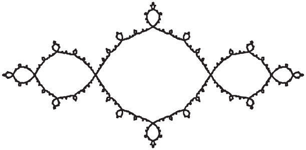
From projective limits to spectral asymptotics, revisited by Bob Strichartz

Alexander Teplyaev
University of Connecticut



June 2025 $*$ Cornell

Plan

- Spectral asymptotics revisited, following Bob Strichartz
- Projective limits, following Luke Rogers and Voldia Nekrashevych
- Monotone limits of Dirichlet forms, with applications
  (joint project with Patricia Alonso-Ruiz and Jun Kigami)
- Spectrum and heat kernels on diamond fractals, a.k.a. Laakso-type spaces
- Brief history of heat kernels on fractals, if time permits

Spectral asymptotics revisited, following Bob Strichartz

If $\{\varphi_j\}_j$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\lambda_j\}$, then

$$K_\lambda(x,y) = \sum_{\lambda_j \leqslant \lambda} \varphi_j(x)\varphi_j(y), \tag{1}$$

is the kernel of the orthogonal projection of $u \in L^2(D_{m,j,x},\mu)$ onto the span of all eigenfunctions with eigenvalues less than or equal to λ.
Generally, define the kernel $K_\lambda(\cdot,\cdot)$ of the spectral projection operator E_λ onto the $[0,\lambda]$ portion of the spectrum:

$$E_\lambda u(x) := \int_{D_{m,j,x}} K_\lambda(x,y)u(y)\,d\mu(y) \tag{2}$$

According to [Strichartz 2012], we define the spectral mass function

$$M(\lambda) := \lim_{k\to\infty} \frac{1}{\mu(B_k)} \int_{B_k} K_\lambda(x,x), d\mu(x), \tag{3}$$

if the limit exists over an increasing sequence of sets B_k. This is, in a sense, a re-definition of the so called integrated density of states in physics.

Setting for direct (inductive) and inverse (projective) limits

Let X be a topological space and let $f\colon X \to X$ be a continuous surjective map.

- **Backward system**: $X \xleftarrow{f} X \xleftarrow{f} X \xleftarrow{f} \ldots$
- **Forward system**: $X \xrightarrow{f} X \xrightarrow{f} X \xrightarrow{f} \ldots$

The aim is to record every backward orbit and every forward orbit inside suitable limit spaces.

Inverse limit (projective)

The inverse limit of the backward system is

$$\varprojlim(X, f) = \left\{(x_0, x_1, x_2, \dots) \in X^{\mathbb{N}} \mid f(x_{n+1}) = x_n \text{ for all } n \geq 0\right\}.$$

- Subspace topology inherited from the product $X^{\mathbb{N}}$.
- Canonical projections $\pi_n\big((x_k)_{k\geq 0}\big) = x_n$ satisfy $\pi_n = f \circ \pi_{n+1}$.
- Universal among spaces Y with maps $\psi_n \colon Y \to X$ such that $\psi_n = f \circ \psi_{n+1}$.

Direct limit (inductive)

Form the coproduct $\bigsqcup_{n\geq 0} X \times \{n\}$ and impose the relation

$$(x, n) \sim \big(f(x), n+1\big) \quad (x \in X,\ n \geq 0).$$

The direct limit is the quotient

$$\varinjlim(X, f) = \Big(\bigsqcup_{n\geq 0} X \times \{n\}\Big) /\!\sim .$$

- Final topology: a set U is open when each inclusion $j_n\colon X \to \varinjlim(X, f)$ has open preimage.
- Maps j_n satisfy $j_{n+1} \circ f = j_n$.
- Universal among spaces receiving a family of maps $\varphi_n\colon X \to Y$ compatible with f.

Selected properties

- $\varprojlim(X, f)$ records all backward orbits inside one compact space when X is compact.
- $\varinjlim(X, f)$ records forward orbits inside one connected space under mild hypotheses.
- When f is a covering map of degree greater than one, $\varprojlim(X, f)$ can be fractal in nature.

