Harmonic analysis and frames for fractal IFS L^2 spaces via Infinite products of projections

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Abstract

We show that an idea, originating initially with a fundamental recursive iteration scheme (usually referred as "the" Kaczmarz algorithm), admits important applications in such infinite-dimensional, and non-commutative, settings as are central to spectral theory of operators in Hilbert space, to optimization, to large sparse systems, to iterated function systems (IFS), and to fractal harmonic analysis. We present a new recursive iteration scheme involving as input a prescribed sequence of selfadjoint projections. Applications include random Kaczmarz recursions, their limits, and their error-estimates.

1. Introduction

Outline

- A formulation of the classical infinite-dimensional Kaczmarz algorithmit in terms of sequences of projections in a Hilbert space $\mathscr{H}$.
- Explicit and algorithmic criteria for convergence of certain *infinite products of projections* in $\mathscr{H}$.
- A *random* Kaczmarz algorithm.
- Applications to stochastic analysis, and to frame-approximation questions in the Hilbert space $L^2(\mu)$, where μ is in a class of *iterated function system* (IFS) measures.

Slice-singular measures

Consider a choice of period interval, $[0,1]$, or $[-\pi,\pi]$, a positive finite measure μ with support in the chosen period interval; and the usual Fourier frequencies realized as complex exponentials e_n, $n \in \mathbb{Z}$. Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$.

Theorem 1 (F&M Riesz). The subset $\{e_n \mid n \in \mathbb{N}_0\}$ is total in $L^2(\mu)$ if and only if μ is singular with respect to Lebesgue measure.

Question. What is a natural extension of F&M Riesz' theorem to higher dimensions, modeling the above formulation? (One of the motivations for this is a certain construction of **frame algorithms** in $L^2(\mu)$; in the form started for $d = 1$ in [DJ07, HJW19, HJW18a, HJW18b].)

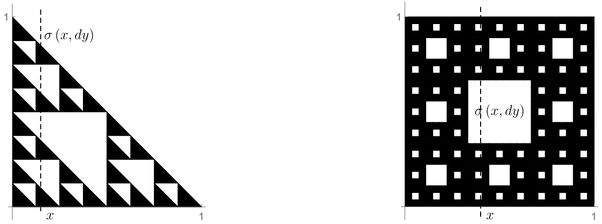
Definition 2. A Borel measure μ on $J^2 := [0,1] \times [0,1]$ is called *slice singular* iff (Def.)

1. $\xi = \mu \circ \pi_1^{-1}$ is singular, where π_1 is the projection onto the first coordinate; and
2. for a.a. x w.r.t. ξ, the measure $\sigma^x(\cdot)$ is singular.

"Singular" is defined relative to Lebesgue measure.

Theorem 3. If μ is slice singular on J^2, then $\{e_n\}_{n\in\mathbb{N}_0^2}$ has dense span in $L^2(\mu)$, where $e_n(x) = e^{i2\pi(n_1x_1+n_2x_2)}$, for all $n = (n_1, n_2) \in \mathbb{N}_0^2$, and $x = (x_1, x_2) \in J^2$.

Example 4 ($d = 2$). $\mu \in \mathcal{M}^+(\mathbb{T}^2)$, W = Sierpinski gasket. Note that, for a.a. x w.r.t. ξ, the measure σ^x on $A(x) = \{y \mid (x, y) \in W\}$ is a fractal measure with variable gap size, for a.a. x, $\sigma^x(dy)$ is singular relative to the Lebesgue measure. Hence we can apply F&M Riesz.



(A) Sierpinski gasket (B) Sierpinski carpet

Figure 1: Examples of slice singular measures.

- While the question of deciding which measures with support in $\mathbb{R}^d$, $d > 1$, are slice singular is quite natural in the present context, very little seems to be known about clean answers: i.e., deciding which measures (in higher dimension) have the property, and which do not. Even when one specializes to the case of standard planar IFS measures with gaps, there seems to be no easy approach to deciding the question; slice singular or not. Hence, at the present stage, the best approach seems to be a case by case study.

