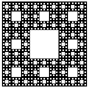
Diffusions on singular spaces

Alexander Teplyaev
University of Connecticut



2025 ∗ Rochester

Plan of the talk:

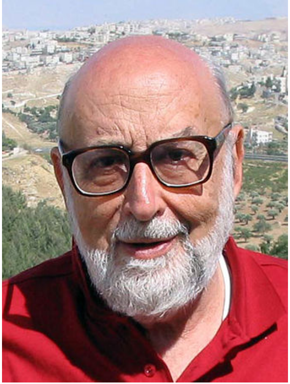
1. Introduction: the spectral dimension of the universe
2. Toy model: Hanoi towers game
3. Existence, uniqueness, heat kernel estimates: geometric renormalization for $\boldsymbol{F}$-invariant Dirichlet forms
   - (Barlow, Bass, Kumagai, T.)
4. Canonical diffusions on the pattern spaces of aperiodic Delone sets
   - (Patricia Alonso-Ruiz, Michael Hinz, Rodrigo Trevino, T.)

François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

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Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

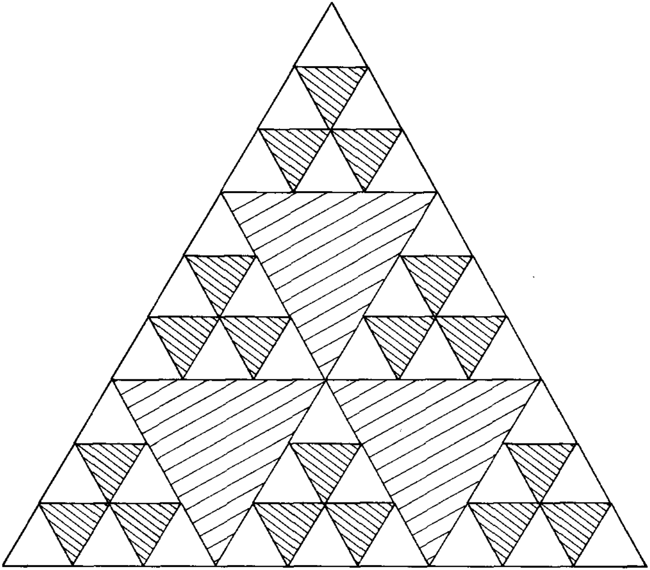

Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

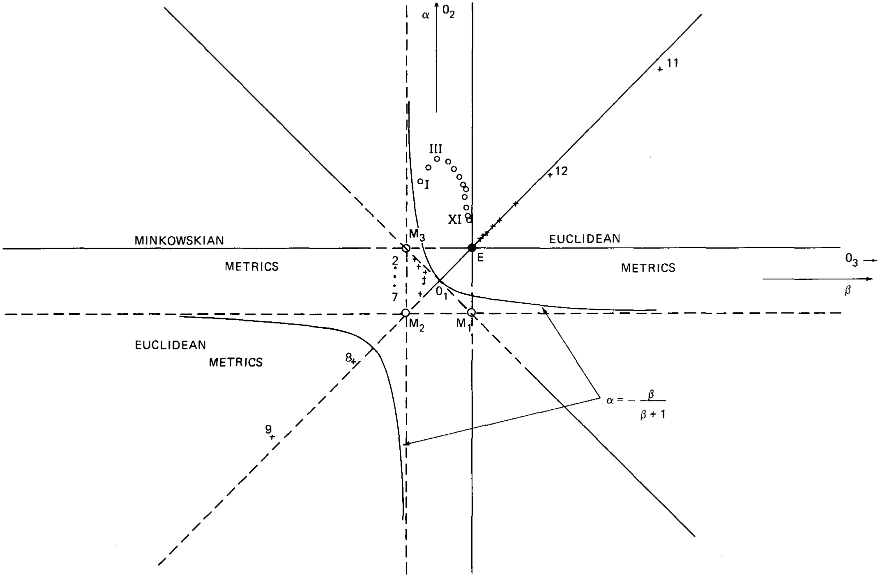

Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

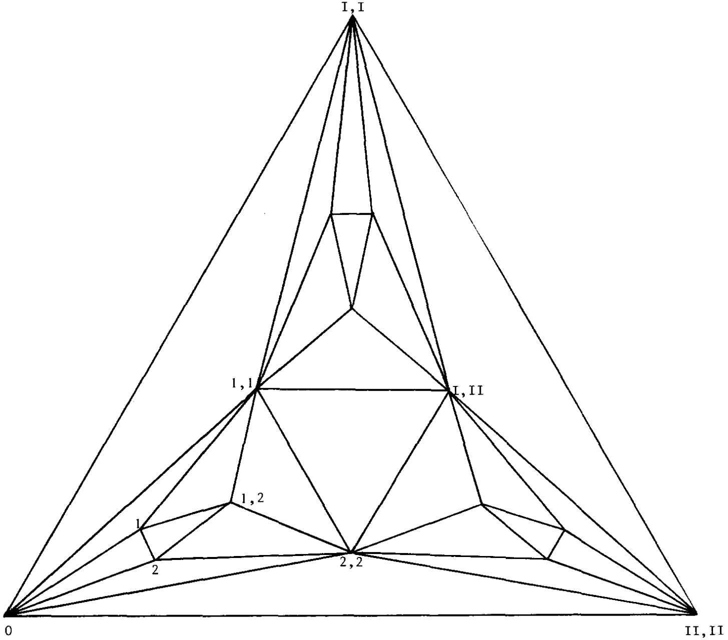

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

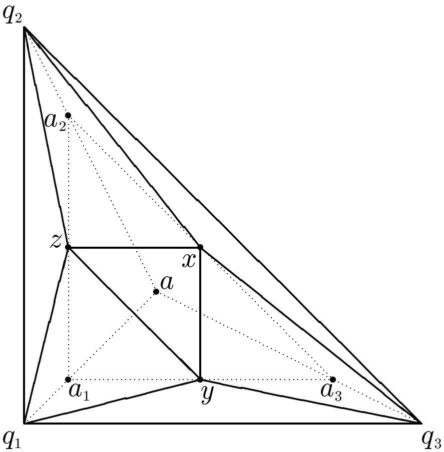

Figure 6.4. Geometric interpretation of Proposition 6.1.

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

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(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the "short-distance spectral dimension," obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \qquad (15)$$

and thus is compatible with the integer value two.

Published for SISSA by Springer

Received: *October 26, 2011*
Accepted: *November 18, 2011*
Published: *December 2, 2011*

Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

JHEP12(2011)012

Toy model: Hanoi towers game

The puzzle was invented by the French mathematician Édouard Lucas in 1883.