Example: doubling map on the circle

Let $X = \mathbb{S}^1$ and $f(z) = z^2$. Then

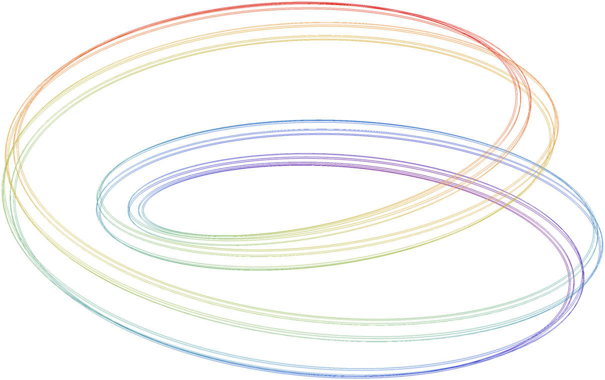
- $\varprojlim(\mathbb{S}^1, f)$ is the dyadic solenoid.
- $\varinjlim(\mathbb{S}^1, f)$ is *exercise* ...

J. Munkres, *Topology*.

A. Hatcher, *Algebraic Topology*.

J. P. May and K. Ponto, *More Concise Algebraic Topology*.

Dyadic solenoid (visualisation)



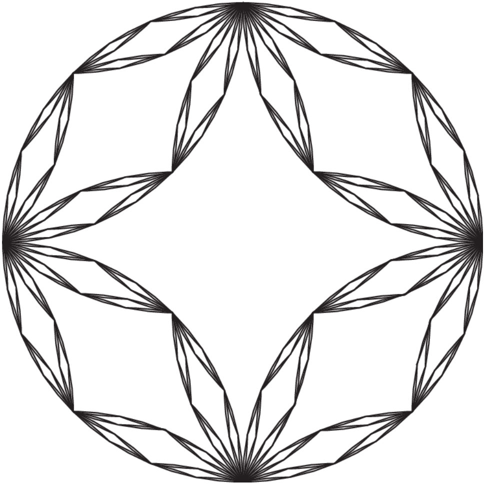


Figure: A diamond fractal.

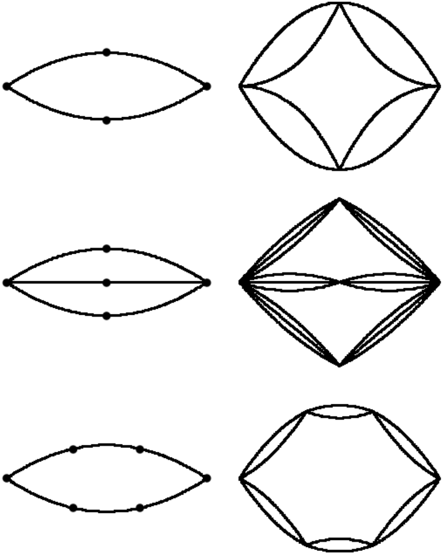


Figure: Schematic constructions of self-similar diamond fractals, $D_{4,2}$, $D_{6,2}$, and $D_{6,3}$, from top to bottom [Akkermans, Dunne, T., 2010]

Spectral mass function, a.k.a. integrated density of states

Theorem (P. A.-R., J. K., A. T., in progress)

The Spectral Mass $M(\lambda)$ on $D_{m,j,x}$ exists and is a pure-jump non-decreasing function

$$N_{D_{m,j,x}}(\lambda) = (b-1) \sum_{l=-\infty}^{\infty} m^l N_{\mathcal{D}}(\lambda / j^{2l}) \tag{4}$$

where $m = jb$ and j is the length division parameter and b is the branching parameter.

Monotone limits of Dirichlet forms, with applications (joint project with Patricia Alonso-Ruiz and Jun Kigami)

Separation property: For any $x, y \in X$ with $x \neq y$, there exists $n \geq 1$ such that $\pi_n(x) \neq \pi_n(y)$.
Monotonicity property: For any $n \geq 1$ and any $u \in \mathcal{F}_{n+1}$

$$\int_{X_n} p_n(u)^2 d\mu_n \leq \int_{X_{n+1}} u^2 d\mu_{n+1}. \tag{5}$$

Theorem (P. A.-R., J. K., A. T., in progress)

