The Riemann-Hilbert problem on extension domains

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Framework: two-sided extension domains [always assumed]

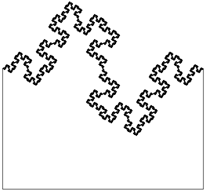
Definition

Ω is a **two-sided extension domain** if:

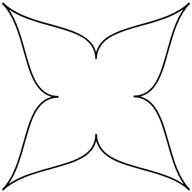
(i) Ω is an H^1-Sobolev extension domain[1];

(ii) $\overline{\Omega}^c := \mathbb{R}^n \backslash \overline{\Omega}$ is an extension domain;

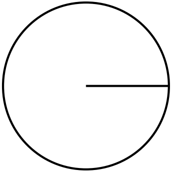
(iii) $\partial\Omega = \partial(\overline{\Omega}^c)$ et $\lambda^{(n)}(\partial\Omega) = 0$.



Two-sided extension domain



(i) and (ii) fail



(iii) fails

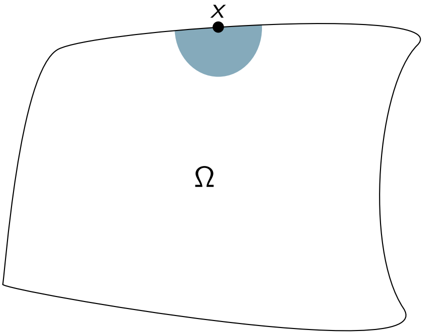
No specified boundary measure.

[1] [Hajłasz, Koskela et Tuominen, 2008]

Boundary conditions on extension domains

- *Dirichlet BC:* **trace operator** [Biegert, 2009]

$$\operatorname{Tr}_i u(x) = \lim_{r\to 0^+} \frac{1}{\lambda^{(n)}(\Omega \cap B_r(x))} \int_{\Omega\cap B_r(x)} u \,\mathrm{d}x, \quad u \in H^1(\Omega),\ x \in \partial\Omega \text{ q.e.}$$



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Theorem (Trace theorem)

- *$\mathcal{B}(\partial\Omega) := \operatorname{Tr}_i(H^1(\Omega))$ is a Hilbert space for*

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid \operatorname{Tr}_i v = f\} = \|\tilde{v}\|_{H^1(\Omega)},$$

where $(-\Delta + 1)\tilde{v} = 0$ weakly on Ω (i.e. $\tilde{v} \in V_1(\Omega)$) and $\operatorname{Tr}_i \tilde{v} = f$.

- $\operatorname{Ker} \operatorname{Tr}_i = H^1_0(\Omega) := \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$.
- $V_1(\Omega) = (H^1_0(\Omega))^\perp$.

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We consider $\mathrm{Tr}_i : H^1(\Omega) \to \mathcal{B}(\partial\Omega)$.

- *Neumann BC:* **weak normal derivative** [Lancia, 2002]

$$\forall v \in H^1(\Omega), \quad \left\langle \frac{\partial_i u}{\partial \nu}, \mathrm{Tr}_i\, v \right\rangle_{\mathcal{B}', \mathcal{B}} = \int_\Omega (\Delta u) v \,\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x,$$

where $u \in H^1(\Omega)$ with $\Delta u \in L^2(\Omega)$.

We consider $\frac{\partial_i}{\partial \nu} : H^1(\Omega) \cap \{\Delta \cdot \in L^2(\Omega)\} \to \mathcal{B}'(\partial\Omega)$.

Boundary conditions on extension domains

- *Dirichlet BC:* **trace operator** [Biegert, 2009]

$$\mathrm{Tr}_e\, u(x) = \lim_{r\to 0^+} \frac{1}{\lambda^{(n)}(\overline{\Omega}^c \cap B_r(x))} \int_{\overline{\Omega}^c \cap B_r(x)} u \,\mathrm{d}x, \;\; u \in H^1(\overline{\Omega}^c), \;\; x \in \partial\Omega \text{ q.e.}$$

