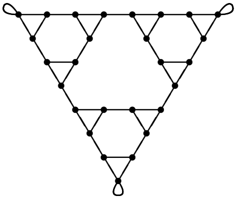


In the previous example $k_j = 0$ for all j.

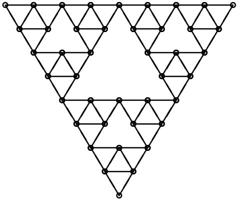
Theorem.
For any sequence $\mathcal{K}$ we have $\sigma(\boldsymbol{\Delta}) = \mathcal{J}_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).

R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups.*





Sierpiński 3-graph
(Hanoi Towers-3 group)



Sierpiński 4-graph
(standard)

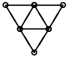
These three polynomials are conjugate:

Sierpiński 3-graph (Hanoi Towers-3 group): $f(x) = x^2 - x - 3$
$f(3) = 3$, $f'(3) = 5$

Sierpiński 4-graph, "adjacency matrix" Laplacian: $P(\lambda) = 5\lambda - \lambda^2$
$P(0) = 0$, $P'(0) = 5$

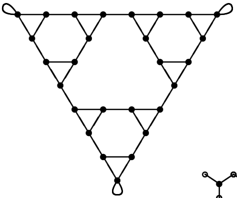
Sierpiński 4-graph, probabilistic Laplacian: $R(z) = 4z^2 + 5z$
$R(0) = 0$, $R'(0) = 5$

Theorem. Eigenvalues and eigenfunctions on the Sierpiński 3-graphs and Sierpiński 4-graphs are in one-to-one correspondence, with the exception of the eigenvalue $z = -\frac{3}{2}$ for the 4-graphs.

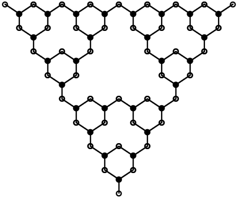
 $\underset{4z^2+5z}{\longleftarrow}$ 

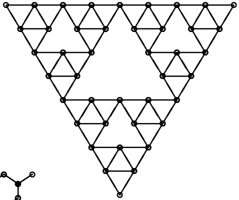
 $\underset{\frac{4}{3}z^2+\frac{8}{3}z}{\longleftarrow}$ 

 $\underset{2z^2+4z}{\longleftarrow}$ 



Sierpiński 3-graph
(Hanoi Towers-3 group)
$R(z) = 2z^2 + 4z$





Sierpiński 4-graph
(standard)
$R(z) = \frac{4}{3}z^2 + \frac{8}{3}z$

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U : \mathcal{H}_0 \to \mathcal{H}$ be an isometry.

Definition. We call an operator H **spectrally similar** to an operator H_0 with functions φ_0 and φ_1 if

$$U^*(H-z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H-z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}.$$

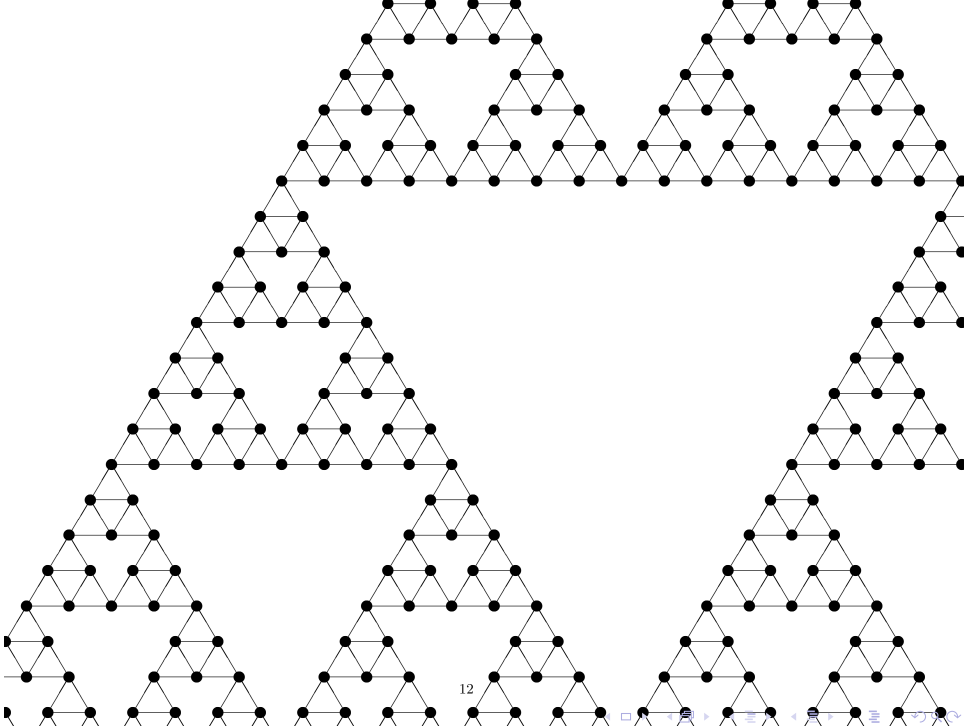
If $H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}$ then