Assume the separation property and the monotonicity property (5). *Then* $(\mathcal{E}, \mathcal{F})$ *is a regular Dirichlet form on* $L^2(X, \mu)$. *Furthermore, let* $\left(\overline{\mathcal{F}_n}\right)_{L^2}$ *be the* $L^2(X, \mu)$*-closure of* $\mathcal{F}_n$ *and let* $\mathcal{G}_{n+1}$ *be the* $L^2(X, \mu)$*-orthogonal complement of* $\left(\overline{\mathcal{F}_n}\right)_{L^2}$ *in* $\left(\overline{\mathcal{F}_{n+1}}\right)_{L^2}$*, i.e.*

$$\left(\overline{\mathcal{F}_{n+1}}\right)_{L^2} = \left(\overline{\mathcal{F}_n}\right)_{L^2} \oplus \mathcal{G}_{n+1} \tag{6}$$

where $\oplus$ *means the* $L^2(X, \mu)$*-orthogonal direct sum, and let* $\mathcal{G}_1 = \left(\overline{\mathcal{F}_1}\right)_{L^2}$. *Then*

$$L^2(X, \mu) = \bigoplus_{n \geq 1} \mathcal{G}_n \tag{7}$$

Theorem (P. A.-R., J. K., A. T., in progress)

Moreover, Let $\tau_n : L^2(X,\mu) \to \mathcal{G}_n$ *be the* $L^2(X,\mu)$*-orthogonal projection and let* H *be the non-negative self-adjoint operator on* $L^2(X,\mu)$ *associated with the Dirichlet form* $(\mathcal{E},\mathcal{F})$*. Then there exists a non-negative self-adjoint operator* H_n *on* $\mathcal{G}_n$ *for each* $n \geq 1$ *such that the following holds.*

(a) If $u \in L^2(X,\mu)$*, then* $u \in \mathrm{Dom}(H)$ *if and only if* $\tau_n(u) \in \mathrm{Dom}\,(H_n)$ *for any* $n \geq 1$ *and* $\sum_{n\geq 1} H_n \tau_n(u)$ *converges in* $L^2(X,\mu)$*.*

(b)

$$H = \bigoplus_{n\geq 1} H_n \tag{8}$$

i.e, for any $u \in \mathrm{Dom}(H)$*,*

$$Hu = \sum_{n\geq 1} H_n\left(\tau_n(u)\right). \tag{9}$$

Spectrum and heat kernels on diamond fractals, a.k.a. Laakso-type spaces

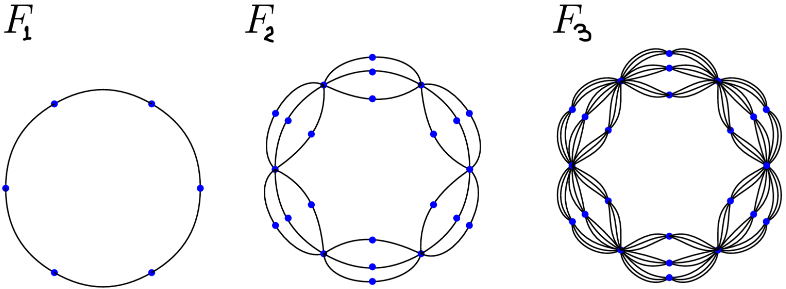


Figure: Approximations of a not-self-similar diamond fractal with $j_1 = 3, n_1 = 2$, $j_2 = 2, n_2 = 3$, $j_3 = 2, n_3 = 3$ [P. Alonso-Ruiz 2018, 2021]

early history of fractals

- in nature, **fractals are everywhere**
- in mathematics and logic, the first mention of fractals is Zeno's Paradox "Achilles and the tortoise" (c. 490–430 BC)

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \ldots$$

  - this is a weakly self-similar set, a zero-dimensional fractal
- the **Cantor set**: Henry John Stephen Smith (1874) and Georg Cantor (1883)

$$0 < \textit{dimension} = \frac{\log 2}{\log 3} < 1$$

- the **Koch snowflake** (1906)
- Gaston Julia (1893 – 1978), Pierre Fatou (1878 – 1929) and Benoit Mandelbrot (1924 – 2010)

ANALYSE MATHÉMATIQUE. — *Sur une courbe dont tout point est un point de ramification.* Note [1] de M. **W. Sierpinski**, présentée par M. Émile Picard.