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- *Neumann BC:* **weak normal derivative** [Lancia, 2002]

$$\forall v \in H^1(\overline{\Omega}^c), \quad \left\langle \frac{\partial_e u}{\partial \nu}, \mathrm{Tr}_e\, v \right\rangle_{\mathcal{B}', \mathcal{B}} = -\int_{\overline{\Omega}^c} (\Delta u) v \,\mathrm{d}x - \int_{\overline{\Omega}^c} \nabla u \cdot \nabla v \,\mathrm{d}x,$$

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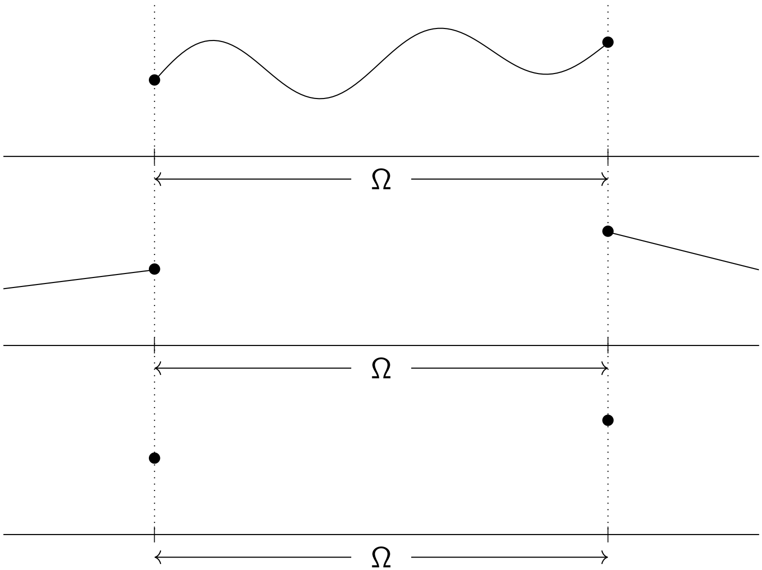
Specific extensions



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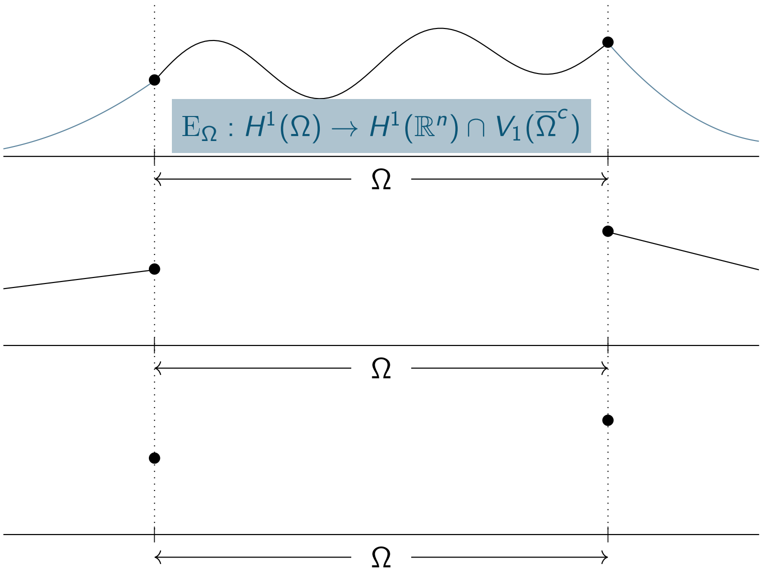

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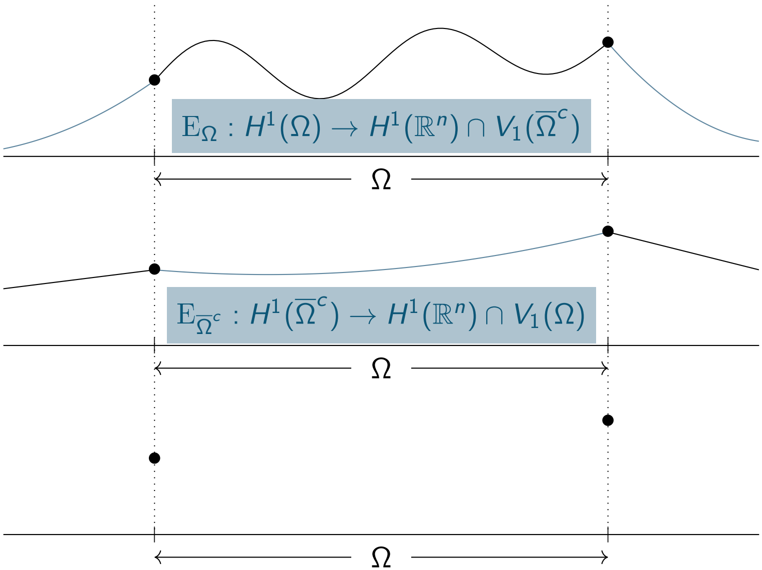

