Spectral theory of Krein-Feller operators

Sze-Man Ngai

Beijing Institut of Mathematical Sciences and Applications

8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

June 17, 2025

Joint with Lei Ouyang and Wen-Quan Zhao.

Part 1. Laplacians defined by measures on $\mathbf{R}^n$: Krein-Feller operators

Basic Assumptions:

1. $\Omega \subseteq \mathbb{R}^n$, bounded open set.
2. $\mu =$ positive finite Borel measure on $\mathbb{R}^n$, $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$.
3. Poincaré inequality (PI): $\exists$ constant $C > 0$ such that $\forall u \in C_c^\infty(\Omega)$,

$$\int_\Omega |u|^2 \, d\mu \le C \int_\Omega |\nabla u|^2 \, dx.$$

$-\Delta_\mu$ is the nonnegative self-adjoint operator in $L^2(\Omega, \mu)$ associated with the closed quadratic form

$$\mathcal{E}(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx.$$

$-\Delta_\mu$ is call the *Krein-Feller operator* or *Dirichlet Laplacian defined by μ.*

Sufficient condition for (PI) and existence of o.n.b. of eigenfunctions

Based mainly on a result of Maz'ja, we obtained:

Theorem 1 (Hu-Lau-N., 2006)

Assume $\underline{\dim}_\infty(\mu) > n-2$.

(a) *(PI) holds.*

(b) $\exists$ *o.n.b.* $\{u_m\}$ *of* $L^2(\Omega, \mu)$ *consisting of eigenfunctions of* $-\Delta_\mu$.

(c) *The eigenvalues satisfy* $0 < \lambda_1 \le \lambda_2 \le \cdots$, *and if* $\dim(\operatorname{dom}\mathcal{E}) = \infty$, *then* $\lim_{m\to\infty} \lambda_m = \infty$.

$$\underline{\dim}_\infty(\mu) = \varliminf_{\delta \to 0^+} \frac{\ln\left(\sup_x \mu(B_\delta(x))\right)}{\ln \delta}.$$

Spectral dimension

The *eigenvalue counting function* for $-\Delta_\mu$ is defined as

$$N(\lambda, -\Delta_\mu) := \#\{k : \lambda_k \le \lambda\}.$$

Define *the spectral dimension* of μ as

$$d_s = d_s(-\Delta_\mu) := 2 \lim_{\lambda \to \infty} \frac{\ln N(\lambda, -\Delta_\mu)}{\ln \lambda},$$

if the limit exists.

Spectral dimension defined by the standard Cantor measure on $[0,1]$ is $\ln 4 / \ln 6 \approx 0.7737$.

Assume that an IFS of contractive similitudes on $\mathbb{R}^n$ satisfying (OSC) has contraction ratios $\{r_i\}_{i=1}^m$ and probability weights $\{p_i\}_{i=1}^m$. Naimark-M. Solomyak (1995) proved that the spectral dimension is given by the following formula:

$$\sum_{i=1}^m (p_i r_i^{2-n})^{d_s/2} = 1.$$

In particular, d_s is independent of the choice of Ω.

IFSs not satisfying (OSC) are said to have *overlaps*.

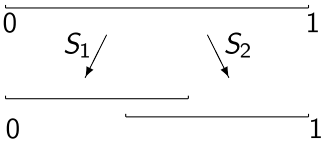
Example 2

$$S_1(x) = rx, \qquad S_2(x) = rx + 1 - r, \qquad 1/2 < r < 1.$$

Let μ_r be the self-similar measure satisfying

$$\mu_r = \frac{1}{2}\mu_r \circ S_1^{-1} + \frac{1}{2}\mu_r \circ S_2^{-1}.$$

μ_r is called an *infinite Bernoulli convolution*.



Theorem 3 (N., 2011)

Let μ be the infinite Bernoulli convolution associated with the golden ratio. Then $d_s(-\Delta_\mu) = 2\alpha \approx 0.998$, where α is the unique positive real number satisfying

$$\sum_{k=0}^{\infty} \sum_{J \in \{0,2\}^k} (\rho^{2k+3} c_J)^\alpha = 1.$$

Moreover, there exist constants $C_1, C_2 > 0$ such that $\forall\ \lambda$ sufficiently large,

$$C_1 \lambda^\alpha \leq N(\lambda, -\Delta_\mu) \leq C_2 \lambda^\alpha.$$

Similar results were obtained for convolutions of Cantor type measures.

Technique: Second-order identities and vector-valued renewal theorem by Lau-Wang-Chu, 1995.

