# Exploring dimensions of infinitely generated self-affine planar sets

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## Overview

- Act 1 Finitely generated self-affine sets
- Act 2 Infinitely generated self-affine sets
- Act 3 Dimensionality results for infinitely generated non-irreducible planar self-affine sets

## Act 1 Finitely generated self-affine sets

Let $X \subset \mathbb{R}^d$ be compact and let $I$ be a finite index set. An iterated function system (IFS) on $X$ is a collection of $\{\phi_i : X \to X : i \in I\}$ of contractions.

**Theorem** [Hutchinson. *Indiana Univ. Math. J.* 1981]

There exists a unique compact non-empty set $F$ such that

$$F = \bigcup_{i \in I} \phi_i(F).$$

- F is self-similar, if $\phi_i$ is a similarity:

$$|\phi_i(x) - \phi_i(y)| = r_i|x - y|$$

- F is self-affine if $\phi_i$ is an affine transformation:

$$\phi_i(x) = L_i(x) + y_i$$

  where $L_i \in \mathrm{GL}_d(\mathbb{R})$ and $y_i \in \mathbb{R}^d$

An IFS satisfies the open set condition if there is a non-empty bounded open set $U$ such that

$$U \supseteq \bigcup_{i \in I} \phi_i(U)$$

with the union disjoint.

**Theorem** [Hutchinson. *Indiana Univ. Math. J.* 1981]

Let $F$ be a self-similar set and let $s$ be the unique solution to $\sum_{i \in I} r_i^s = 1$:

$$\dim_{\mathcal{H}}(F) \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F) \leq s$$

with equality under the open set condition.

- Letting $N_\delta(F)$ denote the smallest number of sets of diameter at most $\delta$ which cover $F$

$$\underline{\dim}_B(F) = \liminf_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)} \quad \text{and} \quad \overline{\dim}_B(F) = \limsup_{\delta \to 0} \frac{\log(N_\delta(F))}{-\log(\delta)}.$$

- $\dim_{\mathcal{H}}(F) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}$

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F) = \lim_{\delta \to 0} \inf \left\{ \sum_j |U_j|^s : \{U_j\} \text{ is a } \delta\text{-cover of } F \right\}$$

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**Theorem** [Falconer. *Math. Proc. Camb. Phil. Soc.* 1988] and [Solomyak. *Math. Proc. Camb. Phil. Soc.* 1998]

Let $F$ be a self-affine set and $s$ be the value

$$\inf \left\{ q \in \mathbb{R}_{>0} : \sum_{i_1 \ldots i_n \in I^*} \phi^q(L_{i_1} \ldots L_{i_n}) < \infty \right\}.$$

$$\dim_{\mathcal{H}}(F) \leq \overline{\dim}_B(F) \leq \min\{d, s\}$$

with equality holding for almost all translation vectors $y_i$, if $\|L_i\| < 1/2$ for all $i \in I$.

The value $s$ is called the affinity dimension of the IFS, and is denoted by $d(L_i \mid i \in I)$.

- For $L \in \mathrm{GL}_d(\mathbb{R})$, let $\alpha_1(L) \geq \cdots \geq \alpha_n(L)$ denote the singular values of $L$, and define the singular value function of $L$ by

$$\phi^r(L) = \begin{cases} \alpha_1(L) \ldots \alpha_{\lceil r \rceil - 1}(L)(\alpha_{\lceil r \rceil}(L))^{r - \lceil r \rceil + 1} & \text{if } 0 < r \leq d, \\ |\det(L)|^{r/2} & \text{if } r > d, \end{cases}$$

- When $d = 2$ we have

$$\phi^r(L) = \begin{cases} \alpha_1(L)^r & \text{if } 0 < r \leq 1, \\ \alpha_1(L)\alpha_2(L)^{r-1} & \text{if } 1 < r \leq 2, \\ |\det(L)|^{r/2} & \text{if } r > 2, \end{cases}$$

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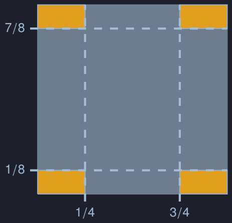
Let $X \subset \mathbb{R}^d$ be compact and let $I$ be a finite index set. An iterated function system (IFS) on $X$ is a collection of $\{\phi_i : X \to X : i \in I\}$ of contractions.