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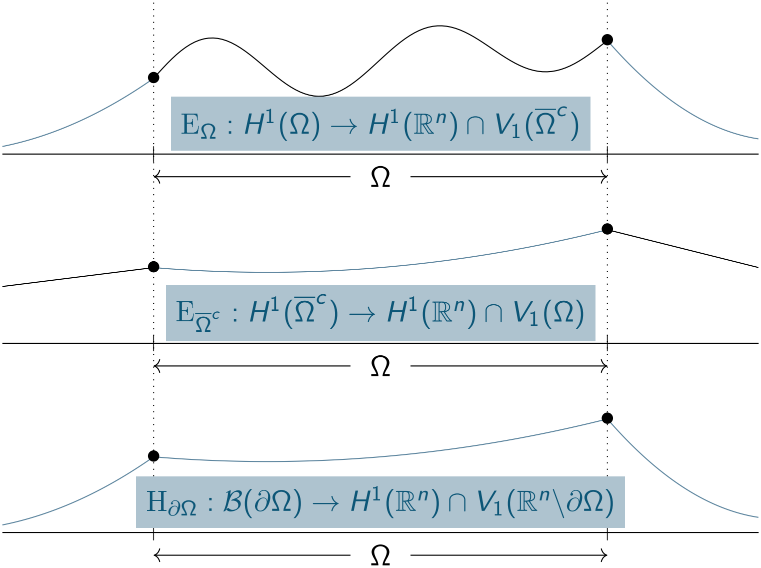

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The Riemann-Hilbert problem on Lipschitz domains

Riemann-Hilbert problem:

$$\begin{cases} u \text{ is holomorphic on } \mathbb{C}\backslash\partial\Omega, \\ \mathrm{Tr}_i^{\partial\Omega} u - \mathrm{Tr}_e^{\partial\Omega} u = f, \\ u(z) \to 0 \text{ as } |z| \to +\infty. \end{cases}$$

[2] [Muskhekishvili, 1977]
[3] [Chapman and Vanden-Broeck, 2006]

The Riemann-Hilbert problem on Lipschitz domains

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If Ω is Lipschitz, then $u = \Phi_{\partial\Omega} f$ where

$$\Phi_{\partial\Omega} f(z) = \frac{1}{2\mathrm{i}\pi} \int_{\partial\Omega} \frac{f(y)}{y - z} \lambda(\mathrm{d}y), \quad z \in \mathbb{C},$$

is **the Cauchy integral**[2].

Used in **signal processing** and for the **computation of gravitational waves**[3].

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Connection to the harmonic transmission problem

Transmission problem:

$$\begin{cases} -\Delta u = 0 \quad \text{on } \mathbb{R}^n \backslash \partial\Omega, \\ \mathrm{Tr}_i^{\partial\Omega} u - \mathrm{Tr}_e^{\partial\Omega} u = f \in \mathcal{B}(\partial\Omega), \\ \dfrac{\partial_i u}{\partial\nu} - \dfrac{\partial_e u}{\partial\nu} = g \in \mathcal{B}'(\partial\Omega). \end{cases}$$

[4] Classical case: [Verchota, 1984], [Costabel, 1988], [McLean, 2000], [Nédélec, 2001], and many more. /Extension domains: [C., Hinz, Rozanova-Pierrat and Teplyaev, 2024].