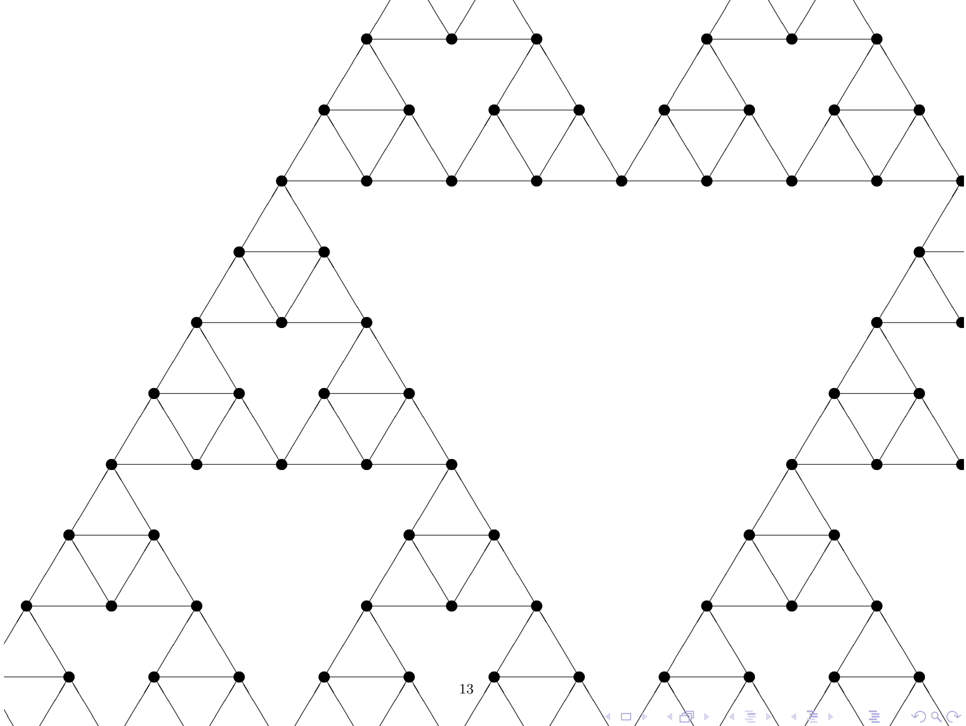
$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0$$

Theorem (Malozemov and T.). If Δ is the graph Laplacian on a self-similar symmetric infinite graph, then

$$\mathcal{J}_R \subseteq \sigma(\Delta_\infty) \subseteq \mathcal{J}_R \cup \mathcal{D}_\infty$$

where $\mathcal{D}_\infty$ is a discrete set and $\mathcal{J}_R$ is the Julia set of the rational function R.





R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984).

S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field.* Phys. Rev. B **29** (1984).

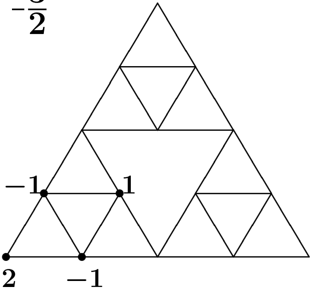
E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984).

Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983–1984).

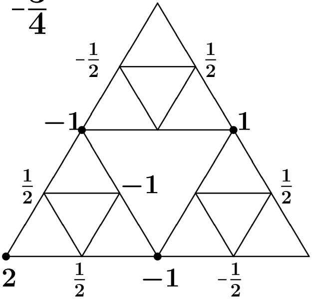
R. B. Stinchcombe, *Fractals, phase transitions and criticality.* Fractals in the natural sciences. Proc. Roy. Soc. London Ser. A **423** (1989), 17–33.