2. Frames, projections, and Kaczmarz algorithms

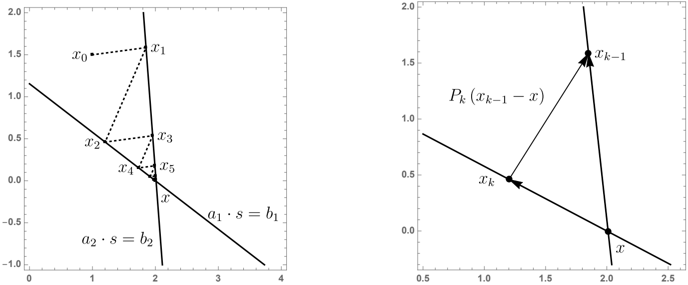
Classical Kaczmarz algorithm

- An iterative method for solving systems of linear equations, for example, $Ax = b$, where A is an $m \times n$ matrix.
- Let x_0 be an arbitrary vector in $\mathbb{R}^n$, and set

$$x_k := argmin_{\langle a_j, x \rangle = b_j} \|x - x_{k-1}\|^2, \; k \in \mathbb{N}. \tag{1}$$

- At each iteration, the minimizer is given by

$$x_k = x_{k-1} + \frac{b_j - \langle a_j, x_{k-1} \rangle}{\|a_j\|^2} a_j. \tag{2}$$



(A) Approximate solution; random starting point x_0

(B) orthogonality relation $\|x_{k-1} - x\|^2 = \|x_{k-1} - x_k\|^2 + \|x_k - x\|^2$

Figure 2: Solution to $Ax = b$ by the Kaczmarz algorithm.

Theorem 5. Let $\{P_j\}_{j\in\mathbb{N}_0}$ be a system of selfadjoint projections in a Hilbert space $\mathscr{H}$. For all $n \in \mathbb{N}_0$, set

$$T_n = (1 - P_n)(1 - P_{n-1}) \cdots (1 - P_0), \text{ and} \tag{3}$$

$$Q_n = P_n (1 - P_{n-1}) \cdots (1 - P_0), \quad Q_0 = P_0. \tag{4}$$

Then,

$$1 - T_n^* T_n = \sum_{j=0}^{n} Q_j^* Q_j, \text{ and} \tag{5}$$

$$1 - T_n = \sum_{j=0}^{n} Q_j. \tag{6}$$

Corollary 6. The following are equivalent:

1. $1 = \sum_{j\in\mathbb{N}_0} Q_j^* Q_j$ in the weak operator topology.
2. $1 = \sum_{j\in\mathbb{N}_0} Q_j$ in the strong operator topology.
3. $T_n \to 0$ in the strong operator topology.

Remark 7. Under suitable conditions on Q_n one can show that the convergence in part (1) of the corollary also holds in the strong operator topology. (See [JST20].)

Definition 8. The system $\{P_j\}_{j\in\mathbb{N}_0}$ is called *effective* if $T_n \to 0$ in the strong operator topology.

Corollary 9. Suppose the system $\{P_j\}_{j\in\mathbb{N}_0}$ is effective. Then, for all $x \in \mathscr{H}$,

$$x = \sum_{j\in\mathbb{N}_0} Q_j x. \tag{7}$$

Moreover, for all $x, y \in \mathscr{H}$,

$$\langle x, y\rangle = \sum_{j\in\mathbb{N}_0} \langle Q_j x, Q_j y\rangle\,; \tag{8}$$

and in particular,

$$\|x\|^2 = \sum_{j\in\mathbb{N}_0} \|Q_j x\|^2\,. \tag{9}$$

The case of rank-1 projections

- Let $\{P_j\}_{j\in\mathbb{N}_0}$ be a system of rank-1 projections, i.e.,

$$P_j = |e_j\rangle\langle e_j|$$

where $\{e_j\}_{j\in\mathbb{N}_0}$ is a set of unit vectors in $\mathscr{H}$.