Le but de cette Note est de donner un exemple d'une courbe cantorienne et jordanienne en même temps, dont tout point est un point de ramification. (Nous appelons *point de ramification* d'une courbe $\mathcal{C}$ un point p de cette courbe, s'il existe trois continus, sous-ensembles de $\mathcal{C}$, ayant deux à deux le point p et seulement ce point commun.)

Soient T un triangle régulier donné; A, B, C respectivement ses sommets : gauche, supérieur et droit. En joignant les milieux des côtés du triangle T, nous obtenons quatre nouveaux triangles réguliers (*fig.* 1), dont trois, T_0, T_1, T_2, contenant respectivement les sommets A, B, C, sont situés parallèlement à T et le quatrième triangle U contient le centre du triangle T; nous exclurons tout l'intérieur du triangle U.

Les sommets des triangles T_0, T_1, T_2 nous les désignerons respectivement :

---

[1] Séance du 1er février 1915.

triangles U_0, U_1, U_2, situés parallèlement à U, dont les intérieurs seront

Fig. 1.

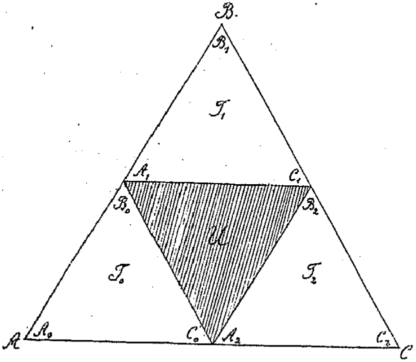
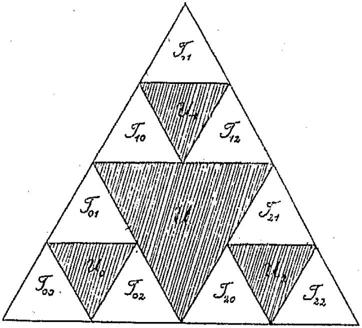


Fig. 2.



exclus (*fig.* 2). Avec chacun des triangles $T_{\lambda_1\lambda_2}$ procédons de même et ainsi

Fig. 3.

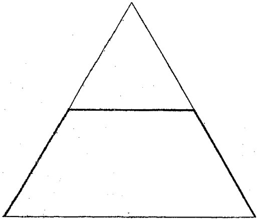
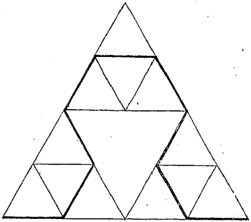


Fig. 4.



d'eux se rencontrent quatre segments différents, situés entièrement sur l'ensemble $\mathfrak{S}$.

Donc, tous les points de la courbe $\mathfrak{S}$, sauf peut-être les points A, B, C, sont ses points de ramification.

Pour obtenir une courbe dont tous les points sans exception sont ses

Fig. 5.

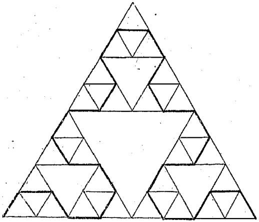
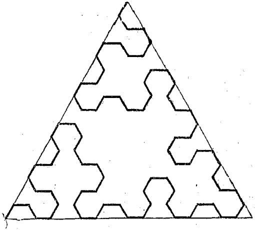


Fig. 6.



points de ramification, il suffit de diviser un hexagone régulier en six triangles réguliers et dans chacun d'eux inscrire une courbe $\mathfrak{S}$.

Newton's law of universal gravitation (April 1686)

$$F = G\frac{m_1 m_2}{r^2}$$

Newton's laws of motion

$$F = ma$$

This is a **space–time** relation.

...