J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988). Cambridge Univ. Press, 1992.

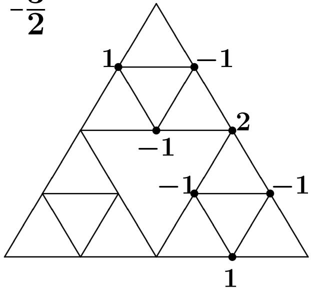
$z = -\frac{3}{2}$



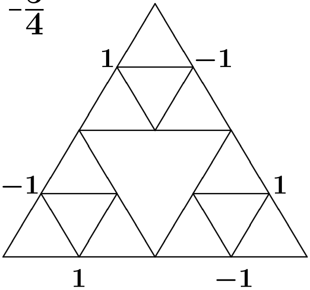
$z = -\frac{3}{4}$



$z = -\frac{3}{2}$



$z = -\frac{5}{4}$

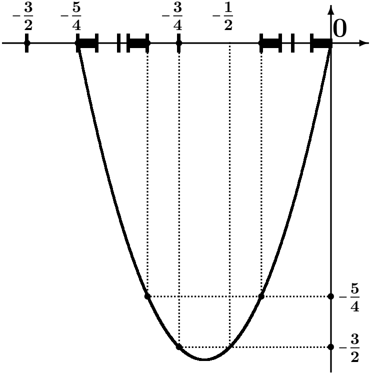


Let Δ be the probabilistic Laplacian (generator of a simple random walk) on the **Sierpiński lattice**. If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z+5)$, then

$$R(z) \in \sigma(\Delta) \iff z \in \sigma(\Delta)$$

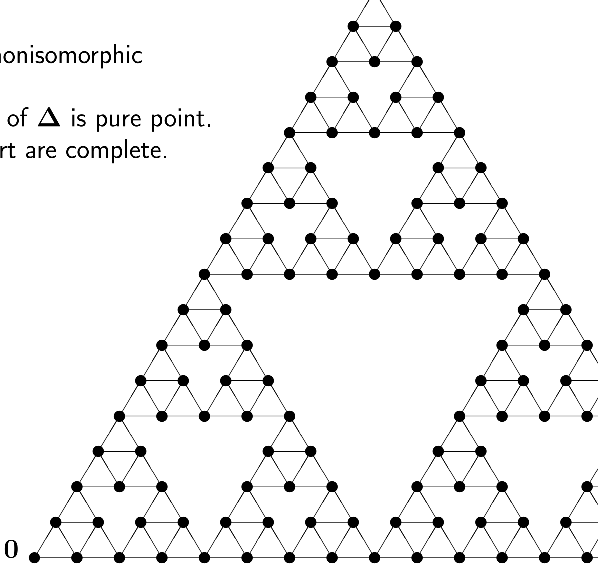
$$\sigma(\Delta) = \mathfrak{J}_R \bigcup \mathfrak{D}$$

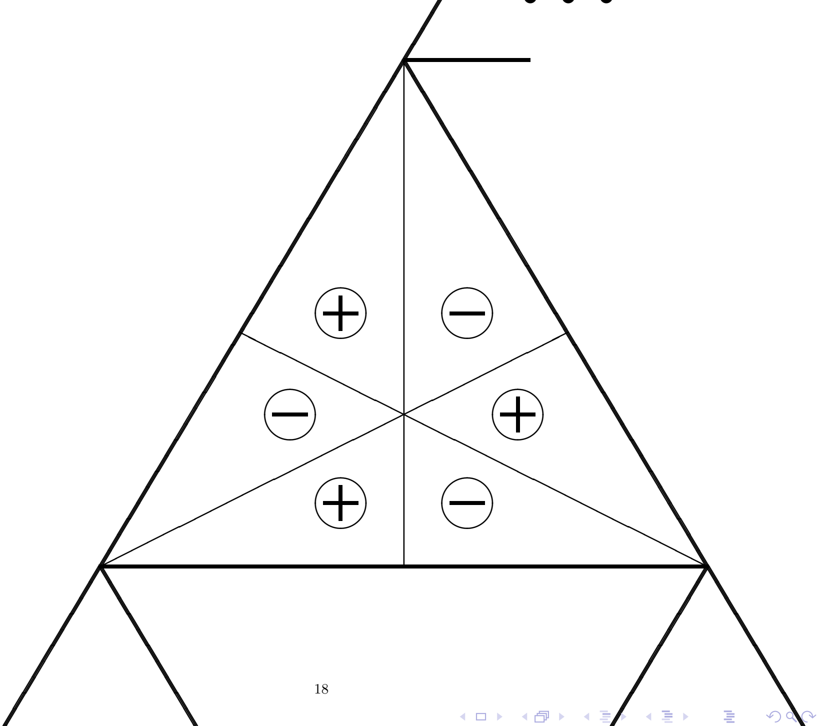
where $\mathfrak{D} \stackrel{\text{def}}{=} \{-\frac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$
and $\mathfrak{J}_R$ is the Julia set of $R(z)$.