Theorem 4

(Kesseböhmer-Niemann, 2022 arxiv). For any self-conformal measure on $\mathbb{R}^n$ (without any separation condition), spectral dimesion $d_s = 2q$, where q is the unique solution of

$$\tau(q) = (n-2)q.$$

Recall that the *L^q-spectrum* is defined as follows:

$$\tau(q) := \underline{\lim}_{\delta \to 0^+} \frac{\ln \sup \sum_i \mu(B_\delta(x_i))^q}{\ln \delta},$$

where $B_\delta(x_i)$ is a disjoint family of δ-balls with center $x_i \in \operatorname{supp}(\mu)$ and the supremum is taken over all such families.

Spectral dimension leads to heat kernel estimates

Theorem 5 (Gu-Hu-N., 2020)

For the infinite Bernoulli convolution associated with the golden ratio and a family of convolutions of Cantor measures, let $\beta := 2/d_s > 2$. *The heat kernel* $p_t(x,y)$ *satisfies the following sub-Gaussian estimate:*

$$p_t(x,y) \asymp \frac{1}{\mu(B_{d_*}(x, t^{1/\beta}))} \exp\left(-c\left(\frac{d_*(x,y)^{\beta/(\beta-1)}}{t^{1/(\beta-1)}}\right)\right),$$

$\forall t \in (0,1)$ *and all* $x, y \in \mathrm{supp}(\mu)$, *where* d_* *is some metric defined in terms of* μ.

Since $1/(\beta-1) < 1$, the heat kernel estimate is sub-Gaussian.

Sub-Gaussian heat kernel estimate leads to infinite wave propagation speed:

(1) Strichartz et al. (1999): conjectured that waves may propagate with infinite speed on certain fractals.

(2) Y.-T. Lee (2012): proved the conjecture for p.c.f. fractals.

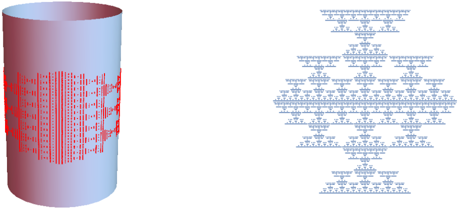
(3) N.-Tang-Xie (2020): proved a general form of Lee's theorem that includes IFSs with overlaps, such as the infinite Bernoulli convolution associated with the golden ratio or a family of convolutions of Cantor-type measures.

Part 3: Extend the theory of Krein-Feller operators to Riemannian manifolds

Joint with Lei Ouyang

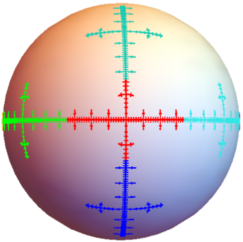
Some Motivations:

- Many interesting fractals can be constructed easily on Riemannian manifolds

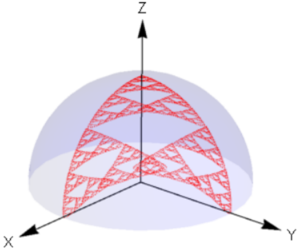


(a) (b)

Figure: (a) On the cylinder with overlaps. (b) On the flat torus with overlaps.



(a)



(b)

Figure: (a) On S^2, pcf. (b) On S^2 with overlaps.

- Yau's conjecture (1982). $(n-1)$-dimensional Hausdorff measure of the nodal set of an eigenfunction Z_u on a Riemannian manifold satisfies the following bounds:

$$c\lambda^{1/2} \leq \mathcal{H}^{n-1}(Z_u) \leq C\lambda^{1/2},$$

where u is a λ-eigenfunction of the Laplacian. Lower bound was proved by Logunov (2018); upper bound remains open.

- In the fractal case, Yau's conjecture probably takes the form

$$c\lambda^{d_s/(2n)} \leq \mathcal{H}^{n-1}(Z_u) \leq C\lambda^{d_s/(2n)}.$$

Laplacians on forms and Hodge's theorem

Historically, Hodge's theorem says that each de Rham cohomology group is isomorphic to a space of harmonic forms, thus establising a relationship between geometry (smooth structure) and analysis. An equivalent form of Hodge's theorem says that there exists a complete o.n.b. for each L^2 space of k-forms (w.r.t. Riemannian volume measure).