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By the superposition principle,

$$u = u_{\mathcal{S}} - u_{\mathcal{D}} \qquad \text{where} \quad \begin{cases} \mathrm{Tr}_i^{\partial\Omega} u_{\mathcal{S}} - \mathrm{Tr}_e^{\partial\Omega} u_{\mathcal{S}} = 0, \\ \dfrac{\partial_i u_{\mathcal{D}}}{\partial\nu} - \dfrac{\partial_e u_{\mathcal{D}}}{\partial\nu} = 0. \end{cases}$$

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We introduce the single and double layer potential operators[4]

$$\mathcal{S}_{\partial\Omega} : g \in \mathcal{B}'(\partial\Omega) \longmapsto u_{\mathcal{S}} \in V_1(\mathbb{R}^n \backslash \partial\Omega),$$
$$\mathcal{D}_{\partial\Omega} : f \in \mathcal{B}(\partial\Omega) \longmapsto u_{\mathcal{D}} \in V_1(\mathbb{R}^n \backslash \partial\Omega).$$

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Outline of the method

- **Solve the R-H problem on two-sided extension domains.**

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Solve the R-H problem on two-sided extension domains.

For Lipschitz boundaries, $\Phi_{\partial\Omega} f$ solves a transmission problem:

$$\Phi_{\partial\Omega} f = \mathcal{S}_{\partial\Omega} g_f - \mathcal{D}_{\partial\Omega} f.$$

$\mathcal{S}_{\partial\Omega}$ and $\mathcal{D}_{\partial\Omega}$ are well-defined for two-sided extension domains.

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💡 **Approximate Ω with smoother $(\Omega_k)_{k\in\mathbb{N}}$**, to prove that the solutions for $\partial\Omega_k$, $\Phi_{\partial\Omega_k} f_k$, **converge** to a solution for $\partial\Omega$, $\Phi_{\partial\Omega} f$.

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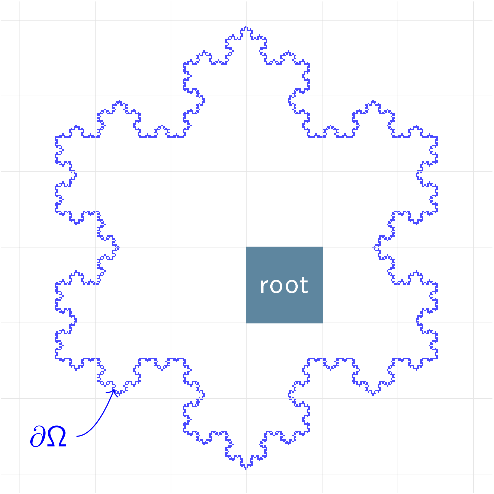
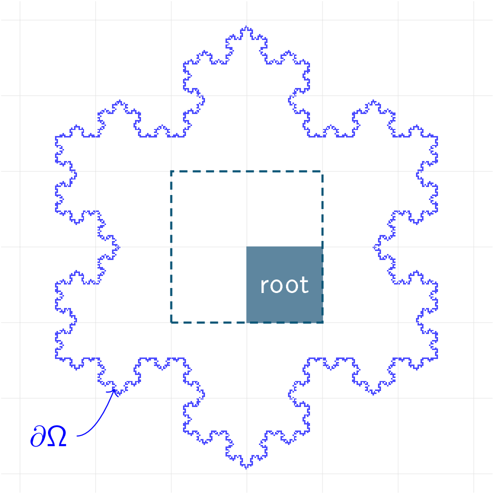
How to approximate Ω?

What does *converge* mean for functions defined on different spaces?

Dyadic approximation



Dyadic approximation



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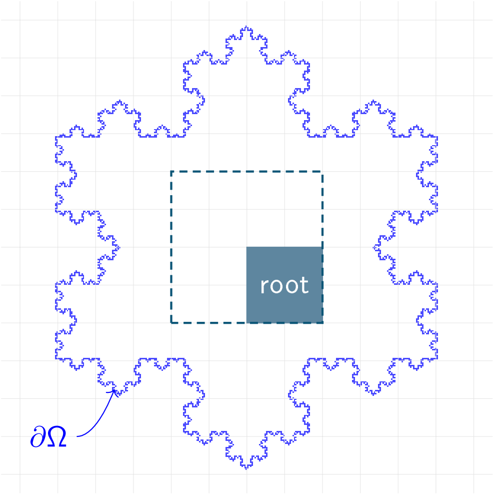