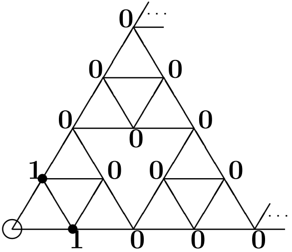
There are uncountably many nonisomorphic Sierpiński lattices.

Theorem (T). The spectrum of $\boldsymbol{\Delta}$ is pure point. Eigenfunctions with finite support are complete.





Let $\Delta^{(0)}$ be the Laplacian with zero (Dirichlet) boundary condition at ∂L. Then the compactly supported eigenfunctions of $\Delta^{(0)}$ are **not** complete (eigenvalues in $\mathcal{E}$ is not the whole spectrum).



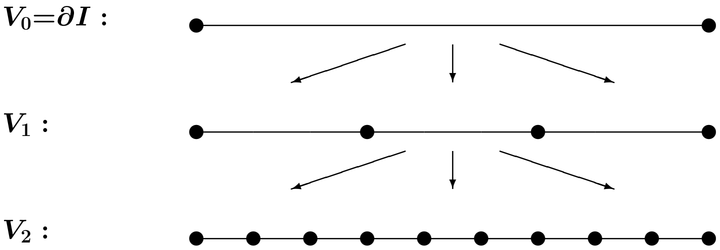
Let $\partial L^{(0)}$ be the set of two points adjacent to ∂L and $\omega_\Delta^{(0)}$ be the spectral measure of $\Delta^{(0)}$ associated with $\mathbb{1}_{\partial L^{(0)}}$. Then $\operatorname{supp}(\omega_\Delta^{(0)}) = \mathfrak{J}_R$ has Lebesgue measure zero and

$$\frac{d(\omega_\Delta^{(0)} \circ R_{1,2})}{d\omega_\Delta^{(0)}}(z) = \frac{(8z+5)(2z+3)}{(2z+1)(4z+5)}$$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^1 \to \mathbb{R}^1$, $F_j(x) = \frac{1}{3}(x+p_j)$, with fixed points $p_j = 0, \frac{1}{2}, 1$. The interval $I{=}[0,1]$ is a unique compact set such that

$$I = \bigcup_{j=1,2,3} F_j(I)$$

The *boundary* of I is $\partial I = V_0 = \{0, 1\}$ and the *discrete approximations* to I are $V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{\frac{k}{3^n}\right\}_{k=0}^{3^n}$



Definition. The $\boldsymbol{discrete\ Dirichlet\ (energy)\ form}$ on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y\in V_n \\ y\sim x}} (f(y)-f(x))^2$$

and the $\boldsymbol{Dirichlet\ (energy)\ form}$ on I is $\mathcal{E}(f) = \lim_{n\to\infty} 3^n \mathcal{E}_n(f) = \int_0^1 |f'(x)|^2 dx$

Definition. A function h is $\boldsymbol{harmonic}$ if it minimizes the energy given the boundary values.

Proposition. $3\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$ and $3\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = 3^{-n}\mathcal{E}(h)$ for a harmonic h.

Proposition. The Dirichlet (energy) form on I is $\boldsymbol{self\text{-}similar}$ in the sense that

$$\mathcal{E}(f) = 3 \sum_{j\,=\,1,2,3} \mathcal{E}(f\circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{2}\sum_{\substack{y\in V_n\\ y\sim x}} f(y)-f(x), \quad x\in V_n\setminus V_0$$

and the Laplacian on I is $\Delta f(x) = \lim_{n\to\infty} 9^n\Delta_n f(x) = f''(x)$

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_0^1 f\Delta f dx + ff'\Big|_0^1$$

Spectral asymptotics: Let $\rho(\lambda)$ be the *eigenvalue counting function* of the Dirichlet or Neumann Laplacian Δ:

$$\rho(\lambda) = \#\{j : \lambda_j < \lambda\}.$$

Then

$$\lim_{\lambda\to\infty}\frac{\rho(\lambda)}{\lambda^{d_s/2}} = \frac{1}{\pi}$$

where $d_s = 1$ is the spectral dimension.