- It follows that every Q_j is a rank-1 operator with range in $span\,\{e_j\}$. Thus there exists a unique $g_j \in \mathscr{H}$ such that

$$Q_j = |e_j\rangle\langle g_j|\,,\ j \in \mathbb{N}_0. \tag{10}$$

Lemma 10. Given $\{P_j\}_{j\in\mathbb{N}_0}$ a sequence of s.a. projections in $\mathscr{H}$; set

$$Q_n := P_n P_{n-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp}, \tag{11}$$

where $P_j^{\perp} := 1 - P_j$; then

$$Q_n = P_n \left(1 - \sum\nolimits_{j=0}^{n-1} Q_j\right). \tag{12}$$

Corollary 11. The vectors $\{g_j\}$ in (10) are determined recursively by

$$g_0 = e_0 \tag{13}$$

$$g_n = e_n - \sum_{j=0}^{n-1} \langle e_j, e_n \rangle \, g_j. \tag{14}$$

Corollary 12. Assume $\{|e_j\rangle\langle e_j|\}_{j\in\mathbb{N}_0}$ is effective, and let $Q_j = |e_j\rangle\langle g_j|$ be as above. Then, for all $x \in \mathscr{H}$, we have

$$x = \sum_{j\in\mathbb{N}_0} \langle g_j, x\rangle\, e_j. \tag{15}$$

In particular, for all $A \in \mathscr{B}(\mathscr{H})$, then

$$Ax = \sum_{j\in\mathbb{N}_0} \langle A^* g_j, x\rangle\, e_j. \tag{16}$$

Moreover, for all $x, y \in \mathscr{H}$,

$$\langle x, y\rangle = \sum_{j\in\mathbb{N}_0} \langle x, g_j\rangle \langle g_j, y\rangle\,, \text{ and}$$

$$\|x\|^2 = \sum_{j\in\mathbb{N}_0} |\langle g_j, x\rangle|^2\,.$$

Corollary 13. The system $\{|e_j\rangle\langle e_j|\}_{j\in\mathbb{N}_0}$ is effective iff $\{g_j\}_{j\in\mathbb{N}_0}$ is a Parseval frame in $\mathscr{H}$.

Remark 14. We note that when μ is slice singular, then the Fourier frequencies $\{e_n\}_{n\in\mathbb{N}_0}$ is effective in $L^2(\mu)$, and every $f \in L^2(\mu)$ has Fourier series expansion.

Random Kaczmarz constructions

- We say $\xi : \Omega \to \mathscr{B}(\mathscr{H})$ is a random variable iff (Def.) for all pairs of vectors $x, y \in \mathscr{H}$, then the functions

$$\Omega \longrightarrow \mathbb{C}, \quad \omega \longmapsto \langle x, \xi(\omega)\, y\rangle_{\mathscr{H}} \tag{17}$$

  are measurable w.r.t. the standard Borel σ-algebra $\mathscr{B}_{\mathbb{C}}$ of subsets of $\mathbb{C}$.
- Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we shall denote the corresponding expectation $\mathbb{E}$, i.e.,

$$\mathbb{E}(\cdots) \overset{\text{Def.}}{=} \int_{\Omega} (\cdots)\, d\mathbb{P}. \tag{18}$$

- Let $\mathscr{H}$ be a Hilbert space. Given a family of selfadjoint projections $\{P_j\}_{j\in\mathbb{N}_0}$ in $\mathscr{H}$, let $\xi : \Omega \to \mathscr{B}(\mathscr{H})$ be a random variable, such that

$$\mathbb{P}(\xi = P_j) = p_j, \; j \in \mathbb{N}_0, \tag{19}$$

where $p_j > 0$, and $\sum_{j\in\mathbb{N}_0} p_j = 1$.

- Suppose further that there exists a constant C, $0 < C < 1$, such that

$$\mathbb{E}\left[\|\xi x\|^2\right] := \sum_{j\in\mathbb{N}_0} p_j \|P_j x\|^2 \geq C\|x\|^2, \; \forall x \in \mathscr{H}. \tag{20}$$

Theorem 15. Let $\{\xi_j\}_{j\in\mathbb{N}_0}$ be an i.i.d. realization of ξ from (19). Fix $\xi_0 = P_0$, and set

$$T_n = (1-\xi_n)(1-\xi_{n-1})\cdots(1-\xi_0), \text{ and} \tag{21}$$

$$Q_n = \xi_n(1-\xi_{n-1})\cdots(1-\xi_0), \ Q_0 = \xi_0. \tag{22}$$

Note that each product in (21) and (22) is an operator-valued random variable.