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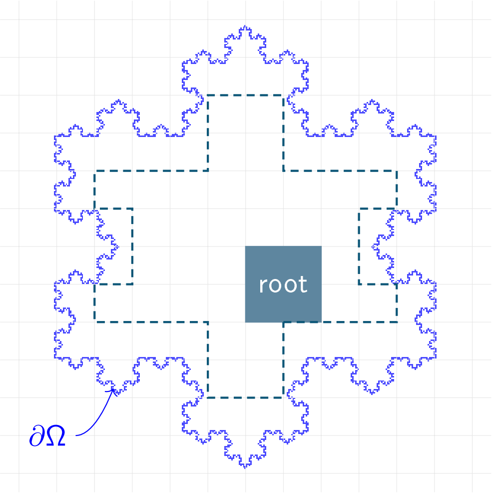

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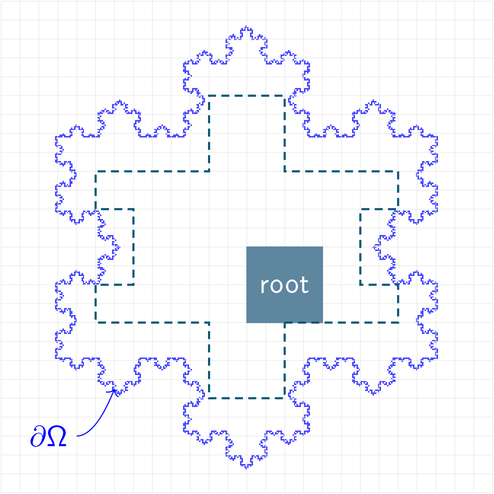

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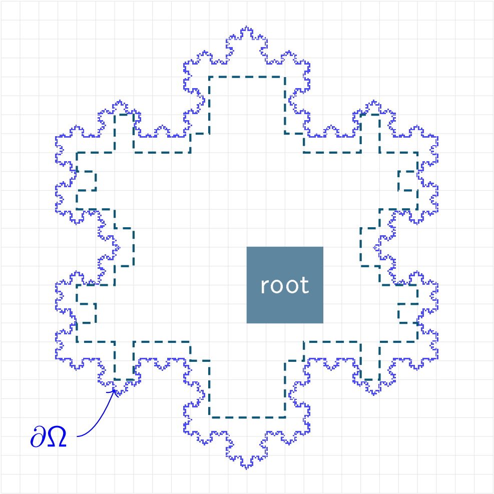

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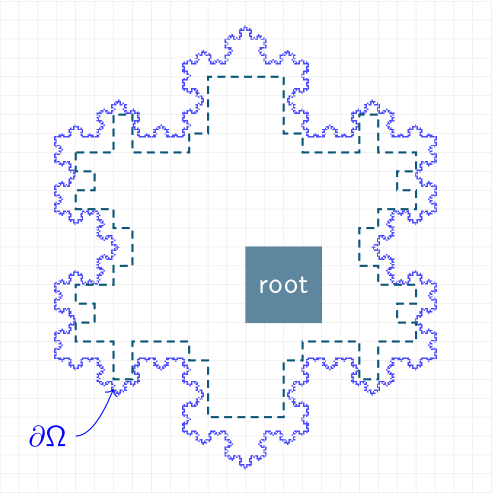

Proposition

Let $(\Omega_k)_{k\in\mathbb{N}}$ be a dyadic approximation of Ω. It holds $\Omega_k \nearrow \Omega$.

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Convergence of and along a sequence of Hilbert spaces

Definition (Convergence of Hilbert spaces [Kuwae-Shioya, 2003])

A sequence (H_k) of Hilbert spaces converges to a Hilbert space H through $(\mathcal{T}_k)_{k\in\mathbb{N}}$, where $(\mathcal{T}_k \in \mathcal{L}(H, H_k))_{k\in\mathbb{N}}$, if it holds

$$\forall u \in H, \quad \|\mathcal{T}_k u\|_{H_k} \xrightarrow[k\to\infty]{} \|u\|_H.$$

Morally, $u \in H$ is represented in H_k by $\mathcal{T}_k u$.

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Morally, $u \in H$ is represented in H_k by $\mathcal{T}_k u$.