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

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Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

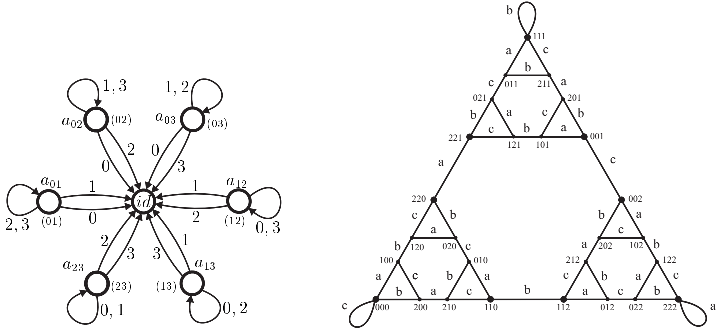
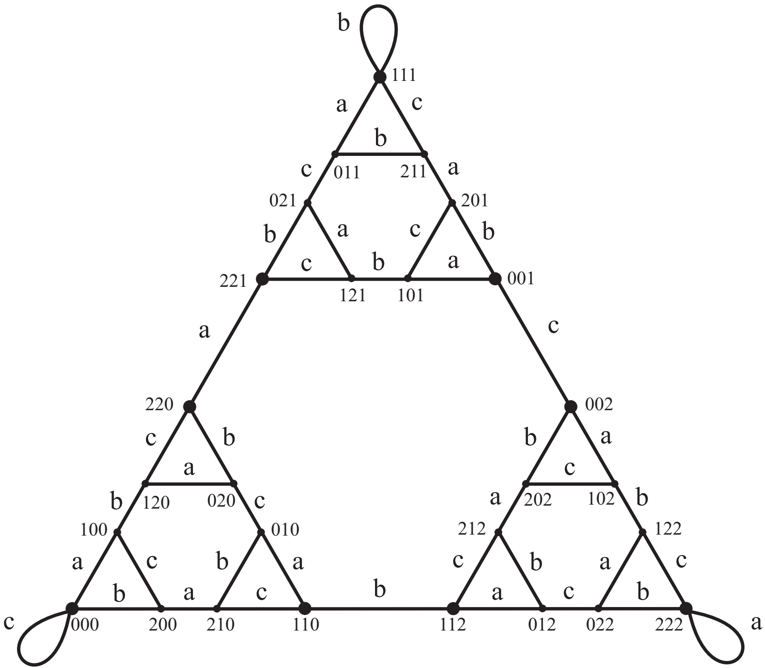


Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3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Initial physics motivation

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Main early mathematical results

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time n is at $X(n)$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension d we show that, as $L \to \infty$, the process $Y_L(t) \equiv X([(d+3)^L t])/2^L$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

$$|X(n)| \sim n^\gamma,$$

where $\gamma = (\ln 2)/\ln(d+3)) \leqslant 2$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

ANALYSE MATHÉMATIQUE. — *Sur une courbe dont tout point est un point de ramification.* Note [1] de M. **W. Sierpinski**, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantorienne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons *point de ramification* d'une courbe $\mathcal{C}$ un point p de cette courbe, s'il existe trois continus, sous-ensembles de $\mathcal{C}$, ayant deux à deux le point p et seulement ce point commun.)

Soient T un triangle régulier donné; A, B, C respectivement ses sommets : gauche, supérieur et droit. En joignant les milieux des côtés du triangle T, nous obtenons quatre nouveaux triangles réguliers (*fig.* 1), dont trois, T_0, T_1, T_2, contenant respectivement les sommets A, B, C, sont situés parallèlement à T et le quatrième triangle U contient le centre du triangle T; nous exclurons tout l'intérieur du triangle U.

Les sommets des triangles T_0, T_1, T_2 nous les désignerons respectivement :

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(1) Séance du 1er février 1915.

triangles U_0, U_1, U_2, situés parallèlement à U, dont les intérieurs seront

Fig. 1.

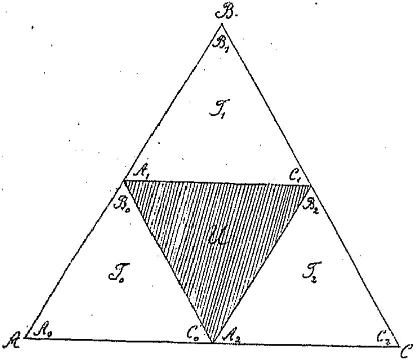
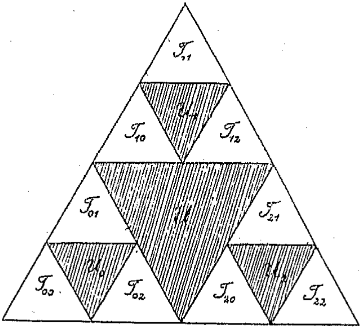


Fig. 2.



exclus (*fig.* 2). Avec chacun des triangles $T_{\lambda_1\lambda_2}$ procédons de même et ainsi

Fig. 3.

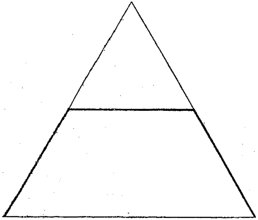
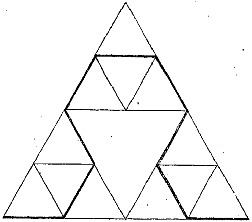


Fig. 4.



d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\mathfrak{C}$.

Donc, tous les points de la courbe $\mathfrak{C}$, sauf peut-être les points A, B, C, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

Fig. 5.

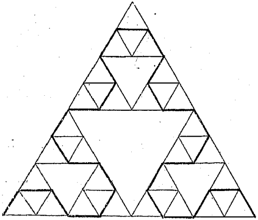
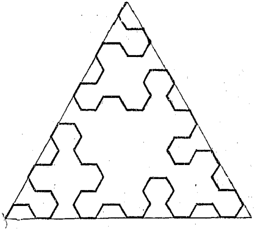


Fig. 6.



points de ramification, il suffit de diviser un hexagone régulier en six triangles réguliers et dans chacun d'eux inscrire une courbe $\mathfrak{C}$.