### Example




$$\dim_{\mathcal{H}}(F) = \dim_B(F) = 3/4 < 1 = d(L_i \mid i \in I)$$

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Let $F$ be a self-affine set and $s$ be the value

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**Question** Can one classify classes of self-affine sets for which Hausdorff and affinity dimensions coincide?

Recall, $I$ is a finite index set.

**Theorem** [Bárány, Hochmann and Rapaport. *Invent. math.* 2019]
Let $F \subset \mathbb{R}^2$ be a self-affine set generated by a finite iterated function system $\Phi = \{\phi_i : i \in I\}$ and let $L_i$ denote the linear part of $\phi_i$. If

- $\{L_i : i \in I\} \subset \mathrm{GL}_2(\mathbb{R})$ generate a non-compact and totally irreducible group, and
- if $\Phi$ satisfies the open set condition with feasible open set $U$ such that $U \cap F \neq \emptyset$

then $\dim_{\mathcal{H}}(F) = \dim_B(F) = \min\{2, d(L_i \mid i \in I)\}$.

**Theorem** [Rapaport. *arXiv:2309.03985*, 2023]
Let $F \subset \mathbb{R}^d$ be a self-affine set generated by a finite iterated function system $\Phi = \{\phi_i : i \in I\}$ and let $L_i$ denote the linear part of $\phi_i$. If

- $L_i = \mathrm{diag}(r_{i,1}, r_{i,2}, \ldots, r_{i,d})$ for all $i \in I$ and not all similarities and
- $\Phi_j = \{\phi_{i,j} : i \in I\}$ is exponentially separated, where $\phi_{i,j}(x) = r_{i,j} + y_{i,j}$,

then $\dim_{\mathcal{H}}(F) = \dim_B(F) = \min\{d, d(L_i \mid i \in I)\}$.

- Non-compact means that not all of the maps $\phi_i$ are similarities. Under this assumption, total irreducibility is equivalent to the property that no line, or union of two lines, is invariant under all of the $L_i$.
- A finite affine IFS $\Psi = \{\psi : i \in I\}$ on $X \subset \mathbb{R}$ is said to be exponentially separated if there exist a constant $c > 0$ and an infinite set $Q \subset \mathbb{N}$ such that $\rho(\psi_{\omega_1} \ldots \psi_{\omega_n}, \psi_{\nu_1} \ldots \psi_{\nu_n}) \geq c^n$ for all $n \in Q$ and distinct $\omega_1 \ldots \omega_n, \nu_1 \ldots \nu_n \in I^n$, where, for two affine maps $\tau_1(x) = r_1 x + c_1$ and $\tau_2(x) = r_2 x + c_2$,

$$\rho(\tau_1, \tau_2) = \begin{cases} \infty & \text{if } r_1 \neq r_2 \\ |c_1 - c_2| & \text{otherwise.} \end{cases}$$

## Act 2 Ininitely generated self-affine sets

Let $X \subset \mathbb{R}^d$ be compact and let $I$ be a countable infinite index set. An iterated function system (IFS) on $X$ is a collection of $\Phi = \{\phi_i : X \to X : i \in I\}$ of contractions.

We define the limit set of an infinite IFS by

$$F = \bigcup_{(i_k)_{k} \in I^{\mathbb{N}}} \bigcap_{k \in \mathbb{N}} \phi_{i_1} \dots \phi_{i_k}(X).$$

Observe that

- $\bigcup_{i \in I} \phi_i(F) = F$, and
- $F$ is not necessarily compact.

A limit set is called self-affine, if $\phi_i$ is an affine transformation for all $i \in I$, namely $\phi_i(x) = L_i(x) + y_i, \quad L_i \in \mathrm{GL}_d(\mathbb{R}), y_i \in \mathbb{R}^d$.

**Theorem** [Käenmäki and Reeve. *J. Frac. Geom.* 2014]

Let $F$ be a self-affine limit set. If $||L_i|| < 1/2$ for all $i \in I$. For almost all translation vectors,

$$\dim_{\mathcal{H}}(F) = \min\{d, d(L_i \mid i \in I)\}$$

The dimension spectrum of an infinite IFS $\Phi$ is $D(\Phi) = \{\dim_{\mathcal{H}}(F_J) : J \subset I \text{ is finite}\}$, where $F_J$ is the attractor of $\{\phi_j : j \in J\}$.