Then, for all $x \in \mathscr{H}$, we have:

$$\lim_{n\to\infty} \mathbb{E}\left[\|T_n x\|^2\right] = 0. \tag{23}$$

Corollary 16. Let T_n and Q_n be as in (21)–(22), then the following hold.

1. For all $x \in \mathscr{H}$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left\|x - \sum\nolimits_{j=0}^{n} Q_j x\right\|^2\right] = 0. \tag{24}$$

2. For all $x, y \in \mathscr{H}$,

$$\lim_{n\to\infty} \mathbb{E}\left[\left|\langle x, y\rangle - \sum\nolimits_{j=0}^{n} \left\langle x, Q_j^* Q_j y\right\rangle\right|^2\right] = 0. \tag{25}$$

Remark 17 (Fusion frames, and measure frames). Our present equation (20) may be viewed as an instance of what is now called *fusion frames*, and developed extensively by Casazza et al. and by others. In addition, we note that our present result (25) is closely related to a formulation of a certain notion of *measure frames*.

Theorem 18. Let the setting be as above, but assume $\xi : \Omega \to \mathscr{B}(\mathscr{H})$ is a random **positive contraction**; i.e., $\mathbb{P}(\xi = P_j) = p_j$, where

$$0 \leq P_j \leq 1, \quad j \in \mathbb{N}_0.$$

Then, the conclusions in Theorem 15 and Corollary 16 hold.

3. System of isometries

Lemma 19. Fix $d > 1$, and let $\mathbb{D}^d$ be the polydisk. Let $H_2\left(\mathbb{D}^d\right)$ be the corresponding Hardy space. Let μ be a Borel probability measure on $\mathbb{T}^d \simeq [0,1]^d$. Then there is a bijective correspondence between:

1. isometries $V : L^2(\mu) \to H_2\left(\mathbb{D}^d\right)$; and
2. Parseval frames $\{g_n\}$ in $L^2(\mu)$.

The correspondence is as follows:

(1)→(2). Given V, isometric; set $g_n := V^*(z^n)$, where $n \in \mathbb{N}_0^d$.
(2)→(1). Given $\{g_n\}$ a fixed Parseval frame in $L^2(\mu)$, set

$$(Vf)(z) = \sum_{n \in \mathbb{N}_0^d} \langle g_n, f \rangle_{L^2(\mu)} z^n, \; z \in \mathbb{D}^d.$$

Definition 20. Fix $d > 1$. For all $x \in \mathbb{T}^d$, and all $z \in \mathbb{D}^d$, let

$$K^* (z, x) = \prod_{j=1}^{d} \frac{1}{1 - z_j \overline{e (x_j)}}. \tag{26}$$

Let $\mu \in \mathcal{M} \left(\mathbb{T}^d\right)$, and set

$$\begin{aligned} (C_\mu f) (z) &= \int_{\mathbb{T}^d} f (x) K^* (z, x) \, d\mu (x) \\ &= \sum_{n \in \mathbb{N}^d} \widehat{f d\mu} (n) z^n. \end{aligned} \tag{27}$$

- In particular,

$$\left(C_{\mu}1\right)(z)=\sum_{n\in\mathbb{N}_{0}^{d}}\widehat{\mu}(n)\,z^{n}, \tag{28}$$

  where $\widehat{\mu}(n)=\int_{\mathbb{T}^{d}}\overline{e_{n}(x)}d\mu(x)$, $n\in\mathbb{N}_{0}^{d}$.
- Let $L^{2}(\mu)\left(=L^{2}\left(\mathbb{T}^{2},\mu\right)\right)$ be as above, where $\mu\in\mathcal{M}^{+}\left(\mathbb{T}^{2}\right)$, $\xi=\mu\circ\pi_{1}^{-1}$, and μ assumes a disintegration