- M.T. Barlow, E.A. Perkins, *Brownian motion on the Sierpinski gasket.* (1988)
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- J. Kigami, *Harmonic calculus on p.c.f. self–similar sets.* Trans. Amer. Math. Soc. **335** (1993)
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Main classes of fractals considered

- $[0,1]$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)

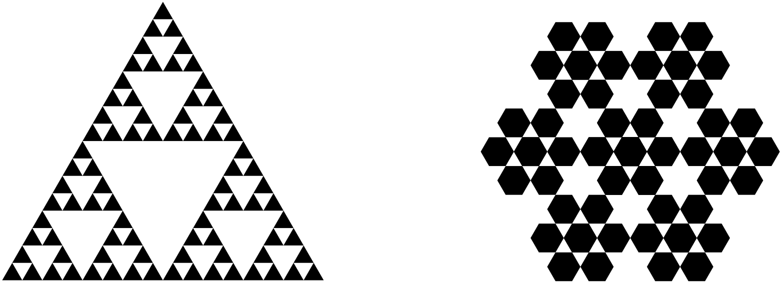


Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

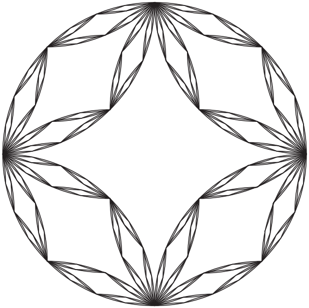


Figure: Diamond fractals, non-p.c.f., but finitely ramified

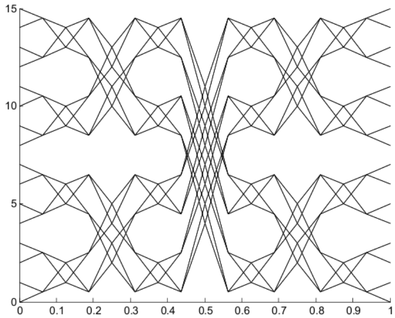


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

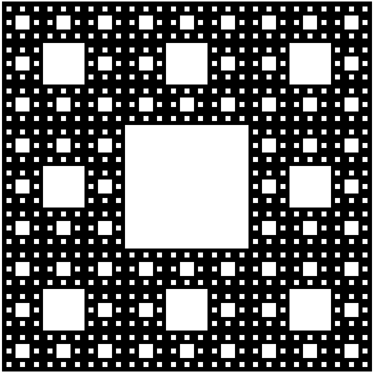


Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates: geometric renormalization for F-invariant Dirichlet forms

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

$$\textit{distance} \sim \sqrt{\textit{time}}$$

"Einstein space–time relation for Brownian motion"

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

- **De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;**
- **Li-Yau (1986) type estimates** on a geodesically complete Riemannian manifold with $\boldsymbol{Ricci} \geqslant \mathbf{0}$:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

$$distance \sim \sqrt{time}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Brownian motion on $\mathbb{R}^d$: $\mathbb{E}|X_t - X_0| = ct^{1/2}$.

Anomalous diffusion: $\mathbb{E}|X_t - X_0| = o(t^{1/2})$, or (in regular enough situations),

$$\mathbb{E}|X_t - X_0| \approx t^{1/d_w}$$

with $d_w > 2$.

Here d_w is the so-called **walk dimension** (should be called **"walk index"** perhaps).

This phenomena was first observed by mathematical physicists working in the transport properties of disordered media, such as (critical) percolation clusters.

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$$d_H = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = d_w = \text{"walk dimension" } (\gamma{=}\text{diffusion index})$$

$$\frac{2d_H}{d_w} = d_S = \text{"spectral dimension" (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)

Theorem (Barlow, Bass, Kumagai (2006)).

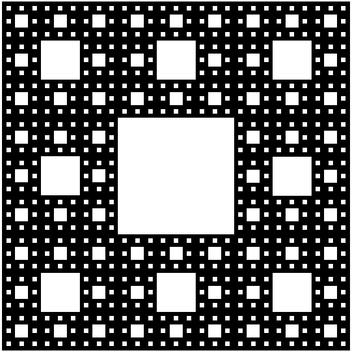
Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the **sub-Gaussian heat kernel estimates are stable under rough isometries**, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any fractal in the class of **generalized Sierpiński carpets there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries**. Therefore there there is a unique corresponding symmetric Markov process and a unique Laplacian. Moreover, the Markov process is Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties: if $d_s < d$, then $d_S < d_H$, $d_w > 2$ and

- **the energy measure and the Hausdorff measure are mutually singular**;
- the domain of the Laplacian is not an algebra;
- if $d(x, y)$ is the shortest path metric, then $d(x, \cdot)$ is not in the domain of the Dirichlet form (not of finite energy) and so methods of Differential geometry are not applicable;
- Lipschitz functions are not of finite energy and, in fact, we can not compute any non-constant functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.



Main geometric tool: the folding map
Main analytic tool: Dirichlet (energy) forms
Main probabilistic tool: coupling

The key result in the center of the proof: the classical elliptic Harnack inequality. Any harmonic function (a local energy minimizer) $u \geqslant 0$ satisfies

$$\sup_{B(x,R/2)} u \leq c_1 \inf_{B(x,R/2)} u$$

where **the constant c_1 is determined only by the geometry of the generalized Sierpiński carpet**.

Remark. This lemma is a hard mix of analysis (commutativity of certain geometric projections and the Laplacian) and probability (coupling).

Corollary. Harmonic functions are quasi-everywhere Hölder continuous (Nash-Moser theory).

BV and Besov spaces on fractals with Dirichlet forms (Patricia Alonso-Ruiz, Fabrice Baudoin, Li Chen, Luke Rogers, Nages Shanmugalingam, T.)

Open question: on the Sierpinski carpet

$$\kappa = d_W - d_H + d_{tH} - 1 = d_W - d_H + \frac{\log 2}{\log 3}$$

would give the optimal Hölder exponent for harmonic functions?
[*Strongly supported by numerical results: L.Rogers et al*]

d_{tH}:= **A new fractal dimension: The topological Hausdorff dimension**
R.Balka, Z.Buczolich, M.Elekes - Adv. Math. 2015

References: **Besov class via heat semigroup on Dirichlet spaces**
I: Sobolev type inequalities
arXiv:1811.04267
II: BV functions and Gaussian heat kernel estimates arXiv:1811.11010
III: BV functions and sub-Gaussian heat kernel estimates arXiv:1903.10078

Theorem. (Grigor'yan and Telcs, also [BBK])

On a MMD space the following are equivalent

- ***(VD)***, ***(EHI)*** and ***(RES)***
- ***(VD)***, ***(EHI)*** and ***(ETE)***
- ***(PHI)***
- ***(HKE)***

and the constants in each implication are effective.

Abbreviations: Metric Measure Dirichlet spaces, Volume Doubling, Elliptic Harnack Inequality, Exit Time Estimates, Parabolic Harnack Inequality, Heat Kernel Estimates.

Theorem 1. Let $(\mathcal{A},\mathcal{F})$, $(\mathcal{B},\mathcal{F})$ be **regular local conservative** irreducible Dirichlet forms on $L^2(F,m)$ and

$$(1+\delta)\mathcal{A}(u,u) \leq \mathcal{B}(u,u) \quad \text{for all } u \in \mathcal{F}$$

where $\delta > 0$. Then $(\mathcal{B}-\mathcal{A},\mathcal{F})$ is a regular local conservative irreducible Dirichlet form on $L^2(F,m)$.