**Theorem** [Chousionis, Leykekhman and Urbański. *Selecta Math.* 2019]

The dimension spectrum of a conformal infinite IFS satisfying the open set condition and bounded distortion property is compact and perfect.

- For $L \in \mathrm{GL}_d(\mathbb{R})$, let $\alpha_1(L) \geq \dots \geq \alpha_n(L)$ denote the singular values of $L$, and define the singular value function of $L$ by

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The dimension spectrum of a conformal infinite IFS satisfying the open set condition and bounded distortion property is compact and perfect.

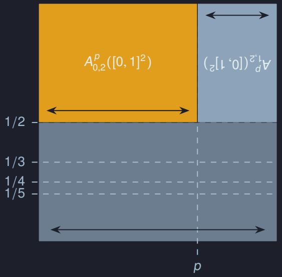
**Theorem** [Jurga. *Selecta Math.* 2021]

The dimension spectrum of an irreducible affine infinite IFS satisfying the strong open set condition, namely $U \cap F \neq \emptyset$, is compact and perfect.

- An IFS is irreducible, if the linear parts of the affine maps do not all preserve a common proper non-trivial linear subspace.

## Act 3 Dimensionality results for infinitely generated non-irreducible planar self-affine sets

Motivated by the question what happens if we have a non irreducible infinite affine IFS and the study of restricted digit sets of signed Lüroth expansions we studied the following class of IFSs.




Let $p \in (0,1)$ and $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 2}$

$$A^p_{s,d} : [0,1]^2 \to [0,1]^2$$

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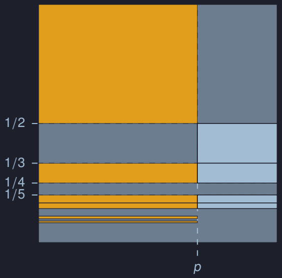


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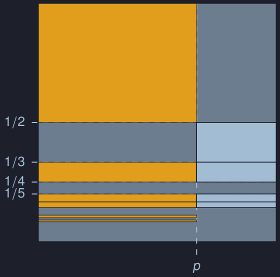


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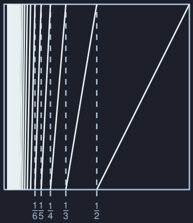
Connections to Lüroth expansions

- Lüroth expansions

$$\lim_{n\to\infty} h^0_{d_1} \ldots h^0_{d_n}(0) = \sum_{n\in\mathbb{N}} \frac{d_n - 1}{\prod_{i=1}^n d_i(d_i - 1)}$$




$$h^0_d(y) = \frac{y + d - 1}{d(d-1)} \qquad d \in \mathbb{N}_{\geq 2}$$

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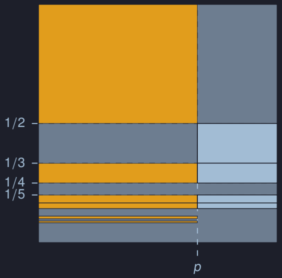


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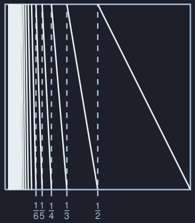


$$h^1_d(y) = \frac{-y+d}{d(d-1)} \qquad d \in \mathbb{N}_{\geq 2}$$

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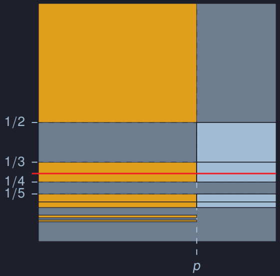


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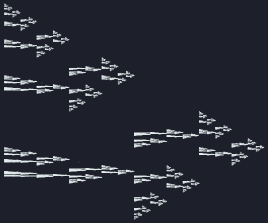
- Signed Lüroth expansions

$$\lim_{n\to\infty} h^{s_1}_{d_1} \dots h^{s_n}_{d_n}(0) = \sum_{n\in\mathbb{N}} (-1)^{\sum_{i=1}^n s_i} \frac{d_n - 1 + s_n}{\prod_{i=1}^n d_i(d_i - 1)}$$