Maxwell's equations (1861) lead to the Einstein's Theory of Relativity (1905)

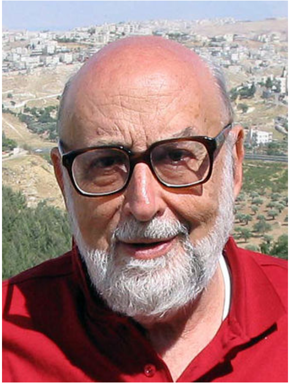
$$E = mc^2$$

François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

F. ENGLERT, J.-M. FRÈRE[1] and M. ROOMAN[2]

Physique Théorique, C.P. 225, Université Libre de Bruxelles, 1050 Brussels, Belgium

Ph. SPINDEL

Faculté des Sciences, Université de l'Etat à Mons, 7000 Mons, Belgium

Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

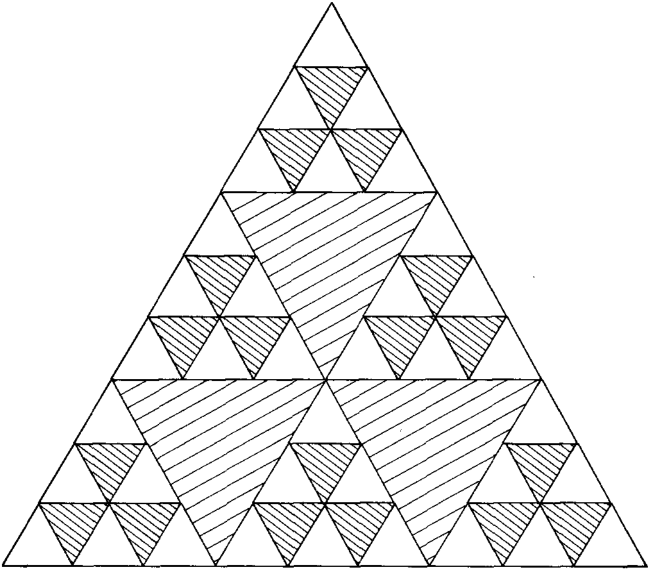


Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

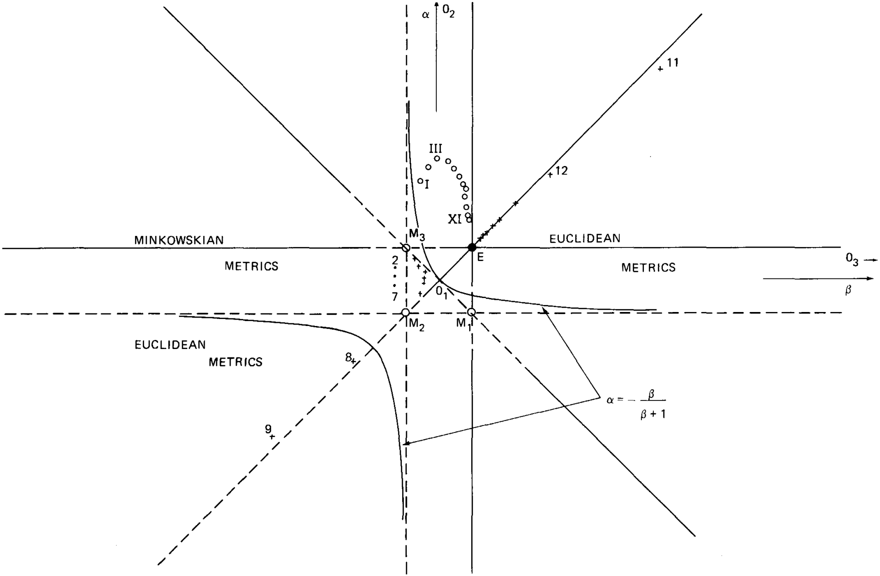


Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

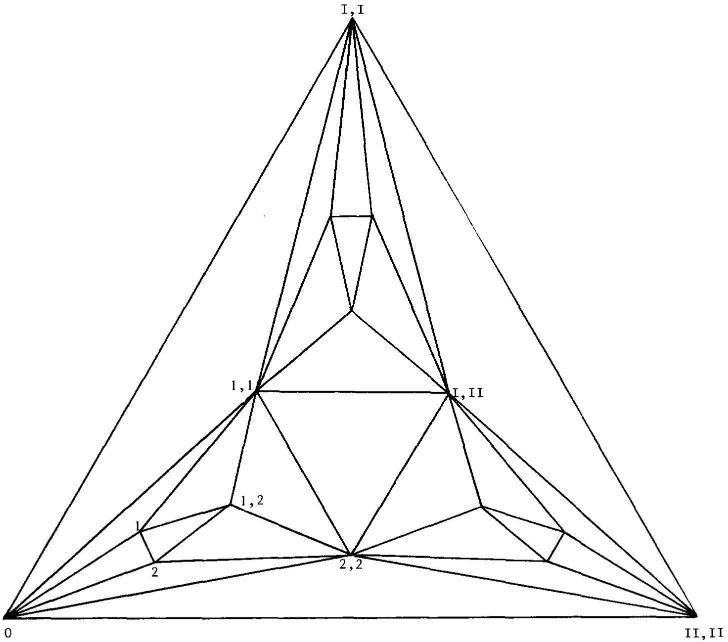


Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

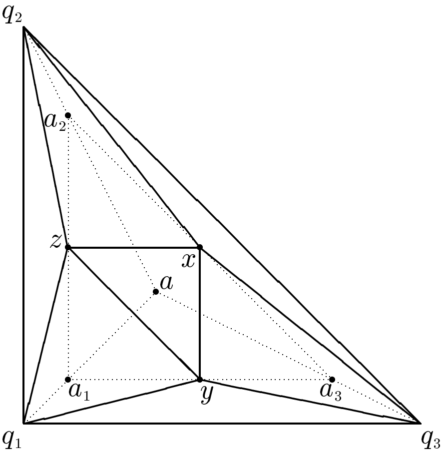


Figure 6.4. Geometric interpretation of Proposition 6.1.

The Spectral Dimension of the Universe is Scale Dependent

J. Ambjørn,[1,3,*] J. Jurkiewicz,[2,†] and R. Loll[3,‡]

[1]The Niels Bohr Institute, Copenhagen University, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