Let $\Gamma^\infty(\bigwedge^k T^*M)$ be the space of smooth k-forms, $d = d_k : \Gamma^\infty(\bigwedge^k T^*M) \to \Gamma^\infty(\bigwedge^{k+1} T^*M)$ be the exterior derivative, and $d^* = d_k^*$ be the adjoint of d_k.

$$\Delta = \Delta^0 = d^*d \quad \text{(Laplacian on 0-forms)}$$

$$\Delta^k = d_{k+1}^* d_k + d_{k-1} d_k^* \quad \text{(Hodge Laplacian on } k\text{-forms, } k \geq 1)$$

Assume that μ satisfies either of the following *Poincaré inequalities*, depending on the boundary condition.

(MPID*) ($\partial\Omega \neq \emptyset$, Dirichlet boundary condition) There exists a constant $C > 0$ such that for all $u \in \Gamma_c^\infty(\bigwedge^k T^*\Omega)$,

$$\int_\Omega |u|^2 \, d\mu \leq C \int_\Omega \left(|du|^2 + |d^* u|^2\right) d\nu.$$

(ν is the Riemmanian volume measure.)

(MPIE*) ($\partial\Omega = \emptyset$) There exists a constant $C > 0$ such that for all $u \in \Gamma_c^\infty(\bigwedge^k T^*\Omega)$,

$$\int_\Omega |u|^2 \, d\mu \leq C \int_\Omega \left(|u|^2 + |du|^2 + |d^* u|^2\right) d\nu.$$

Define bilinear forms

$$\mathcal{E}_D(u,v) = \mathcal{E}_E(u,v) = \int_\Omega \langle du, dv \rangle \, d\nu + \int_\Omega \langle d^* u, d^* v \rangle \, d\nu$$

These bilinear forms define Laplacians on k-forms, Δ_μ^D and Δ_μ^E, with Dirichlet and empty boundary conditions, respectively.

Theorem 6 (N.-Ouyang, 2024, arxiv)

(Hodge's Theorem for Krein-Feller operators on k-forms) Assume that $\underline{\dim}_\infty(\mu) > n-2$. *Then there exists an orthonormal basis of* $L^2(\bigwedge^k T^*\Omega, \mu)$ *consisting of eigenforms of* Δ_μ^k*, where* $0 \le k \le n$. *The eigenvalues* $\{\lambda_m\}_{m=1}^\infty$ *satisfy* $0 \le \lambda_1 \le \lambda_2 \le \cdots$*, where*

(a) $\lambda_1 > 0$ *if* $\partial\Omega \neq \emptyset$*, and*

(b) $\lambda_1 = 0$ *if* $\partial\Omega = \emptyset$.

Moreover, in both cases, if $\dim(\mathrm{dom}(\mathcal{E})) = \infty$*, then* $\lim_{m\to\infty} \lambda_m = \infty$*, and each eigenspace is finite-dimensional.*

Part 3. Continuity of Eigenfunctions of Krein-Feller Operators. Joint with Wen-Quan Zhao

Theorem 7 (N.-Zhao, 2024, arxiv)

Let $n \geq 2$, M be a smooth complete Riemannian n-manifold, and $\Omega \subseteq M$ be a bounded domain with smooth boundary. Let μ be a positive finite Borel measure on M with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Assume $\underline{\dim}_\infty(\mu) > n - 2$. Then the eigenfunctions of $-\Delta_\mu$ are continuous on Ω.

Ideas of proof for the case $\partial M = \emptyset$

- Step 1. Define Green's operator G_μ on $L^2(M,\mu)$ as

$$(G_\mu f)(x) := \int_M G_y(x) f(y)\, d\mu(y).$$

  G_μ is bounded on $L^2(M,\mu)$.
  $f \in L^2(M,\mu) \Rightarrow G_\mu f \in W^{1,2}(M)$.

- Step 2. Let $\widetilde{\mathcal{H}}_\mu := \left\{u \in L^2(M,\mu) : \int_M u\, d\mu = 0\right\}$ and $D_\mu := G_\mu(\widetilde{\mathcal{H}}_\mu)$. Let $f \in \widetilde{\mathcal{H}}_\mu$. Then $G_\mu f \in \mathrm{dom}(\Delta_\mu)$ and $\Delta_\mu(G_\mu f) = f$. Consequently, $D_\mu \subseteq \mathrm{dom}(\Delta_\mu)$ and $G_\mu|_{\widetilde{\mathcal{H}}_\mu} = \left(\Delta_\mu|_{D_\mu}\right)^{-1}$.

- Step 3. Consequently, u is a λ-eigenfunction of Δ_μ iff $u = \lambda G_\mu u + C$ for some constant C.

- Step 4. Prove that $G_\mu(u)$ is continuous by using properties of the Green function and the continuity of μ.

Thank you!