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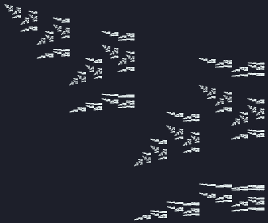

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$J = \{(0,2),(1,3),(0,4),(1,4),(0,6),(1,6),(0,7),(1,7),(0,10),(0,12)\} \quad p = 2/3$

**Theorem** [van Golden, Kalle, Kombrink and Samuel. *Nonlinearity* 2025]

- $\dim_{\mathcal{H}}(F^p_J) \le \min\left\{2, d(L^p_{s,d} \mid (s,d) \in J)\right\}$
- If there exists a non-empty finite alphabet $J_1 \subset J$ such that, for all $J_1 \subset J_2 \subset J$, $\dim_B(F^p_{J_2}) = d(A^p_{s,d} : (s,d) \in J_2)$, then

$$d(A^p_{s,d} : (s,d) \in J) \le \underline{\dim}_B(F^p_J) \text{ and } d(A^p_{s,d} : (s,d) \in J) = \sup_{J' \subset J \text{ finite}} d(A^p_{s,d} : (s,d) \in J').$$

**Examples for the lower bound on $\underline{\dim}_B(F^p_J)$**

- $\dim_B(\pi_1(F^p_{J_1})) = \dim_B(\pi_2(F^p_{J_1})) = 1$

  For example when $(0,2),(1,2) \in J_1$
- $\dim_B(\pi_1(F^p_{J_1})) = 1$ and $\min\{p, 1-p\} \ge (d-s)^{-1}$ for all $(s,d) \in J$.

  For example when $p = 1/2$, $\pi_1(J) = \{0,1\}$ and $\pi_2(J) \subset \mathbb{N}_{\ge 3}$

Recall $p \in (0,1)$ and for $(s,d) \in \{0,1\} \times \mathbb{N}_{\ge 2}$

$$A^p_{s,d}(x) = L^p_{s,d}(x) + \begin{pmatrix} sp \\ \frac{1}{d-s} \end{pmatrix} \quad \text{and} \quad L^p_{s,d}(x) = \begin{pmatrix} p^{1-s}(1-p)^s & 0 \\ 0 & \frac{(-1)^s}{d(d-1)} \end{pmatrix}$$

We set $J$ to be an infinite subset of $\{0,1\} \times \mathbb{N}_{\ge 2}$,

$$d(L^p_{s,d} \mid (s,d) \in J) = \inf\left\{q \in \mathbb{R}_{>0} : \sum_{(s_1,d_1)\dots(s_n,d_n) \in J^*} \phi^q(L^p_{s_1,d_1} \cdots L^p_{s_n,d_n}) < \infty\right\},$$

and $F^p_J$ denotes the limit set of $\{A^p_{s,d} : (s,d) \in J\}$.

For $(s,d) \in J$ let $a_{s,d} = p^{1-s}(1-p)^s$ and $b_{s,d} = (d(d-1))^{-1}$ and set

$$P_J(r) = \begin{cases} \max\left\{\sum_{(s,d)\in J} a_{s,d}^r, \sum_{(s,d)\in J} b_{s,d}^r\right\} & \text{if } 0 < r \leq 1 \\ \max\left\{\sum_{(s,d)\in J} a_{s,d} b_{s,d}^{r-1}, \sum_{(s,d)\in J} a_{s,d}^{r-1} b_{s,d}\right\} & \text{if } 1 < r \leq 2 \\ \sum_{(s,d)\in J} (a_{s,d} b_{s,d})^{r/2} & \text{if } r > 2 \end{cases}$$

**Theorem** [van Golden, Kalle, Kombrink and Samuel. *Nonlinearity* 2025]

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf\{r \in \mathbb{R}_{>0} : P_J(r) \leq 1\}$$

If $\pi_1(J) = \{0,1\}$, then

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf\left\{r \in [1,2] : \max\left\{\sum_{(s,d)\in J} a_{s,d} b_{s,d}^{r-1}, \sum_{(s,d)\in J} a_{s,d}^{r-1} b_{s,d}\right\} \leq 1\right\}$$