Technical lemma. If $\mathcal{E}$ is a local regular Dirichlet form with domain $\mathcal{F}$, then for any $f \in \mathcal{F} \cap L^\infty(F)$ we have $\Gamma(f,f)(A) = 0$, if $A = \{x{\in}F : f(x){=}0\}$ where $\Gamma(f,f)$ is the energy measure or the "square field operator"

$$\int_F g d\Gamma(f,f) = 2\mathcal{E}(f,fg) - \mathcal{E}(f^2,g), \; g \in \mathcal{F}_b.$$

Definition

Let $(\mathcal{E}, \mathcal{F})$ be a Dirichlet form on $L^2(F, \mu)$. We say that $\mathcal{E}$ **is invariant with respect to all the local symmetries of F** (F-invariant or $\mathcal{E} \in \mathfrak{E}$) if

- (1) If $S \in \mathcal{S}_n(F)$, then $U_S R_S f \in \mathcal{F}$ for any $f \in \mathcal{F}$.
- (2) Let $n \geq 0$ and S_1, S_2 be any two elements of $\mathcal{S}_n$, and let Φ be any isometry of $\mathbb{R}^d$ which maps S_1 onto S_2. If $f \in \mathcal{F}^{S_2}$, then $f \circ \Phi \in \mathcal{F}^{S_1}$ and $\mathcal{E}^{S_1}(f \circ \Phi, f \circ \Phi) = \mathcal{E}^{S_2}(f, f)$ where

$$\mathcal{E}^S(g, g) = \frac{1}{m_F^n} \mathcal{E}(U_S g, U_S g)$$

  and $Dom(\mathcal{E}^S) = \{g : g \text{ maps } S \text{ to } \mathbb{R}, U_S g \in \mathcal{F}\}$.
- (3) $\mathcal{E}(f, f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$ for all $f \in \mathcal{F}$

Lemma

Let $(\mathcal{A}, \mathcal{F}_1)$, $(\mathcal{B}, \mathcal{F}_2) \in \mathfrak{E}$ with $\mathcal{F}_1 = \mathcal{F}_2$ and $\mathcal{A} \geq \mathcal{B}$. Then $\mathcal{C} = (1 + \delta)\mathcal{A} - \mathcal{B} \in \mathfrak{E}$ for any $\delta > 0$. Hence we can use the Hilbert projective metric on $\mathfrak{E}$.

$$\Theta f = \frac{1}{m_F^n} \sum_{S \in \mathcal{S}_n(F)} U_S R_S f.$$

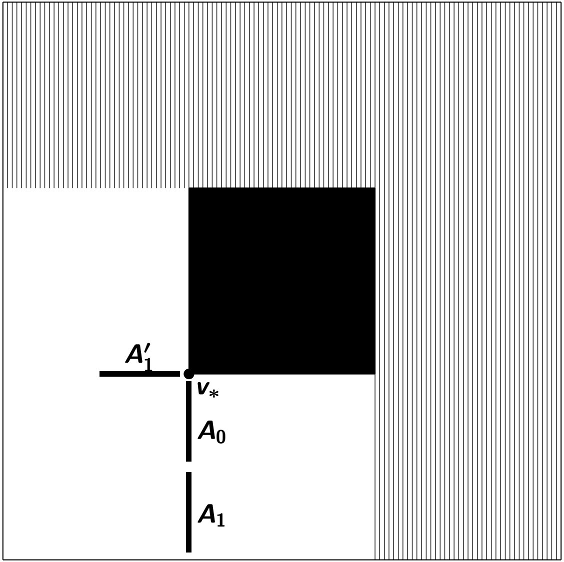
Note that Θ is a projection operator because $\Theta^2 = \Theta$. It is bounded on $C(F)$ and is an orthogonal projection on $L^2(F, \mu)$.

Lemma

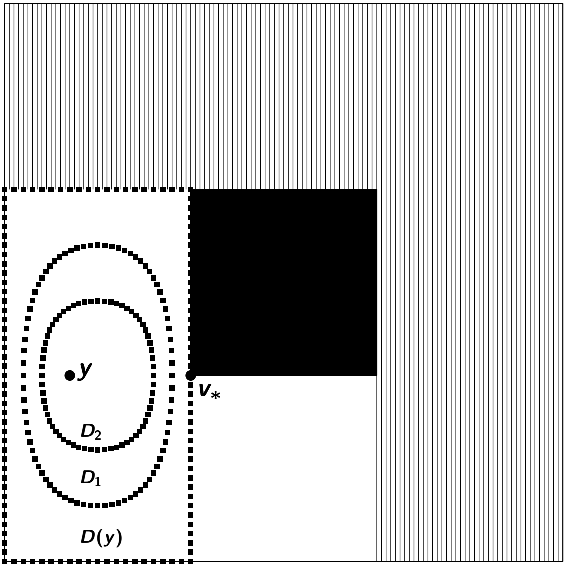
Assume that $\mathcal{E}$ is a local regular Dirichlet form on F, T_t is its semigroup, and $U_S R_S f \in \mathcal{F}$ whenever $S \in \mathcal{S}_n(F)$ and $f \in \mathcal{F}$. Then the following, for all $f, g \in \mathcal{F}$, are equivalent:

$$\text{(a):} \quad \mathcal{E}(f,f) = \sum_{S \in \mathcal{S}_n(F)} \mathcal{E}^S(R_S f, R_S f)$$

(b): $\mathcal{E}(\Theta f, g) = \mathcal{E}(f, \Theta g)$ **(c):** $T_t \Theta f = \Theta T_t f$



The half-face A_1 corresponds to a "slide move",
and the half-face A_1' corresponds to a "corner move",
analogues of the "corner" and "knight's" moves in [BB89].


y
v_*
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$D(y)$

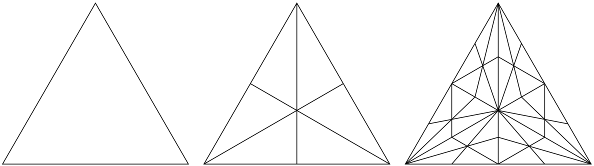


FIGURE 1. Barycentric subdivision of a 2-simplex, the graphs G_0^T, G_1^T and G_2^T.

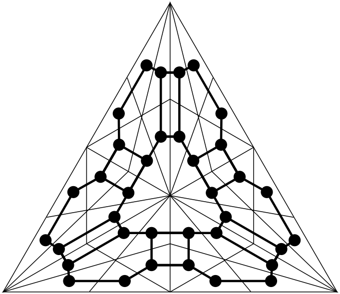


FIGURE 2. Adjacency (dual) graph G_2, in bold, and the barycentric subdivision graph pictured together with the thin image of G_2^T.

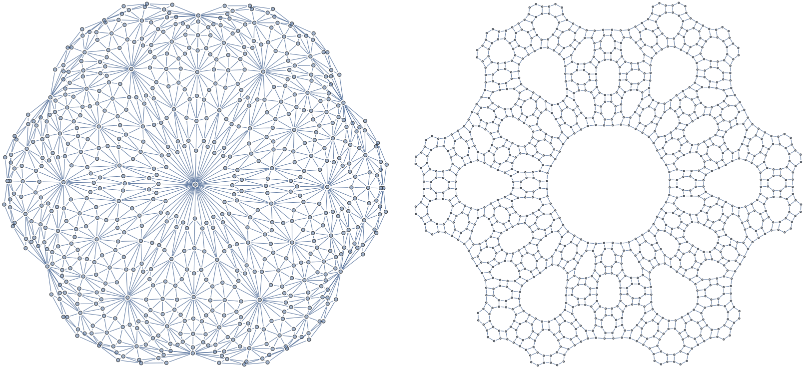


Figure 3. On the left: the graph G_4^T for barycentric subdivision of a 2-simplex. On the right: the adjacency (dual) graph G_4.