[2]Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland

[3]Institute for Theoretical Physics, Utrecht University, Leuvenlaan 4, NL-3584 CE Utrecht, The Netherlands

(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \tag{15}$$

and thus is compatible with the integer value two.

Published for SISSA by Springer

Received: October 26, 2011
Accepted: November 18, 2011
Published: December 2, 2011

Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

Martin Reuter and Frank Saueressig

Institute of Physics, University of Mainz,
Staudingerweg 7, D-55099 Mainz, Germany
E-mail: reuter@thep.physik.uni-mainz.de,
saueressig@thep.physik.uni-mainz.de

Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

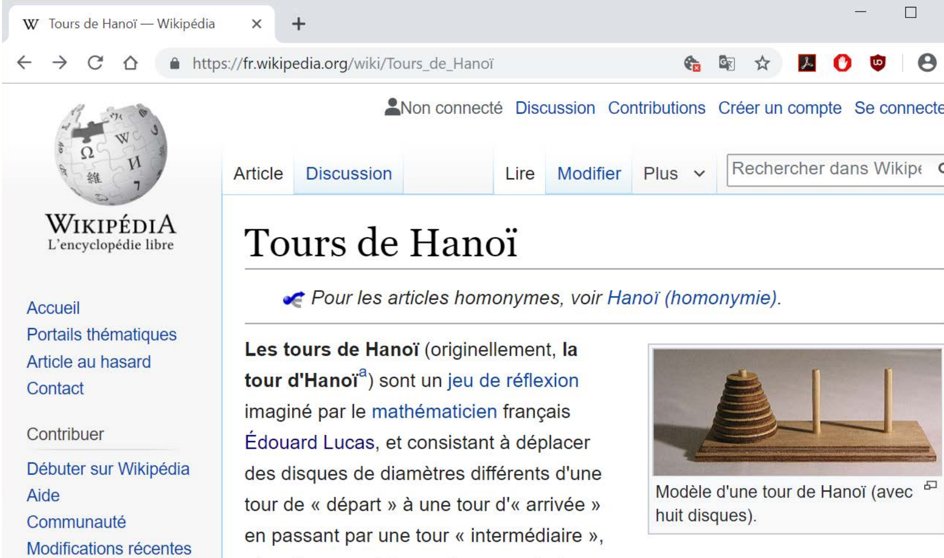
Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data

Martin Reuter and Frank Saueressig

a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2 + d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered

Toy model: Hanoi towers game



The puzzle was invented by the French mathematician Édouard Lucas in 1883.