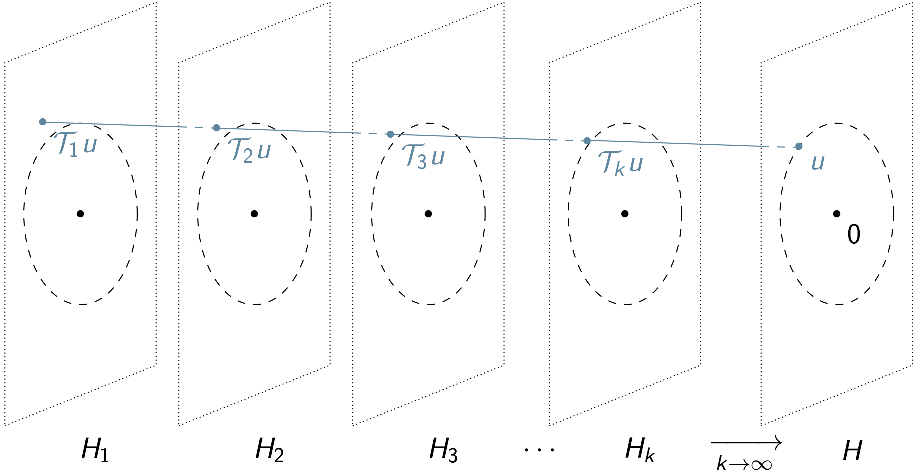
Definition (Convergence of vectors)

Assume $H_k \to H$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$. A sequence $(u_k \in H_k)_{k\in\mathbb{N}}$ is said to converge to $u \in H$ if it holds

$$\|u_k - \mathcal{T}_k u\|_{H_k} \xrightarrow[k\to\infty]{} 0.$$

Converging to u means becoming arbitrarily close to its representatives $\mathcal{T}_k u$.

Illustration of the convergence of Hilbert spaces



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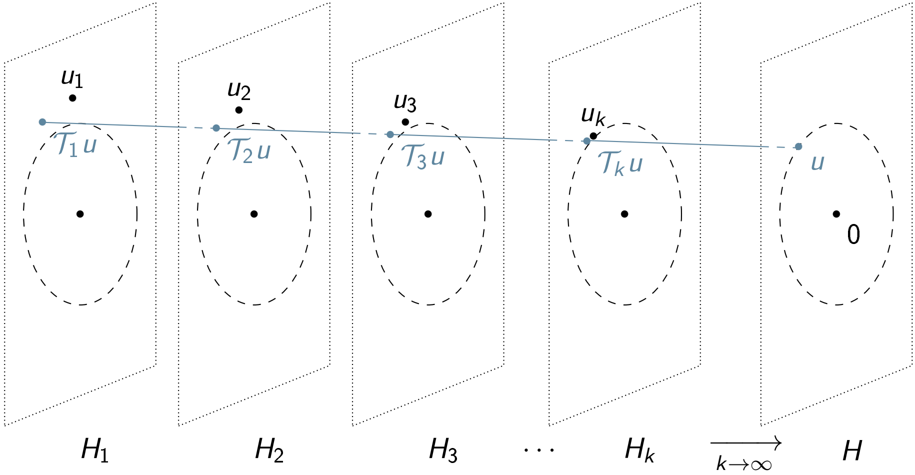

$u_k \to u$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$...

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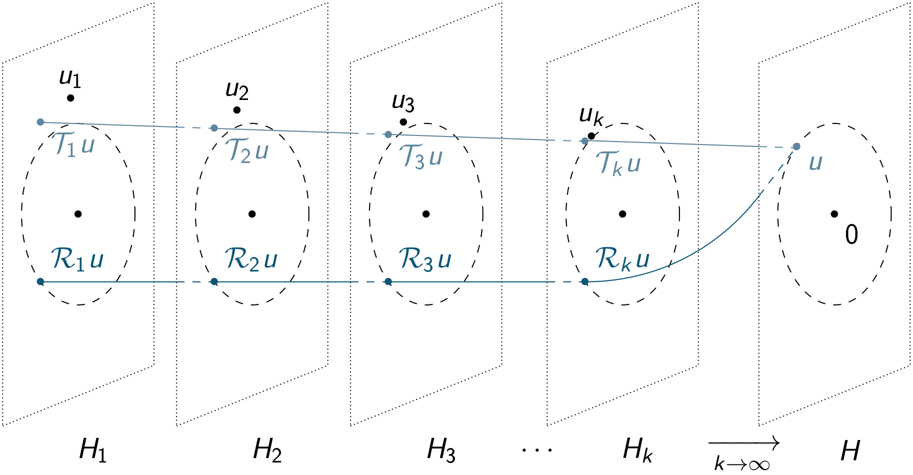