Theorem 1.1. *The resistances across graphs G_n^T and G_n^H (defined in Subsection 2.2) are reciprocals, that is $R_n^T = 1/R_n$, and the asymptotic limits*

$$\log \rho^T = \lim_{n\to\infty} \frac{1}{n} \log R_n^T \qquad \textit{and} \qquad \log \rho = \lim_{n\to\infty} \frac{1}{n} \log R_n$$

exist (and $\rho^T = 1/\rho$). Furthermore, $2/3 \le \rho^T \le 4/5$ and $5/4 \le \rho \le 3/2$.

These estimates agree with the numerical experiments from [12], which suggest that there exists a limiting Dirichlet form on these fractals and estimates $\rho \approx 1.306$, and hence $\rho^T \approx 0.7655$.

Conjecture 1. *In the case $5/4 \le \rho \le 3/2$ ($\rho \approx 1.306$), we conjecture that the recent results of A. Grigor'yan, J. Hu, K.-S. Lau and M. Yang in [24–26, 28] can imply existence of the Dirichlet form.*

Conjecture 2. *Since $2/3 \le \rho^T \le 4/5 < 5/4 \le \rho \le 3/2$, we conjecture that there is essentially no uniqueness of the Dirichlet forms, spectral dimensions, resistance scaling factors etc for repeated barycentric subdivisions.*

Diffusions on the pattern spaces of aperiodic Delone sets (Patricia Alonso-Ruiz, Michael Hinz, Rodrigo Trevino, T.)

A subset $\boldsymbol{\Lambda} \subset \mathbb{R}^d$ is a **Delone set** if it is **uniformly discrete**:

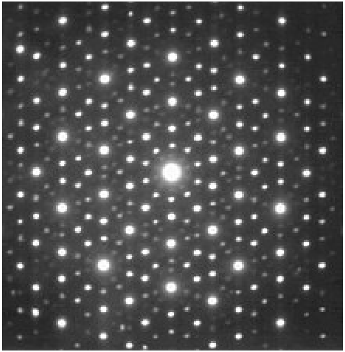
$$\exists \varepsilon > 0 : \ |\vec{x} - \vec{y}| > \varepsilon \quad \forall \vec{x}, \vec{y} \in \boldsymbol{\Lambda}$$

and relatively dense:

$$\exists R > 0 : \ \boldsymbol{\Lambda} \cap B_R(\vec{x}) \neq \varnothing \quad \forall \vec{x} \in \mathbb{R}^d.$$

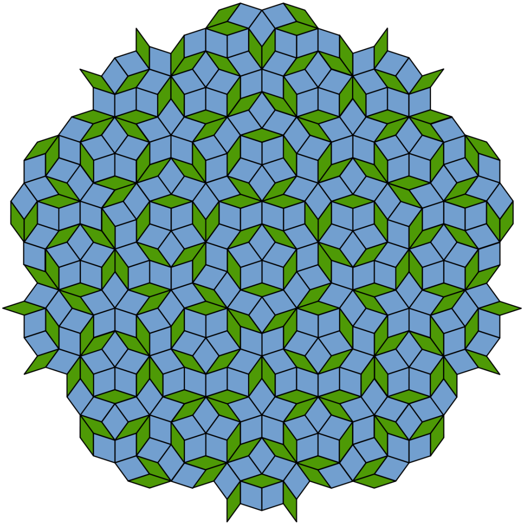
A Delone set has **finite local complexity** if $\forall R > 0 \exists$ finitely many clusters $P_1, \dots, P_{n_R}$ such that for any $\vec{x} \in \mathbb{R}^d$ there is an i such that the set $B_R(\vec{x}) \cap \boldsymbol{\Lambda}$ is translation-equivalent to P_i. A Delone set $\boldsymbol{\Lambda}$ is **aperiodic** if $\boldsymbol{\Lambda} - \vec{t} = \boldsymbol{\Lambda}$ implies $\vec{t} = \vec{0}$. It is **repetitive** if for any cluster $P \subset \boldsymbol{\Lambda}$ there exists $R_P > 0$ such that for any $\vec{x} \in \mathbb{R}^d$ the cluster $B_{R_P}(\vec{x}) \cap \boldsymbol{\Lambda}$ contains a cluster which is translation-equivalent to P. These sets have applications in crystallography (≈ 1920), coding theory, approximation algorithms, and the theory of quasicrystals.

Electron diffraction picture of a Zn-Mg-Ho quasicrystal



Aperiodic tilings were discovered by mathematicians in the early 1960s, and, some twenty years later, they were found to apply to the study of natural quasicrystals (1982 Dan Shechtman, 2011 Nobel Prize in Chemistry).

Penrose tiling



pattern space of a Delone set

Let $\Lambda_0 \subset \mathbb{R}^d$ be a **Delone set**. The **pattern space (hull)** of Λ_0 is the closure of the set of translates of Λ_0 with respect to the metric ϱ, i.e.

$$\Omega_{\Lambda_0} = \overline{\left\{\varphi_{\vec{t}}\,(\Lambda_0) : \vec{t} \in \mathbb{R}^d\right\}}.$$

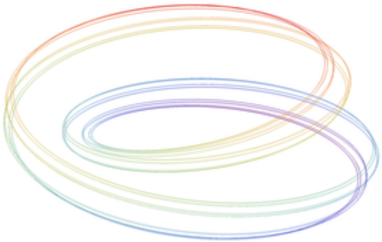
Definition

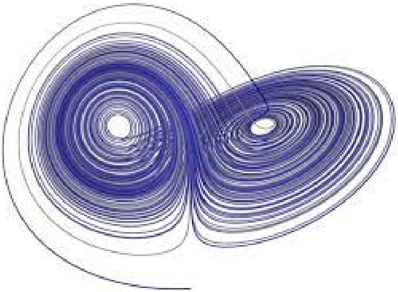
Let $\Lambda_0 \subset \mathbb{R}^d$ be a Delone set and denote by $\varphi_{\vec{t}}\,(\Lambda_0) = \Lambda_0 - \vec{t}$ its translation by the vector $\vec{t} \in \mathbb{R}^d$. For any two translates Λ_1 and Λ_2 of Λ_0 define $\varrho(\Lambda_1, \Lambda_2) = \inf\{\varepsilon > 0 : \ \exists\, \vec{s}, \vec{t} \in B_\varepsilon(\vec{0}) : \ B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{s}}(\Lambda_1) = B_{\frac{1}{\varepsilon}}(\vec{0}) \cap \varphi_{\vec{t}}(\Lambda_2)\} \wedge 2^{-1/2}$

Assumption

The action of $\mathbb{R}^d$ on Ω is uniquely ergodic:
Ω is a compact metric space with the unique $\mathbb{R}^d$-invariant probability measure μ.