Recall $p \in (0,1)$ and for $(s,d) \in \{0,1\} \times \mathbb{N}_{\geq 2}$

$$A_{s,d}^p(x) = L_{s,d}^p(x) + \begin{pmatrix} sp \\ \frac{1}{d-s} \end{pmatrix} \quad \text{and} \quad L_{s,d}^p(x) = \begin{pmatrix} p^{1-s}(1-p)^s & 0 \\ 0 & \frac{(-1)^s}{d(d-1)} \end{pmatrix}$$

We set $J$ to be an infinite subset of $\{0,1\} \times \mathbb{N}_{\geq 2}$,

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf\left\{q \in \mathbb{R}_{>0} : \sum_{(s_1,d_1)\ldots(s_n,d_n)\in J^*} \phi^q(L_{s_1,d_1}^p \cdots L_{s_n,d_n}^p) < \infty\right\},$$

and $F_J^p$ denotes the limit set of $\{A_{s,d}^p : (s,d) \in J\}$.

For $(s,d) \in J$ let $a_{s,d} = p^{1-s}(1-p)^s$ and $b_{s,d} = (d(d-1))^{-1}$ and set

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Moreover, if (a) $\sum_{(s,d)\in J} b_{s,d} \leq 1$, (b) $J = \{0,1\} \times I$ for some $I \subset \mathbb{N}_{\geq 2}$, or (c) $p = 1/2$, then

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf\left\{r \in [1,2] : \sum_{(s,d)\in J} a_{s,d} b_{s,d}^{r-1} \leq 1\right\}$$

We set $J$ to be an infinite subset of $\{0,1\} \times \mathbb{N}_{\geq 2}$,

$$d(L_{s,d}^p \mid (s,d) \in J) = \inf\left\{q \in \mathbb{R}_{>0} : \sum_{(s_1,d_1)\dots(s_n,d_n)\in J^*} \phi^q(L_{s_1,d_1}^p \cdots L_{s_n,d_n}^p) < \infty\right\},$$

and $F_J^p$ denotes the limit set of $\{A_{s,d}^p : (s,d) \in J\}$.

**Theorem** [van Golden, Kalle, Kombrink and Samuel. *Nonlinearity* 2025]

- $\dim_{\mathcal{H}}(F_J^p) \le \min\left\{2, d(L^p_{s,d} \mid (s,d) \in J)\right\}$
- If there exists a non-empty finite alphabet $J_1 \subset J$ such that, for all $J_1 \subset J_2 \subset J$, $\dim_B(F^p_{J_2}) = d(A^p_{s,d} : (s,d) \in J_2)$, then

$$d(A^p_{s,d} : (s,d) \in J) \le \underline{\dim}_B(F_J^p) \text{ and } d(A^p_{s,d} : (s,d) \in J) = \sup_{J' \subset J \text{ finite}} d(A^p_{s,d} : (s,d) \in J').$$

**Theorem** [van Golden, Kalle, Kombrink and Samuel. *Nonlinearity* 2025]

If $J = \{0\} \times I_1 \cup \{1\} \times I_2$, for some $I_1, I_2 \subseteq \mathbb{N}_{\ge 2}$, then

$$\dim_{\mathcal{H}}(F_J^p) \ge 1 + \inf\left\{ r \in (0,1] : \left( \sum_{d_1 \in I_1} \frac{1}{(d_1(d_1-1))^r} \right)^p \left( \sum_{d_2 \in I_2} \frac{1}{(d_2(d_2-1))^r} \right)^{1-p} \le 1 \right\}.$$

Further, if $I_1 = I_2 = I$, then

$$\dim_{\mathcal{H}}(F_J^p) = d(L^p_{s,d} \mid (s,d) \in J) = 1 + \inf\left\{ r \in (0,1] : \sum_{d \in I} \frac{1}{(d(d-1))^r} \le 1 \right\}.$$

**Theorem** [van Golden, Kalle, Kombrink and Samuel. *Nonlinearity* 2025]

$$\{\dim_{\mathcal{H}}(F_J^p) : J \subseteq \{0,1\} \times \mathbb{N}_{\ge 2}\} = [0,2].$$