Definition. The *spectral zeta function* is $\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} (-\lambda_j)^{-s/2}$
Its poles are the *complex spectral dimensions.*

Let $R(z)$ be a polynomial of degree N such that its Julia set $\mathfrak{J}_R \subset (-\infty, 0]$, $R(0) = 0$ and $c = R'(0) > 1$.

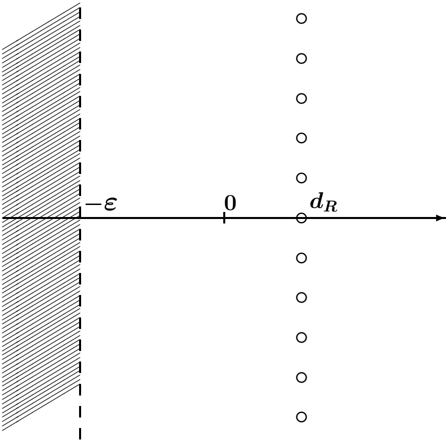
Definition. The *zeta function of* $R(z)$ for $\mathrm{Re}(s) > d_R = \frac{2\log N}{\log c}$ is

$$\zeta_R^{z_0}(s) = \lim_{n\to\infty} \sum_{z \in R^{-n}\{z_0\}} (-c^n z)^{-s/2} = \sum \lambda_j^{-s/2}$$

Theorem. $\zeta_R^{z_0}(s) = \dfrac{f_1(s)}{1-Nc^{-s/2}} + f_2^{z_0}(s)$, where $f_1(s)$ and $f_2^{z_0}(s)$ are analytic for $\mathrm{Re}(s)>0$. If $\mathfrak{J}_R$ is totally disconnected, then this meromorphic continuation extends to $\mathrm{Re}(s)>-\varepsilon$, where $\varepsilon>0$.

In the case of polynomials this theorem has been improved by Grabner et al.

$d_R \in$ the poles of $\zeta_R^{z_0} \subseteq \left\{\frac{2\log N + 4in\pi}{\log c} : n \in \mathbb{Z}\right\}$



Theorem. $\zeta_\Delta(s) = \zeta^0_R(s)$ where $R(z) = z(4z^2+12z+9)$.

The Riemann zeta function $\zeta(s)$ satisfies $\zeta(s) = \pi^s \zeta^0_R(s)$ The only complex spectral dimension is the pole at $s = 1$.

<u>A sketch of the proof:</u> If $z \neq -\frac{1}{2}, -\frac{3}{2}$, then

$$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$$

and so $\zeta_\Delta(s) = \zeta^0_R(s)$ since the eigenvalues λ_j of Δ are limits of the eigenvalues of $9^n \Delta_n$.

Also $\lambda_j = -\pi^2 j^2$ and so

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \left(\pi^2 j^2\right)^{-s/2} = \pi^{-s}\zeta(s)$$

where $\zeta(s)$ is the Riemann zeta function. *Q.E.D.*

$$\zeta(s) = \pi^s \lim_{n\to\infty} \sum_{\substack{z \in R^{-n}\{0\} \\ z \neq 0}} \left(-9^n z\right)^{-s/2}$$

Definition. Δ_μ is μ*–Laplacian* if

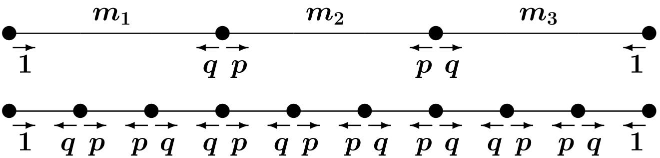
$$\mathcal{E}(f) = \int_0^1 |f'(x)|^2 dx = -\int_0^1 f \Delta_\mu f d\mu + f f' \Big|_0^1.$$

Definition. A probability measure μ is *self-similar* with weights m_1, m_2, m_3 if $\mu = \sum\limits_{j=1,2,3} m_j \mu \circ F_j$.