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

Rostislav Grigorchuk [1], Zoran Šunić

Department of Mathematics, Texas A&M University, MS-3368, College Station, TX, 77843-3368, USA

Received 23 January, 2006; accepted after revision +++++

Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*

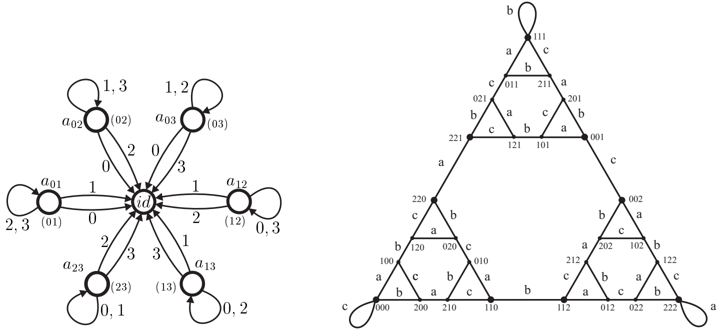
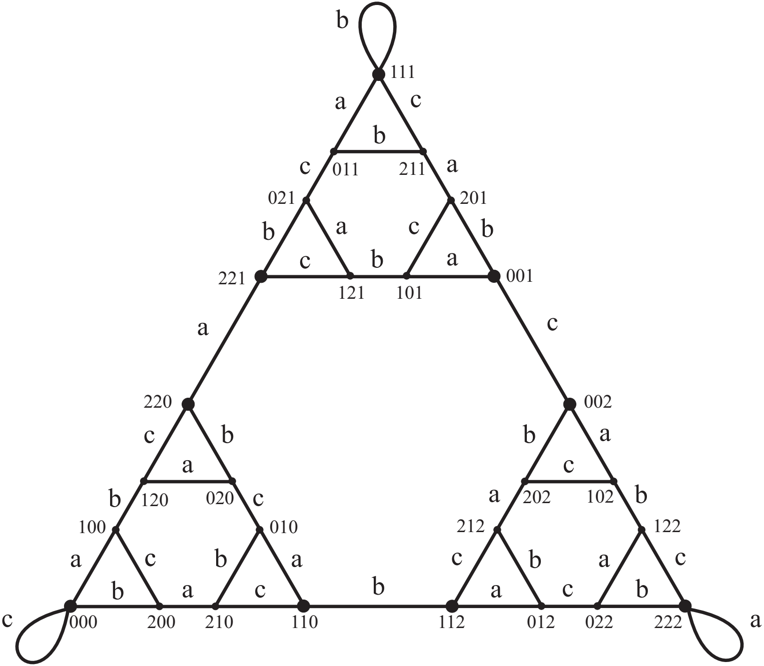


Figure 1. The automaton generating $H^{(4)}$ and the Schreier graph of $H^{(3)}$ at level 3 / L'automate engendrant $H^{(4)}$ et le graphe de Schreier de $H^{(3)}$ au niveau 3


111
011
211
021
201
221
121
101
001
220
002
120
020
202
102
100
010
212
122
000
200
210
110
112
012
022
222

Initial physics motivation

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Main early mathematical results

Sheldon Goldstein, *Random walks and diffusions on fractals.* Percolation theory and ergodic theory of infinite particle systems (Minneapolis, Minn., 1984–1985), IMA Vol. Math. Appl., 8, Springer

Summary: we investigate the asymptotic motion of a random walker, which at time n is at $X(n)$, on certain 'fractal lattices'. For the 'Sierpiński lattice' in dimension d we show that, as $L \to \infty$, the process $Y_L(t) \equiv X([(d+3)^L t])/2^L$ converges in distribution to a diffusion on the Sierpin'ski gasket, a Cantor set of Lebesgue measure zero. The analysis is based on a simple 'renormalization group' type argument, involving self-similarity and 'decimation invariance'. In particular,

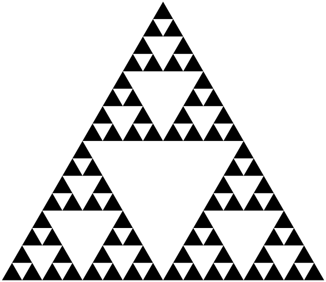
$$|X(n)| \sim n^{\gamma},$$

where $\gamma = (\ln 2)/\ln(d+3)) \leqslant 2$.