$u_k \to u$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$... but not through $(\mathcal{R}_k)_{k\in\mathbb{N}}$!

Decompositions of the space of solutions

The space of solutions to $(-\Delta+1)u=0$ on $\mathbb{R}^n\backslash\partial\Omega$ can be described as

$$\begin{aligned} V_1(\mathbb{R}^n\backslash\partial\Omega) &= V_1(\Omega) \quad \oplus \quad V_1(\overline{\Omega}^c) && \text{(geographic)} \\ &= \underbrace{V_{1,\mathcal{S}}(\mathbb{R}^n\backslash\partial\Omega)}_{\text{null jump in Tr}} \quad \oplus \quad \underbrace{V_{1,\mathcal{D}}(\mathbb{R}^n\backslash\partial\Omega)}_{\text{null jump in } \frac{\partial}{\partial\nu}} && \text{(in potentials)} \end{aligned}$$

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The problem is connected to the decomposition in potentials...

but the geographic decomposition is more tangible.

Convergence framework for V_1 functions inside

If $\Omega_k \nearrow \Omega$, which $u_k \in V_1(\Omega_k)$ represents a given $u \in V_1(\Omega)$ best?

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$$\|u|_{\Omega_k}\|^2_{H^1(\Omega_k)} = \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2)\mathbb{1}_{\Omega_k}\,\mathrm{d}x \xrightarrow[k\to\infty]{} \int_{\mathbb{R}^n} (|\nabla u|^2 + |u|^2)\mathbb{1}_{\Omega}\,\mathrm{d}x = \|u\|^2_{H^1(\Omega)}$$

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Proposition

If $\Omega_k \nearrow \Omega$, then $V_1(\Omega_k) \longrightarrow V_1(\Omega)$ through $(\cdot|_{\Omega_k})_{k\in\mathbb{N}}$.

Convergence framework at the boundary: traces

If $\Omega_k \nearrow \Omega$, which $f_k \in \mathcal{B}(\partial\Omega_k)$ represents a given $f \in \mathcal{B}(\partial\Omega)$ best?

Ex: $\Omega =]0,1[$ and $\Omega_k =]2^{-k}, 1-2^{-k}[$, $k \geq 2$.

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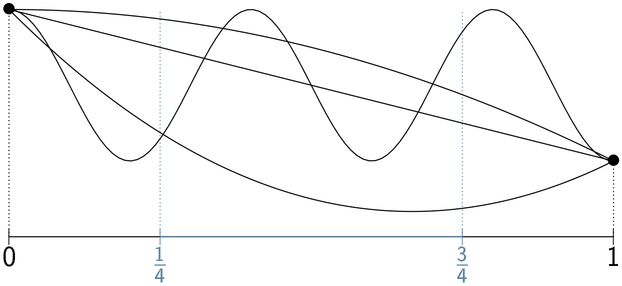

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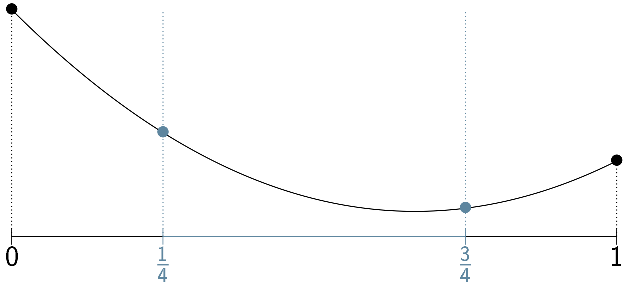

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Extend using a **solution to the problem** (here, $(-\Delta+1)u = 0$).