$$d\mu=\int\sigma^{x}(dy)\,d\xi(x)$$

Theorem 21 (see e.g., [Sar94, BS13]). Assume μ is slice singular. There are then two associated isometries:

$$L^2(\xi) \xrightarrow{V_\xi} H_2(\mathbb{D}), \quad (V_\xi f)(z) = \frac{(C_\xi f)(z)}{(C_\xi 1)(z)}, \tag{29}$$

and

$$L^2(\sigma^x) \xrightarrow{V_{\sigma^x}} H_2(\mathbb{D}), \quad (V_{\sigma^x} f)(z) = \frac{(C_{\sigma^x} f)(z)}{(C_{\sigma^x} 1)(z)}. \tag{30}$$

Corollary 22. The mapping

$$V_\mu : L^2(\mu) \longrightarrow H_2\left(\mathbb{D}^2\right) \left(= H_2(\mathbb{D}) \otimes H_2(\mathbb{D})\right)$$

given by

$$\left(V_\mu F\right)(z_1, z_2) = V_\xi\left(\left(V_{\sigma^x(\cdot)} F(x, \cdot)\right)(z_2)\right)(z_1) \tag{31}$$

is isometric. It follows that $\left\{g_n := V_\mu^*(z^n)\right\}_{n \in \mathbb{N}_0^2}$ is a Parseval frame in $L^2(\mu)$.

Remark 23. From the above discussion, we see that if $V : L^2(\mu) \to H_2(\mathbb{D}^2)$ is an isometry, then $\{g_n := V^*(z^n)\}_{n \in \mathbb{N}_0^2}$ is a Parseval frame in $L^2(\mu)$. This implication holds in general. Since there are “many” such isometries, it follows that there are “many” Parseval frames. For more details, see [HJW19, HJW18a, HJW18b] and the reference therein.

4. General iterated function system (IFS)-theory

- Let (M, d) be a complete metric space. Fix an alphabet $B = \{b_1, \cdots, b_N\}$, $N \geq 2$, and let $\{\tau_b\}_{b \in B}$ be a contractive IFS with attractor $W \subset M$, i.e.,

$$W = \bigcup_b \tau_b(W). \tag{32}$$

  In fact, W is uniquely determined by (32).
- Let $\{p_b\}_{b \in B}$, $p_b > 0$, $\sum_{b \in B} p_b = 1$, be fixed. Set $\Omega = B^{\mathbb{N}}$, equipped with the product topology. Let

$$\mathbb{P} = \times_1^\infty p = \underbrace{p \times p \times p \cdots \cdots}_{\aleph_0 \text{ product measure}} \tag{33}$$

  be the infinite-product measure on Ω (see [Kak43, Hid80]).

- We construct a random variable $X : \Omega \to M$ with value in M (a measure space $(M, \mathscr{B}_M)$), such that the distribution $\mu := \mathbb{P} \circ X^{-1}$ is a Borel probability measure supported on W, satisfying

$$\mu = \sum_{b \in B} p_b \, \mu \circ \tau_b^{-1}. \tag{34}$$

That is, μ is the IFS measure.

Theorem 24. For points $\omega = (b_{i_1}, b_{i_2}, b_{i_3}, \cdots) \in \Omega$ and $k \in \mathbb{N}$, set

$$\omega\big|_k = (b_{i_1}, b_{i_2}, \cdots, b_{i_k}), \text{ and} \tag{35}$$

$$\tau_{\omega|_k} = \tau_{b_{i_k}} \circ \cdots \circ \tau_{b_{i_2}} \circ \tau_{b_{i_1}}. \tag{36}$$

Then $\bigcap_{k=1}^{\infty} \tau_{\omega|_k}(M)$ is a singleton, say $\{x(\omega)\}$. Set $X(\omega) = x(\omega)$, i.e.,

$$\{X(\omega)\} = \bigcap_{k=1}^{\infty} \tau_{\omega|_k}(M); \tag{37}$$

then:

- $X : \Omega \to M$ is an (M, d)-valued random variable.
- The distribution of X, i.e., the measure

$$\mu = \mathbb{P} \circ X^{-1} \tag{38}$$

is the unique Borel probability measure on (M, d) satisfying:

$$\mu = \sum_{b \in B} p_b \, \mu \circ \tau_b^{-1}; \tag{39}$$

equivalently,

$$\int_M f \, d\mu = \sum_{b \in B} p_b \int_M (f \circ \tau_b) \, d\mu, \tag{40}$$

holds for all Borel functions f on M.