Proposition. $\Delta_\mu f(x) = \frac{f''}{\mu} = \lim\limits_{n\to\infty} \left(1+\frac{2}{pq}\right)^n \Delta_n f(x)$.

$$\Delta_n f\left(\tfrac{k}{3^n}\right) = \begin{cases} p f\left(\frac{k-1}{3^n}\right) + q f\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \\ q f\left(\frac{k-1}{3^n}\right) + p f\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \end{cases}$$

where $m_1 = m_3$, $p = \frac{m_2}{m_1+m_2}$, $q = \frac{m_1}{m_1+m_2}$, and



Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leqslant \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$d_s = \frac{\log 9}{\log(1+\frac{2}{pq})} \leqslant 1.$$

All the inequalities are strict if and only if $p \neq q$.

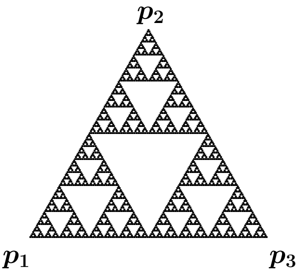
Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

where $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$.

Note that $R'(0) = 1 + \frac{2}{pq}$, and $d_s = d_R$.

Theorem. $\zeta_{\Delta_\mu}(s) = \zeta^0_R(s)$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{R}^2$,
$F_j(x) = \frac{1}{2}(x+p_j)$, with fixed points p_1, p_2, p_3.



The **Sierpiński gasket** is a unique compact set S such that

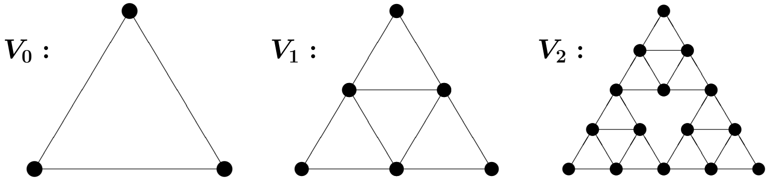
$$S = \bigcup_{j=1,\,2,\,3} F_j(S)$$

Definition. The *boundary* of S is

$$\partial S = V_0 = \{p_1, p_2, p_3\}$$

and *discrete approximations* to S are

$$V_n = \bigcup_{j=1,2,3} F_j(V_{n-1})$$



Definition. The *discrete Dirichlet (energy) form* on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y \in V_n \\ y \sim x}} (f(y) - f(x))^2$$

and the *Dirichlet (energy) form* on S is

$$\mathcal{E}(f) = \lim_{n \to \infty} \left(\tfrac{5}{3}\right)^n \mathcal{E}_n(f)$$

Definition. A function h is *harmonic* if it minimizes the energy given the boundary values.

Proposition. $\frac{5}{3}\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$
$\frac{5}{3}\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = \left(\frac{5}{3}\right)^{-n}\mathcal{E}(h)$ for a harmonic h.

Theorem (Kigami). $\mathcal{E}$ is a local regular Dirichlet form on S which is self-similar in the sense that

$$\mathcal{E}(f) = \frac{5}{3} \sum_{j = 1,2,3} \mathcal{E}(f \circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{4} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on S is

$$\Delta_\mu f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)$$

if this limit exists and $\Delta_\mu f$ is continuous.

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_S f \Delta_\mu f d\mu + \sum_{p \in \partial S} f(p) \partial_n f(p)$$

where μ is the normalized Hausdorff measure, which is self-similar with weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$:

$$\mu = \frac{1}{3} \sum_{j=1,2,3} \mu \circ F_j.$$

Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$1 < d_s = \tfrac{\log 9}{\log 5} < 2.$$

Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$ where $z \neq -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{4}$ and $R(z) = z(5+4z)$.

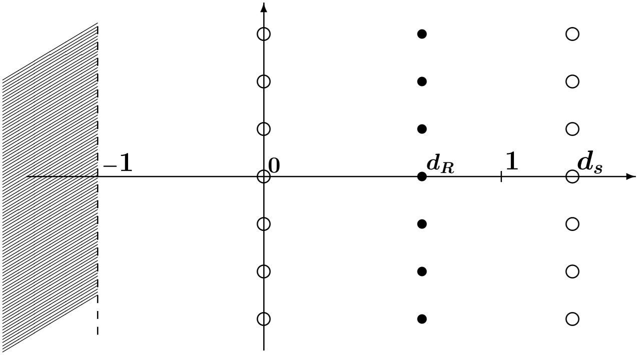
Theorem (Fukushima, Shima). Every eigenvalue of Δ_μ has a form

$$\lambda = 5^m \lim_{n\to\infty} 5^n R^{-n}(z_0)$$

where $R^{-n}(z_0)$ is a preimage of $z_0 = -\frac{3}{4}, -\frac{5}{4}$ under the n-th iteration power of the polynomial $R(z)$. The multiplicity of such an eigenvalue is $C_1 3^m + C_2$.

