Counting functions of the Laplacian on domains with fractal boundary

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joint work with Lucas Schmidt

Fractals 8 – Cornell University

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$\Omega \subset \mathbb{R}^d$ non-empty, open, bounded $\quad (d \geq 1)$

$$\begin{cases} -\Delta u = \eta u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

$-\Delta := -\sum_{k=1}^{d} \frac{\partial^2}{\partial x_k^2}$ Laplacian

Solutions ordered:

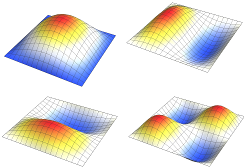
$$0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n \leq \cdots ,$$

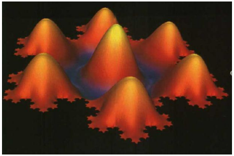
where $\eta_n \to \infty$ for $n \to \infty$.

Physical interpretation:

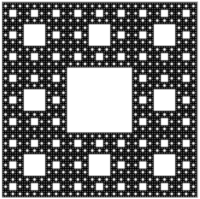
(1) wave equation of function periodic in time

η_k acoustic frequency of a wave mode.





(Lapidus)

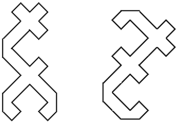


Can one hear the shape of a drum?

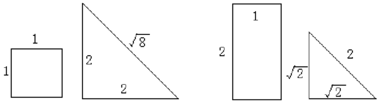
Inverse Problem – *Compute the drum given its spectrum*

Unsolvable for domains with piecewise smooth boundary:

(Gordon, Webb, Wolpert 1992)



(Buser, Conway, Doyle, Semmler 1994)



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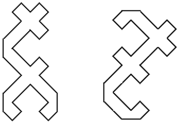

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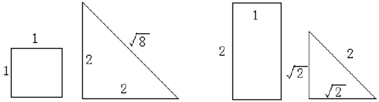
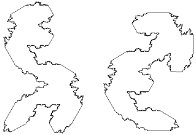
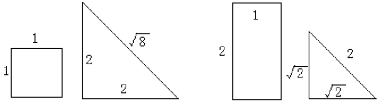


Unsolvable for domains with fractal boundary:

(Chen, Sleemann 2000)



(Lapidus)
Fractal sprays – countable unions of



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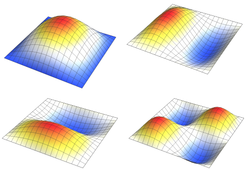

Theorem (Weyl 1912, Ivrii 1980)

If Ω has sufficiently smooth boundary, then as $\eta \to \infty$

$$N_\Omega(\eta) := \#\{\eta_n \leq \eta\} = c_d \mathrm{vol}_d(\Omega) \cdot \eta^{\frac{d}{2}} + c_{d-1} \mathrm{vol}_{d-1}(\partial\Omega) \cdot \eta^{\frac{d-1}{2}} + o(\eta^{\frac{d-1}{2}}).$$

Can one hear the shape of a drum?

Forward Problem – *Influence of a domain's geometry on its Laplace spectrum*

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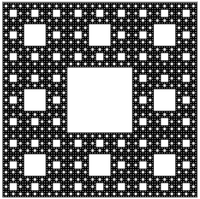
Solutions ordered:

$$0 < \eta_1 \leq \eta_2 \leq \cdots \leq \eta_n \leq \cdots,$$

where $\eta_n \to \infty$ for $n \to \infty$.

$\Omega = [0,1]^2 \setminus S$

$\partial\Omega = S$



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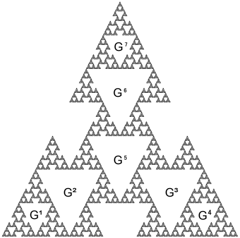
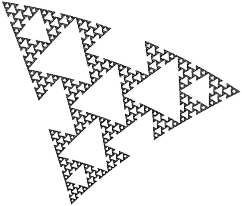
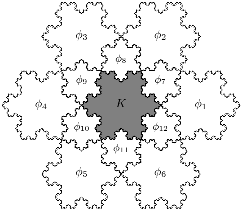
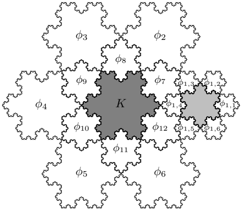
Domains of interest





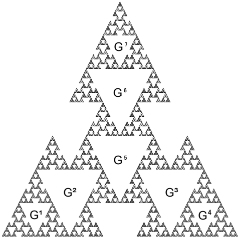


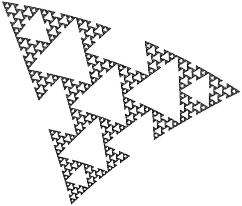


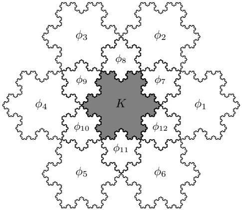
- Γ itself has fractal boundary
- Contraction ratios not all the same
- Conformal contractions
- Infinitely many contractions
- Not all concatenations allowed