- The support $W_\mu = supp\,(\mu)$ is the minimal closed set (IFS), $\neq \emptyset$, satisfying

$$W_\mu = \bigcup_{b \in B} \tau_b\,(W_\mu)\,. \tag{41}$$

In general, the random variable $X : \Omega \to W$ (see (37)) is not 1-1, but it is always onto. It is 1-1 when the IFS is non-overlap; see Definition 25 below.

Definition 25. We say that (τ_b, W) is "non-overlap" iff for all $b, b' \in B$, with $b \neq b'$, we have $\tau_b(W) \cap \tau_{b'}(W) = \emptyset$.

Corollary 26. Assume $p \neq p'$, i.e., $p_b \neq p'_b$, for some $b \in B$. (Recall that $\sum_{b \in B} p_b = \sum_{b \in B} p'_b = 1$, p_b, $p'_b > 0$.) Let $\mathbb{P} = \times_1^\infty p$, and $\mathbb{P}' = \times_1^\infty p'$ be the corresponding infinite product measures; and let $\mu = \mathbb{P} \circ X^{-1}$, $\mu' = \mathbb{P}' \circ X^{-1}$ be the respective distributions. Then μ and μ' are mutually singular.

5. Sierpinski and random power series

- Given a system of contractive mappings, affine or conformal, there are then two associated fixed-point problems, one for compact sets, and the other for probability measures: The case of the sets W is discussed in (41), and the measures μ in (39). For a fixed IFS, the set in question arises as the support of an associated IFS-measure μ.
- Probabilistic features of these constructions are outlined. In particular, we show that these planar Sierpinski measures μ are slice-singular.

- Given a probability measure μ on I^d where $I = [0, 1]$, a key property that μ may, or may not, have is that the Fourier frequencies $\{e_n\}_{n \in \mathbb{N}^d}$ are *total* in $L^2(\mu)$, i.e., that the closed span of $\{e_n\}_{n \in \mathbb{N}^d}$ is $L^2(\mu)$.
- The result in $d = 1$, that, if ν on I is singular, then the set $\{e_n\}_{n \in \mathbb{N}_0}$ is total in $L^2(\nu)$, fails for $d = 2$. There are examples when μ on I^2 is positive, singular w.r.t. the 2D Lebesgue measure, but $\{e_n\}_{n \in \mathbb{N}_0^2}$ is *not* total in $L^2(\mu)$.

Example 27. Take $\mu = \lambda_1 \times \nu$ (see Figure 3), where λ_1 is Lebesgue measure and ν is a singular measure in I, then $\{e_n\}_{n \in \mathbb{N}_0^2}$ is *not* total in $L^2(\mu)$.

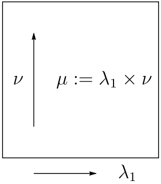


Figure 3: $\lambda_1 =$ Lebesgue, $\nu \perp \lambda_1$

For the Sierpinski case (affine IFS), with the Sierpinski measure μ, we shall show below that $\{e_n\}_{n\in\mathbb{N}_0^2}$ is indeed total in $L^2(\mu)$. Let the alphabets be

$$B = \{b_0, b_1, b_2\} := \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \tag{42}$$

Set

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \text{ and } \quad \tau_j(x) = M^{-1}(x + b_j). \tag{43}$$

The Sierpinski gasket (Figure 4) is the IFS attractor W satisfying

$$W = \bigcup_{j=0}^{2} \tau_j(W).$$

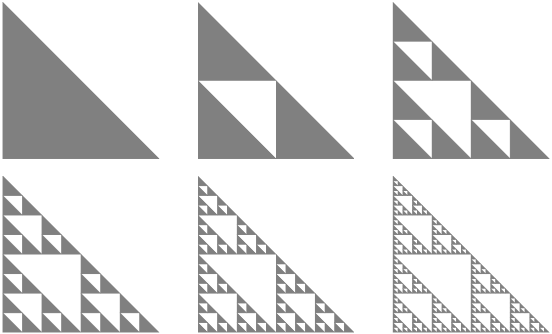


Figure 4: Construction of the Sierpinski gasket.