Theorem. Zeta function of the Laplacian on the Sierpiński gasket is

$$\zeta_{\Delta_\mu}(s) \;=\; \tfrac{1}{2}\,\zeta_R^{-\frac{3}{4}}(s)\left(\frac{1}{5^{s/2}-3}+\frac{3}{5^{s/2}-1}\right) \;+\; \tfrac{1}{2}\,\zeta_R^{-\frac{5}{4}}(s)\left(\frac{3\cdot 5^{-s/2}}{5^{s/2}-3}-\frac{5^{-s/2}}{5^{s/2}-1}\right)$$



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

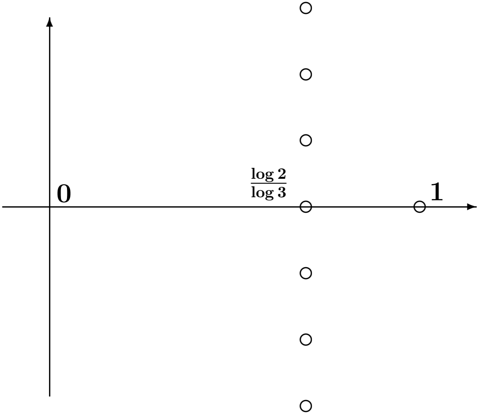
Definition. If $\mathcal{L}$ is a fractal string, that is, a disjoint collection of intervals of lengths l_j, then its *geometric zeta function* is $\zeta_{\mathcal{L}}(s) = \sum l_j^s$.

Theorem (Lapidus). If $A = -\frac{d^2}{dx^2}$ is a Neumann or Dirichlet Laplacian on $\mathcal{L}$, then $\zeta_A(s) = \pi^{-s}\zeta(s)\zeta_{\mathcal{L}}(s)$.

<u>**Example: Cantor self-similar fractal string.**</u>

If $\mathcal{L}$ is the complement of the middle third Cantor set in $[0,1]$, then the complex spectral dimensions are 1 and $\{\frac{\log 2 + 2in\pi}{\log 3} : n \in \mathbb{Z}\}$,

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1 - 2\cdot 3^{-s}}, \quad \zeta_A(s) = \zeta(s)\frac{\pi^{-s}}{1 - 2\cdot 3^{-s}}$$


0
log 2 / log 3
1

Definition. A post critically finite (p.c.f.) self-similar set F is a compact connected metric space with a finite boundary $\partial F \subset F$ and contractive injections $\psi_i : F \to F$ such that

$$F = \Psi(F) = \bigcup_{i=1}^{k} \psi_i(F)$$

and

$$\psi_v(F) \bigcap \psi_w(F) \subseteq \psi_v(\partial F) \bigcap \psi_w(\partial F),$$

for any two different words v and w of the same length. Here for a finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_m}$.

We assume that ∂F is a minimal such subset of F. We call $\psi_w(F)$ an m-cell. ***The p.c.f. assumption is that every boundary point is contained in a single 1-cell.***

Theorem (Kigami, Lapidus). The spectral dimension of the Laplacian Δ_μ is the unique solution of the equation

$$\sum_{i=1}^{k} (r_i \mu_i)^{d_s/2} = 1$$

Conjecture. On every p.c.f. fractal F there exists a local regular Dirichlet form $\mathcal{E}$ which gives positive capacity to the boundary points and is self-similar in the sense that

$$\mathcal{E}(f) = \sum_{i=1}^{k} \rho_i \mathcal{E}(f \circ \psi_i)$$

for a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$.

Definition. The group G of acts on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geqslant 0$.

Proposition. Suppose a group G of acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permutation group of V_0. Then there exists a unique, up to a constant, G-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights ρ_i and

$$\mathcal{E}_0(f, f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2.$$

Moreover, for any G-invariant self-similar measure μ the Laplacian Δ_μ has the spectral self-similarity property (a.k.a. spectral decimation).

end of the talk :-)

Thank you!