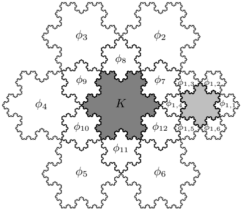
Influence of a domain's geometry on its Laplace spectrum

Domains of interest









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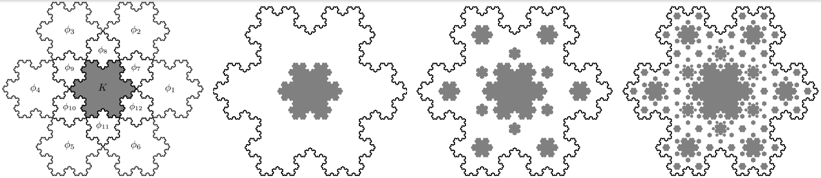
Fractal Sprays

Role of the generator



$$N_{\bigcup_\omega \phi_\omega K}(\eta) = \sum_\omega N_{\phi_\omega K}(\eta) = \sum_\omega N_K(r_\omega^2 \eta), \qquad r_\omega = |\phi'_\omega|$$

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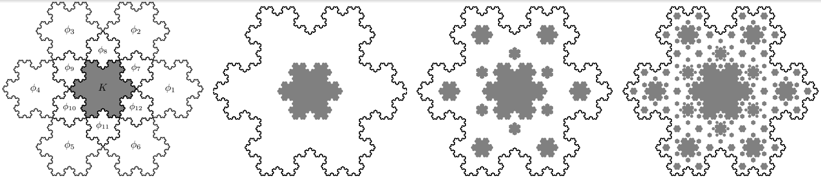

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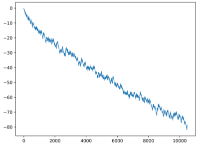
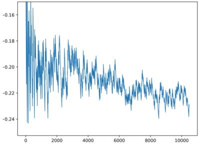
$\rightarrow$ Lapidus 1991: $\quad N_K(\eta) = c_2 \mathrm{vol}_2(K)\eta + \mathcal{O}(\eta^{\frac{\log 4}{2\log 3}})$

$\rightarrow$ Netrusov, Safarov 2005: boundary locally a graph

Strichartz, Wiese 2022:



$N_K(\eta)$
$-c_2\mathrm{vol}_2(K)\eta$



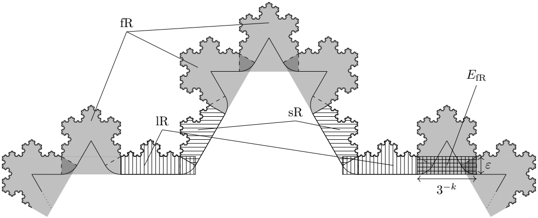
$$\frac{N_K(\eta) - c_2\mathrm{vol}_2(K)\eta}{\eta^{\frac{\log 4}{2\log 3}}}$$

Theorem (K., Schmidt 2024 & van den Berg, Lianantonakis 2001)

$\Omega \subseteq \mathbb{R}^d$ well-covered, each covering domain has well-behaved foliation, $\partial\Omega$ has upper inner Minkowski dim δ. $\exists C_\pm \in \mathbb{R}$: $\forall \eta \geq \eta_0$

$$C_-\eta^{\delta/2} \leq N_\Omega(\eta) - c_d \mathrm{vol}_d(\Omega)\eta^{\frac{d}{2}} \leq C_+\eta^{\delta/2}$$

well-covered:



- Cover $\{D_i^\epsilon \subset \Omega\}_{i\in I_\epsilon}$ of $\Omega_{-\epsilon}$
- $\#I_\epsilon \leq C(\Omega)\epsilon^{-\delta}$
- D_i^ϵ have uniformly comparable diamater
- Each D_i^ϵ well-foliated

The generator

Explicit error bounds

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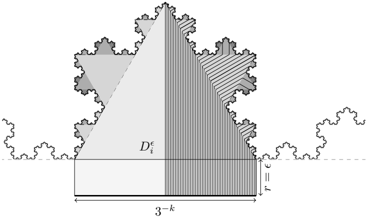
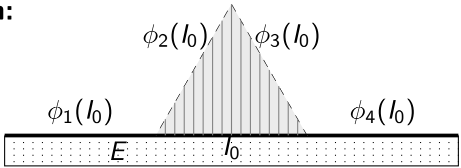
well-foliated:

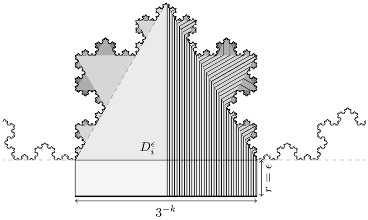


- parameterise fibers $\phi\colon (q,t) \mapsto \gamma_q(t)$
- $\beta := |\det D\phi|$
- $\operatorname{ess}\inf_{(q,t)\in D} \beta(q,t) > 0$
- $\sup_{q\in I_0} \int_{\gamma_q} \beta(q,t)\mathrm{d}t < \infty$

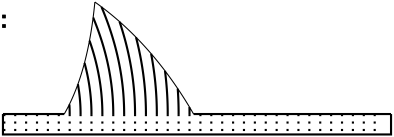
Seed-foliation:



Foliation:



Conformal Seed:



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- $N_\Omega^N(\eta) \leq \sum_i N_{D_i^\epsilon}^N(2\eta) + N^N(\mathrm{int} \bigcup_{Q \in \mathcal{W}_\epsilon} \overline{Q}, \eta)$

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- 2-term asymptotics for polygons $\rightarrow$

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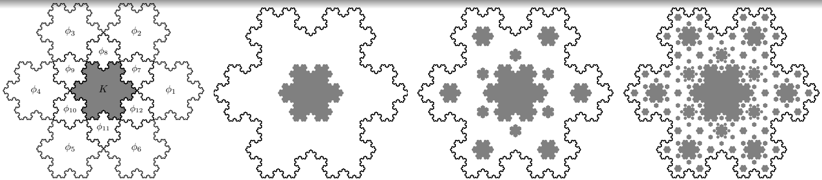
- foliation gives lower bound on η_2^N on D_i^ϵ $\Rightarrow N_{D_i^\epsilon}^N(\eta) = 1\ \forall \eta \leq \eta_0,\ \eta_0 \sim \epsilon^{-2}$
- 2-term asymptotics for polygons $\to$ $c_d \mathrm{vol}_d(\Omega)\eta^{d/2} + A_\Omega \epsilon^{d-1-\delta}\eta^{(d-1)/2}$
- For large η can choose ϵ: $\eta \approx \epsilon^{-2}$

Higher order asymptotics
Transforms



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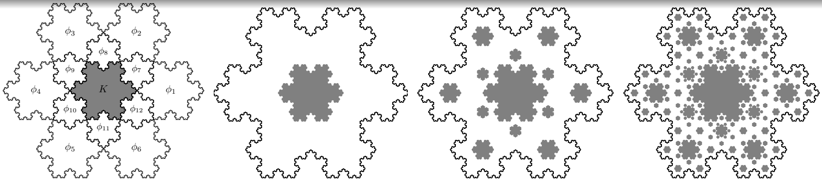

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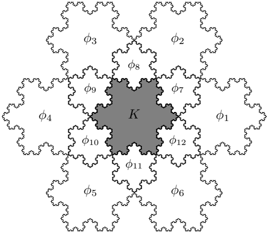
$\rightarrow$ Fourier-Laplace transform
$\rightarrow$ meromorphic extension
$\rightarrow$ poles, residues at poles
$\rightarrow$ recover information on the original problem

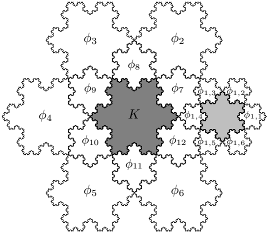
Influence of a domain's geometry on its Laplace spectrum

Steiner-Formulae & Weyl-Asymptotics



$$N_\Omega(\eta) = \sum_{z \in \widetilde{\mathcal{Z}}} \mathcal{W}(\eta)\eta^{z} + \mathcal{W}_2(\eta)\eta^{1} + \mathcal{O}(\eta^{\frac{\log 4}{2\log 3}})$$

$$\widetilde{\mathcal{Z}} = \left\{ \Re(z) < \frac{\log 4}{2\log 3} \mid 1 = \sum_{i \in \Sigma} r_i^{-2z} \right\}$$



$$\mathrm{vol}_2((\partial\Omega)_\varepsilon) = \sum_{z \in \mathcal{Z}} \mathcal{M}(\varepsilon)\varepsilon^{z} + \sum_{z \in \mathcal{S}} \mathcal{M}_2(\varepsilon)\varepsilon^{z}$$

$$\mathcal{Z} = \left\{ \Re(z) \mid 1 = \sum_{i \in \Sigma} r_i^{2-z} \right\}$$

$$\mathcal{S} = \left\{ 2, 2 - \frac{\log 4}{\log 3} \right\}$$