We have the random variable $B^{\mathbb{N}} \xrightarrow{X} W$, given by

$$\omega = (b_{i_1}, b_{i_2}, b_{i_3}, \cdots) \longmapsto x = \sum_{k=1}^{\infty} M^{-k} b_{i_k}. \tag{44}$$

As a Cantor set, W (the Sierpinski gasket) is the boundary of the tree symbol representation; see Figure 5.

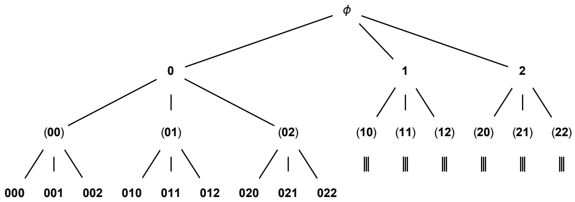


Figure 5: Symbol representations of infinite words.

- Recall that every $\omega \in B^{\mathbb{N}}$ is an infinite word $\omega = (b_{i_1}, b_{i_2}, b_{i_3}, \cdots)$, with $i_k \in \{0, 1, 2\}$. Setting $\omega\big|_n = (b_{i_1}, \cdots, b_{i_n})$, a finite truncated word, and $\tau_{\omega|_n} = \tau_{i_n} \circ \cdots \circ \tau_{i_1}$; then $\bigcap_n \tau_{\omega|_n}(W) = \{x\}$, i.e., the intersection is a singleton. We set $X(\omega) = x$.
- Let p be the probability distribution on B, where

$$p = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right). \tag{45}$$

  Let $\mathbb{P} = \times_1^{\infty} p$, and $\mu = \mathbb{P} \circ X^{-1}$ be the corresponding IFS measure, i.e., μ is the unique Borel probability measure on W, s.t.

$$d\mu = \frac{1}{3} \sum_{j=0}^{2} \mu \circ \tau_j^{-1}. \tag{46}$$

Lemma 28. Let W be the Sierpinski gasket, and μ be the corresponding IFS measure. Let $\widehat{\mu}$ be the Fourier transform of μ, i.e., $\widehat{\mu}(\lambda) := \int_W e^{i2\pi\lambda\cdot x} d\mu(x)$. Then

$$\widehat{\mu}(\lambda) = \frac{1}{3}\left[1 + e^{i\pi\lambda_1} + e^{i\pi\lambda_2}\right]\widehat{\mu}(\lambda/2), \tag{47}$$

where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Lemma 29. Let W be the Sierpinski gasket. Then points in W are represented as random power series

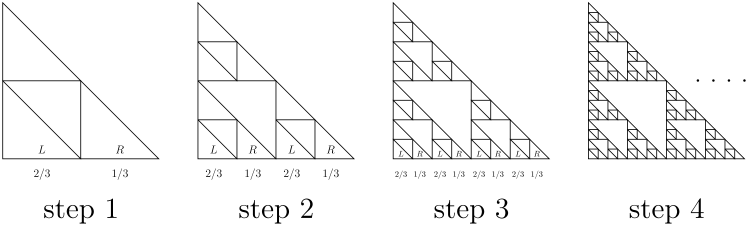
$$\begin{bmatrix} x \\ y \end{bmatrix} \in W \Longleftrightarrow \begin{cases} x = \sum_{k=1}^{\infty} \varepsilon_k 2^{-k} \\ y = \sum_{k=1}^{\infty} \eta_k 2^{-k} \end{cases} \tag{48}$$

where $(\varepsilon_k), (\eta_k)$ are defined on $\Omega = \{0,1\}^{\mathbb{N}}$, i.e., the binary probability space.