Topological solenoids (similar topological features as the pattern space $\mathbf{\Omega}$):





The harmonic measures of Lucy Garnett A.Candel, Adv. Math, 2003

Foliations, the ergodic theorem and Brownian motion L.Garnett, JFA 1983

Theorem

(i) *If* $\vec{W} = (\vec{W}_t)_{t\geq 0}$ *is the standard Gaussian Brownian motion on* $\mathbb{R}^d$, *then for any* $\Lambda \in \Omega$ *the process* $X_t^\Lambda := \varphi_{\vec{W}_t}(\Lambda) = \Lambda - \vec{W}_t$ *is a conservative Feller diffusion on* (Ω, ϱ).

(ii) *The semigroup* $P_t f(\Lambda) = \mathbb{E}[f(X_t^\Lambda)]$ *is*

self-adjoint on L^2_μ, Feller but not strong Feller.

Its associated Dirichlet form is regular, strongly local, irreducible, recurrent, and has Kusuoka-Hino dimension d.

(iii) *The semigroup* $(P_t)_{t>0}$ **does not admit heat kernels with respect to μ**. *It does have Gaussian heat kernel with respect to the not-σ-finite (no Radon-Nykodim theorem) pushforward measure* λ_Ω^d

$$p_\Omega(t, \Lambda_1, \Lambda_2) = \begin{cases} p_{\mathbb{R}^d}(t, h_{\Lambda_1}^{-1}(\Lambda_2)) & \text{if } \Lambda_2 \in \mathbf{orb}(\Lambda_1), \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

(iv) **There are no semi-bounded or L^1 harmonic functions (Liouville-type)**.

no classical inequalities

Useful versions of the Poincare, Nash, Sobolev, Harnack inequalities DO NOT HOLD,
except in orbit-wise sense.

spectral properties

Theorem

The unitary **Koopman operators** $U_{\vec{t}}$ *on* $L^2(\Omega, \mu)$ *defined by* $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ *commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

hence commute with the Laplacian Δ*, and all spectral operators, such as the unitary Schrödinger semigroup.*

... hence we may have continuous spectrum (no eigenvalues) under some assumptions even though μ is a probability measure on the compact set Ω.

Under special conditions P_t may be connected to the evolution of a **Phason**: "Phason is a quasiparticle existing in quasicrystals due to their specific, quasiperiodic lattice structure. Similar to phonon, phason is associated with atomic motion. However, whereas phonons are related to translation of atoms, phasons are associated with atomic rearrangements. As a result of these rearrangements, waves, describing the position of atoms in crystal, change phase, thus the term "phason" (from the wikipedia)".

Phason evolution

Corollary

The unitary **Koopman operators** $U_{\vec{t}}$ *on* $L^2(\Omega,\mu)$ *defined by* $U_{\vec{t}} f = f \circ \varphi_{\vec{t}}$ *commute with the heat semigroup*

$$\boxed{U_{\vec{t}} P_t = P_t U_{\vec{t}}}$$

hence commute with the Laplacian $\boldsymbol{\Delta}$, *and all spectral operators, including the unitary* **Schrödinger semigroup** $e^{i\Delta t}$

$$\boxed{U_{\vec{t}} e^{i\Delta t} = e^{i\Delta t} U_{\vec{t}}}$$

Recent physics work on phason ("accounts for the freedom to choose the origin"):
Topological Properties of Quasiperiodic Tilings
(Yaroslav Don, Dor Gitelman, Eli Levy and Eric Akkermans
Technion Department of Physics)
`https://phsites.technion.ac.il/eric/talks/`
J. Bellissard, A. Bovier, and J.-M. Chez, Rev. Math. Phys. 04, 1 (1992).

Topological Properties of Quasiperiodic Tilings

Yaroslav Don, Eli Levy and Eric Akkermans
Department of Physics, Technion – Israel Institute of Technology, Haifa, Israel

Abstract

Topological properties of finite quasiperiodic tilings are examined. We study two specific physical quantities: (a) the structure factor related to the Fourier transform of the structure; (b) spectral properties (using scattering matrix formalism) of the corresponding quasiperiodic Hamiltonian. We show that both quantities involve a phase, whose windings describe topological numbers. We link these two phases, thus establishing a "Bloch theorem" for specific types of quasiperiodic tilings.

Substitution Rules – 1D Tilings

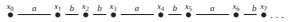
Define a *binary* substitution rule by

$$\begin{matrix}\sigma(a) = a^{\alpha} b^{\beta} \\ \sigma(b) = a^{\gamma} b^{\delta}\end{matrix} \iff \begin{matrix} a \mapsto a^{\alpha} b^{\beta} \\ b \mapsto a^{\gamma} b^{\delta}.\end{matrix}$$

Associate occurrence matrix: $M = \left(\begin{smallmatrix}\alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right)$ Consider only primitive matrices:

- Largest eigenvalue $\lambda_1 > 1$ (Perron-Frobenius)
- Left and right first eigenvectors are strictly positive

Distribution of letters underlies distribution of atoms:



Define atomic density

$$\rho(x) = \sum\nolimits_k \delta(x - x_k),$$

with distances for a and b given by $\delta_k = x_{k+1} - x_k = d_{a,b}$.
Let $\bar{d}$ be the mean distance and u_k the deviations from the mean. Define

$$x_k = \bar{d}\,k + \delta\,u_k, \quad \delta \equiv d_a - d_b.$$

Let $g(\xi) = \sum_k e^{-i\xi x_k}$ be the diffraction pattern, and $S(\xi) = |g(\xi)|^2$ the structure factor. Using $\xi_0 = 2\pi/\bar{d}$, the Bragg peaks are located at [1]

$$\xi_{m,N} = m\,\lambda_1^{-N}\,\xi_0, \quad m, N \in \mathbb{Z}.$$

We consider the following families:

Pisot. The second eigenvalue $|\lambda_2| < 1$.
Non-Pisot. The second eigenvalue $|\lambda_2| \geq 1$.
Fluctuations u_k are unbounded [2]; there are no Bragg peaks [3].