Shigeo Kusuoka, *A diffusion process on a fractal.* Probabilistic methods in mathematical physics (Katata/Kyoto, 1985), 1987.

Main classes of fractals considered

- $[0, 1]$
- Sierpiński gasket
- nested fractals
- p.c.f. self-similar sets, possibly with various symmetries
- finitely ramified self-similar sets, possibly with various symmetries
- infinitely ramified self-similar sets, with local symmetries, and with heat kernel estimates (such as the Generalized Sierpiński carpets)
- metric measure Dirichlet spaces, possibly with heat kernel estimates (MMD+HKE)



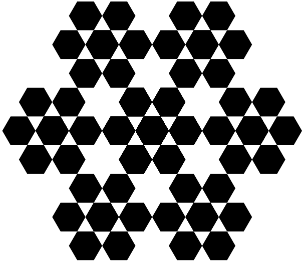


Figure: Sierpiński gasket and Lindstrøm snowflake (nested fractals), p.c.f., finitely ramified)

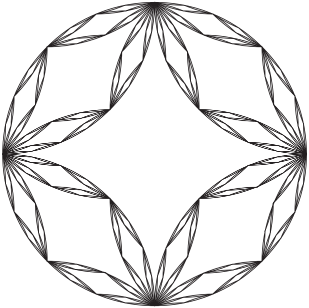


Figure: Diamond fractals, non-p.c.f., but finitely ramified

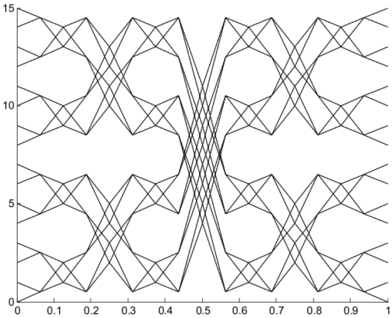


Figure: Laakso Spaces (Ben Steinhurst), infinitely ramified

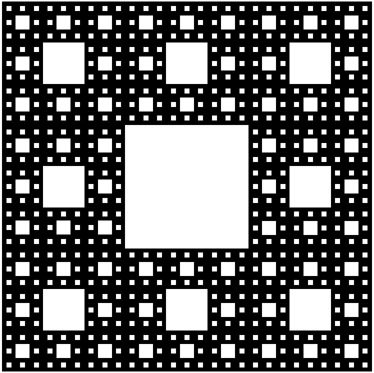


Figure: Sierpiński carpet, infinitely ramified

Existence, uniqueness, heat kernel estimates: geometric renormalization for F-invariant Dirichlet forms

Brownian motion:
Thiele (1880), Bachelier (1900)
Einstein (1905), Smoluchowski (1906)
Wiener (1920'), Doob, Feller, Levy, Kolmogorov (1930'),
Doeblin, Dynkin, Hunt, Ito ...

$$\textit{distance} \sim \sqrt{\textit{time}}$$

"Einstein space–time relation for Brownian motion"

Wiener process in $\mathbb{R}^n$ satisfies $\frac{1}{n}\mathbb{E}|W_t|^2 = t$ and has a Gaussian transition density:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

- **De Giorgi-Nash-Moser estimates for elliptic and parabolic PDEs;**
- **Li-Yau (1986) type estimates** on a geodesically complete Riemannian manifold with $\boldsymbol{Ricci} \geqslant \mathbf{0}$:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

$$distance \sim \sqrt{time}$$

Gaussian:

$$p_t(x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$$

Li-Yau Gaussian-type:

$$p_t(x,y) \sim \frac{1}{V(x,\sqrt{t})} \exp\left(-c\frac{d(x,y)^2}{t}\right)$$

Sub-Gaussian:

$$p_t(x,y) \sim \frac{1}{t^{d_H/d_w}} \exp\left(-c\left(\frac{d(x,y)^{d_w}}{t}\right)^{\frac{1}{d_w-1}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

Thank you for your attention!