Moreover, ε_k is i.i.d. on $\{0,1\}$, $k \in \mathbb{N}$, with distribution $(2/3, 1/3)$. That is, $Prob\,(\varepsilon_k = 0) = 2/3$, and $Prob\,(\varepsilon_k = 1) = 1/3$. The same conclusion holds for η_k as well.

Lemma 30. Let μ be the IFS measure of the Sierpinski gasket as above, and $\xi = \mu \circ \pi_1^{-1}$, so that μ has the disintegration.

1. Then the measure ξ is singular and non-atomic. More precisely, ξ is the product measure $\bigtimes_1^\infty \{2/3, 1/3\}$ defined on $\{0,1\}^{\mathbb{N}}$.
2. For a.a. x w.r.t ξ, the measure $\sigma^x(dy)$ (in the y-variable) is singular. Hence μ is slice singular, and $\{e_n\}_{n\in\mathbb{N}_0^2}$ is total in $L^2(\mu)$.



$$\xi = \mu \circ \pi_1^{-1} = \bigtimes_1^{\infty} \{2/3, 1/3\}$$
$$Pr\,(\varepsilon_k \in L) = 2/3, \quad Pr\,(\varepsilon_k \in R) = 1/3$$

Figure 6: The measure ξ, or $d\xi\,(x)$ as an infinite product measure.

Transition probabilities

- There is a Markov chain associated with the transition probabilities:

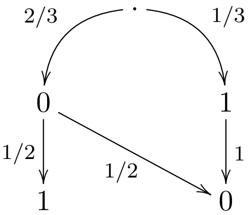


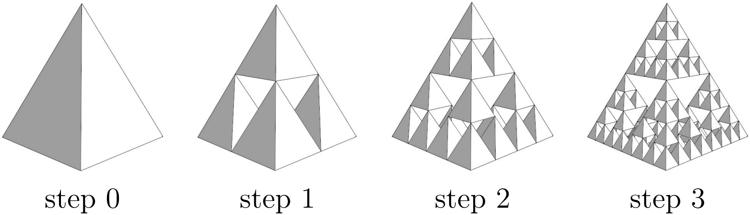
Figure 7: transition probabilities

- Note that

$$\begin{bmatrix} 2/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2/3 & 1/3 \end{bmatrix},$$

so the conditional expectation can be expressed as a Perron-Frobenius problem with the row vector $\begin{bmatrix} 2/3 & 1/3 \end{bmatrix}$ as a left Perron-Frobenius vector.

- As another example, consider the fractal Eiffel Tower W_{Ei} (see Figure 8).



$$\xi = \mu \circ \pi_1^{-1} = \times_1^{\infty} \{3/4, 1/4\}$$
$$Pr(\varepsilon_k = 0) = 3/4, \quad Pr(\varepsilon_k = 1) = 1/4$$

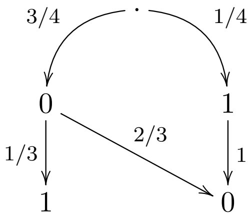
Figure 8: Construction of the fractal Eiffel Tower.

- In this case, we have

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \quad B = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\},$$

  and $p = (1/4, 1/4, 1/4, 1/4)$. It follows that each coordinate of points in W_{Ei} has representation $\sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$, where $\{\varepsilon_k\}$ is i.i.d. with $Pr\,(\varepsilon_k = 0) = 3/4$, and $Pr\,(\varepsilon_k = 1) = 1/4$.

- The transition probabilities are given by the diagram below.



One checks that

$$\begin{bmatrix} 3/4 & 1/4 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 \end{bmatrix}.$$

Conjecture 31. Given an affine contractive IFS measure μ supported in $[0,1]^d$, let $T = (T_{ij})$ be the corresponding Markov transition matrix. Then the following are equivalent:

1. The Fourier frequencies $\{e_n\}_{n \in \mathbb{N}_0^d}$ are total in $L^2(\mu)$.
2. The Perron-Frobenius vector v ($vT = v$, or $\sum_j v_j T_{ji} = v_i$) is non-constant, i.e., not proportional to $(1, 1, \cdots, 1)$.

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