Examine the following examples:

Fibonacci. $a \mapsto ab$, $b \mapsto a$. It is Pisot, $M = \left(\begin{smallmatrix}1 & 1 \\ 1 & 0\end{smallmatrix}\right)$, $\lambda_1 = (\sqrt{5}+1)/2 \equiv \tau$ the golden ratio and $\lambda_2 = -\tau^{-1}$. Bragg peaks are located at $\xi_{p,q} = (p + q/\tau)\,\xi_0$.
In C&P language, $s = 1/\tau$ and

$$\boxed{\xi_{p,q}/\xi_0 = p + qs} \quad p, q \in \mathbb{Z}.$$

Thue-Morse. $a \mapsto ab$, $b \mapsto ba$. Here it is Pisot, $M = \left(\begin{smallmatrix}1 & 1 \\ 1 & 1\end{smallmatrix}\right)$, $\lambda_1 = 2$ and $\lambda_2 = 0$. Bragg peaks: $\xi_{m,N} = m\,2^{-N}\,\xi_0$, $\quad m, N \in \mathbb{Z}$.

Spectral Properties of Tilings

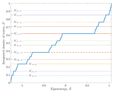
Consider a $1D$ tight-binding equation,

$$-(\psi_{k+1} + \psi_{k-1}) + V_k \psi_k = 2E\psi_k.$$

The gaps in the integrated density of states are given by the gap labeling theorem [4],

$$\mathcal{N}_{m,N} = \frac{1}{c} m\,\lambda_1^{-N} \pmod 1, \quad m, N \in \mathbb{Z}.$$

Here, c is the gcd of λ_1 and its corresponding eigenvectors in both M and the collared M_2.



In C&P sequences,

$$\boxed{\mathcal{N}_{p,q} = p + qs \pmod 1} \quad p, q \in \mathbb{Z}.$$

Structural Phase – Phason as a Gauge Field

Another way to define a tiling is by using a characteristic function. We consider the following choice [5, 6]:

$$\chi(n, \phi) = \operatorname{sign}[\cos(2\pi n s + \phi) - \cos(\pi s)].$$

with $s \to s_N = c_N/d_N$ the slope of the C&P scheme, and $n = 0 \ldots d_N - 1$.

The phase ϕ, called a phason, accounts for the gauge freedom to choose the origin. It is taken discretely as $\phi \to \phi_\ell = 2\pi\ell/d_N$.

Let $s_0(n) = \chi(n, 0)$. Let $\mathcal{T}[s_0(n)] = s_0(n+1)$ be the translation operator. Define

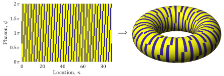
$$\Sigma_0 = \begin{pmatrix} s_0 \\ \mathcal{T}[s_0] \\ \cdots \\ \mathcal{T}^{d_N - 1}[s_0] \end{pmatrix} \implies \Sigma_0(n, \ell) = \mathcal{T}^{\ell}[s_0(n)].$$

Consider now the row permuted Σ_1

$$\Sigma_1(n, \ell) = \mathcal{T}^{m(\ell)}[s_0(n)], \qquad m(\ell) = \ell\,c_N^{-1} \pmod{d_N}.$$

Lemma. For $\phi_\ell = 2\pi\ell/d_N$ with $n, \ell = 0 \ldots d_N - 1$ one has $\chi(n, \phi_\ell) = \Sigma_1(n, \ell)$.
This defines a *discrete* phason ϕ_ℓ for the structure.

The structure of Σ_1 ($d_N \times d_N$) is that of a torus:



The discrete Fourier transform of Σ_1 about n reads

$$G(\xi, \ell) \equiv \sum\nolimits_{n=0}^{d_N - 1} \omega^{-\xi n}\,\Sigma_1(n, \ell) = \omega^{m(\ell)\xi}\,\varsigma_0(\xi).$$

- The structure factor $S(\xi, \phi) = |\varsigma_0(\xi)|^2$ is ϕ-independent.
- The phase of $G(\xi, \ell)$ reads

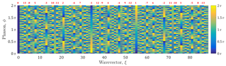
$$\boxed{\Theta(\xi, \ell) \equiv \arg \omega^{m(\ell)\xi} = \phi_\ell\,\xi/c_N \pmod{2\pi}.}$$

Corollary. For any diffraction peak (discrete Bragg peak) $\xi_{p,q} = q c_N$ one has the (discrete) winding number at $\xi_{p,q}$,

$$\Theta(\xi_{p,q}) = \frac{2\pi}{d_N}\ell\,q,$$

hence

$$\boxed{W_{\xi_{p,q}} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \Theta(\xi = \xi_{p,q}, \phi)}{\partial \phi}\,d\phi = q.}$$



Here we used the Fibonacci sequence ($s = 1/\tau$) with $d_N = 89$ sites. The winding numbers are indicated by the red numbers above.

Remark. The analysis above for the winding numbers is done for rational approximations $s_N = c_N/d_N$. It holds by construction for the irrational case $s_N \to s$.

Spectral Phase: Scattering Matrix Approach

Spectral properties are also accessible from the continuous wave equation,

$$-\psi''(x) - k_0^2\,v(x)\,\psi(x) = k_0^2\,\psi(x),$$

with scattering boundary conditions.

The scattering $\mathcal{S}$-matrix is defined by $\left(\begin{smallmatrix}\overleftarrow{o} \\ \overrightarrow{o}\end{smallmatrix}\right) = \left(\begin{smallmatrix}\overleftarrow{\tau}(k) & \overrightarrow{r}(k) \\ \overleftarrow{r}(k) & \overrightarrow{\tau}(k)\end{smallmatrix}\right)\left(\begin{smallmatrix}\overrightarrow{\iota} \\ \overleftarrow{\iota}\end{smallmatrix}\right) \equiv \mathcal{S}\left(\begin{smallmatrix}\overrightarrow{\iota} \\ \overleftarrow{\iota}\end{smallmatrix}\right)$, with $\overrightarrow{r} = \overrightarrow{R}\,e^{i\overrightarrow{\theta}}$ and $\overleftarrow{r} = \overleftarrow{R}\,e^{i\overleftarrow{\theta}}$. It is unitary and can be diagonalized to $\mathcal{S} \mapsto \left(\begin{smallmatrix}e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2}\end{smallmatrix}\right)$ so that $\det \mathcal{S} = e^{2i\delta(k)}$ with the total phase shift $\delta(k) = (\gamma_1(k) + \gamma_2(k))/2$ independent of ϕ. We are interested in the chiral phase,

$$\boxed{\alpha(k, \phi) = \overrightarrow{\theta}(k, \phi) - \overleftarrow{\theta}(k, \phi).}$$

Using the Krein-Schwinger formula [7] allows to relate the change of density of states to the scattering data,

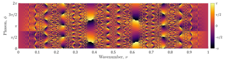
$$\varrho(k) - \varrho_0(k) = \frac{1}{2\pi}\operatorname{Im}\frac{d}{dk}\ln\det\mathcal{S}(k).$$

So that the integrated density of states is

$$\mathcal{N}(k) - \mathcal{N}_0(k) = \delta(k)/\pi.$$

The total phase shift $\delta(k)$ is independent of the phason ϕ unlike the *chiral* phase $\alpha(k, \phi)$, whose winding for values of k inside the gaps is given by [8],

$$\boxed{W_{\alpha_q} = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial\alpha(k = k_{p,q}, \phi)}{\partial\phi}\,d\phi = 2q.}$$



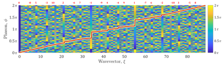
Here we used the Fibonacci sequence ($s = 1/\tau$) with $d_N = 233$ sites.

Relation between both Phases: a "Bloch Theorem"

In $1D$ C&P structures, The locations of Bragg peaks for a diffraction spectrum correspond to the spectral density of states,

$$\boxed{\xi_{p,q}/d_N = p + qs = \mathcal{N}_{p,q}} \quad p, q \in \mathbb{Z}.$$

Drawing the integrated density of states $\mathcal{N}$ (red line) on top the structural phase $\Theta(\xi, \phi)$, shows numerically the relation between $\xi_{p,q}$ and $\mathcal{N}_{p,q}$.



Both $\xi_{p,q}$ and $\mathcal{N}_{p,q}$ are isomorphic to $\mathbb{Z} \oplus s\mathbb{Z}$. The second $\mathbb{Z}$, corresponding to q, can be derived independently from the windings of both the structural phase $\Theta(\phi)$ and the chiral phase $\alpha(\phi)$. Since these phases account for windings, they are isomorphic

$$\boxed{\Theta(\phi) \cong \alpha(\phi).}$$

The winding gives a topological interpretation to these phases. This result can be viewed as a Bloch theorem for quasiperiodic tilings [9].

Cut and Project Scheme

An alternative method to build quasiperiodic tilings is the Cut & Project scheme. The procedure is as follows [10].

Cut.

1. Start with an n-dimensional space $R = \mathbb{R}^n$.
2. Insert "atoms" on the integer lattice $Z = \mathbb{Z}^n$.
3. Divide R into the *physical space* E and the *internal space* $E_\perp$ such that $E \oplus E_\perp = R$ and $E \cap E_\perp = \emptyset$.
4. To resolve ambiguity for E, choose an initial location $\boldsymbol{c} \in R$ such that E passes through $\boldsymbol{c}$. There is no such requirement for $E_\perp$.

Project.

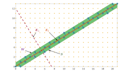
1. Inspect the hypercube $\mathbb{I}_n = [-0.5, 0.5)^n$.
2. The *window* is its projection on the internal space $W = \pi_\perp(\mathbb{I}n)$.
3. The *strip* is the product with the physical space $S = W \otimes E$.
4. Choose only the points inside the strip $S \cap Z$, and project them onto the physical space, $Y = \pi(S \cap Z)$.
5. The *atomic density* is given by $\rho(\boldsymbol{x}) \equiv \rho_{\boldsymbol{c}}(\boldsymbol{x}) = \sum_{\boldsymbol{y} \in Y}\delta(\boldsymbol{x} - \boldsymbol{y})$ with $\boldsymbol{x} \in E$. Note the implicit dependency of Y on $\boldsymbol{c}$.

For $1D$ systems, define the phason

$$\phi = 2\pi\,b/W \qquad b \in E_\perp,$$

where W is the window above.
The slope s is given by

$$1/s = 1 + \cot\beta.$$



Useful Tools

In periodic structures, topological numbers are described as Chern numbers. This does not happen in quasiperiodic tilings, since there exists no notion of a Brillouin zone. But alternative tools exist to describe topological properties of quasiperiodic tilings. We now enumerate some of them.

- Tiling space T (dependent on λ_1 or s) and its hull Ω_T.
- Čech cohomology $\check{H}^1(\Omega_T)$, simplicial cohomology $H^1(\Gamma_n)$ and Bratteli graphs [11, 12].
- K-theory, $K_0(\Omega_T)$ group and the abstract gap labeling theorem [4, 13].
- The Bloch theorem described before can be given an interpolation for $1D$ C&P tilings (for an irrational slope $s \notin \mathbb{Q}$) by means of the "commutative diagram":

$$\begin{array}{ccc} \Theta(\phi) & \xrightarrow{\overset{!}{\cong}} & \alpha(\phi) \\ \uparrow & & \uparrow \\ \mathbb{Z}^2 \cong \check{H}^1(\Omega_{C\&P}) & \xrightarrow{\psi} & K_0(\Omega_{C\&P}) \cong \mathbb{Z} \oplus \mathbb{Z} \\ \downarrow C_{\mu(s)} & & \downarrow \tau_* \\ \mathbb{Z} \oplus s\mathbb{Z} & \xrightarrow{\cong} & \mathbb{Z} \oplus s\mathbb{Z} \end{array}$$

The topological features are contained in $\check{H}^1$ or K_0 groups.

Conclusions

- We have defined two types of phases—a structural and spectral one—whose windings unveil topological features of quasiperiodic tilings.
- We found a relation between these two phases, which can be interpreted as a Bloch-like theorem.
- We have considered here a subset of tilings, which are known as Sturmian (C&P) words. Our results can be extended to a broader families of tilings in one dimension, and to tiles in higher dimensions ($D > 1$).
- All these features have been observed experimentally [5, 6].

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This work was supported by the Israel Science Foundation Grant No. 924/09.
We thank C. Schochet for useful discussions.

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Topological Properties of Qu

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We study two
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ties involve a
two phases,
ic tilings.

The Phason – Structural Phase

Another way to define a tiling is by using a characteristic function. We consider the following choice [4, 5]:

$$\chi(n,\phi) = \operatorname{sign}\left[\cos\left(2\pi n\,\lambda_1^{-1} + \phi\right) - \cos\left(\pi\lambda_1^{-1}\right)\right]$$

with $n = 0\ldots F_N - 1$ and $[0,2\pi] \ni \phi \to \phi_\ell = 2\pi F_N^{-1}\,\ell$. The phase ϕ—called a phason—accounts for the freedom to choose the origin.
Let $s_0(n) = \chi(n,0)$. Let $\mathcal{T}[s_0(n)] = s_0(n+1)$ be the translation operator. Define

$$\Sigma_0 = \begin{pmatrix} s_0 \\ \mathcal{T}[s_0] \\ \cdots \\ \mathcal{T}^{F_N-1}[s_0] \end{pmatrix} \implies \Sigma_0(n,\ell) = \mathcal{T}^{\ell}[s_0(n)]$$

Scatter

Spectral pro

with scatter

The scatter
$\vec{r} = \vec{R}\,e^{i\vec{\vartheta}}$

Helmholtz, Hodge and de Rham

Theorem

Assume $d = 1$. *Then the space* $L^2(\Omega, \mu, \mathbb{R}^1)$ *admits the orthogonal decomposition*

$$L^2(\Omega, \mu, \mathbb{R}^1) = \operatorname{Im} \nabla \oplus \mathbb{R}(dx). \tag{2}$$

In other words, the **L^2-cohomology is 1-dimensional**, which is surprising because the **de Rham cohomology is not one dimensional**.

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Thank you!

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