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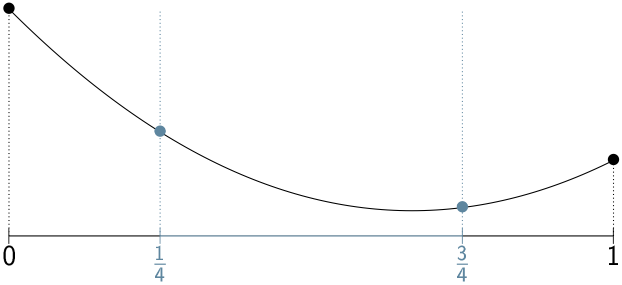

Proposition

If $\Omega_k \nearrow \Omega$, then $\mathcal{B}(\partial\Omega_k) \longrightarrow \mathcal{B}(\partial\Omega)$ through $(\mathrm{Tr}^{\partial\Omega_k} \circ \mathrm{H}_{\partial\Omega})_{k\in\mathbb{N}}$.

Did we do things right?

We chose those convergence frameworks for the V_1 and $\mathcal{B}$ spaces because they felt ‘natural’. **Do they work together?**

Did we do things right?

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Proposition

Assume $\Omega_k \nearrow \Omega$. *If* $(u_k \in V_1(\Omega_k))_{k\in\mathbb{N}}$ *and* $u \in V_1(\Omega)$, *then*

$$\begin{aligned} & u_k \xrightarrow[k\to\infty]{} u \quad \textit{through } (\cdot|_{\Omega_k})_{k\in\mathbb{N}} \\ \iff \quad & \mathrm{Tr}_i^{\partial\Omega_k} u_k \xrightarrow[k\to\infty]{} \mathrm{Tr}_i^{\partial\Omega} u \quad \textit{through } (\mathrm{Tr}^{\partial\Omega_k} \circ \mathrm{H}_{\partial\Omega})_{k\in\mathbb{N}}. \end{aligned}$$

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The convergence frameworks for V_1 and $\mathcal{B}$ are indeed **compatibles**.

Are those convergence frameworks relevant?

A priori, a convergence across Hilbert spaces is very weak...

Proposition

Assume $\Omega_k \nearrow \Omega$ *and* $(\mathrm{E}_{\Omega_k})_{k\in\mathbb{N}}$ *is uniformly bounded. If* $(u_k \in V_1(\Omega_k))_{k\in\mathbb{N}}$ *and* $u \in V_1(\Omega)$*, then*

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The convergence across spaces for V_1 functions can be strengthened into a (standard) convergence on $H^1(\mathbb{R}^n)$: **we did things in the 'right way'!**

Consequently, the convergence framework for $\mathcal{B}$ is also 'right'.

Convergence framework for V_1 functions outside

If $\Omega_k \nearrow \Omega$, which $v_k \in V_1(\overline{\Omega}_k^c)$ represents a given $v \in V_1(\overline{\Omega}^c)$ best?

Ex: $\Omega =]0,1[$ and $\Omega_k =]2^{-k}, 1-2^{-k}[$, $k \geq 2$.

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If $\Omega_k \nearrow \Omega$, then $V_1(\overline{\Omega}_k^c) \longrightarrow V_1(\overline{\Omega}^c)$ through $(\qquad)_{k\in\mathbb{N}}$.

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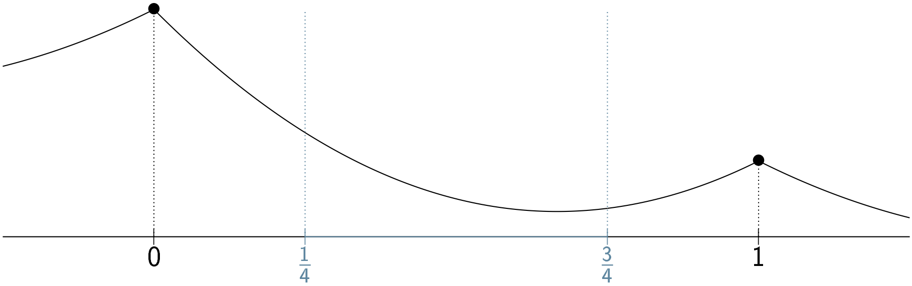

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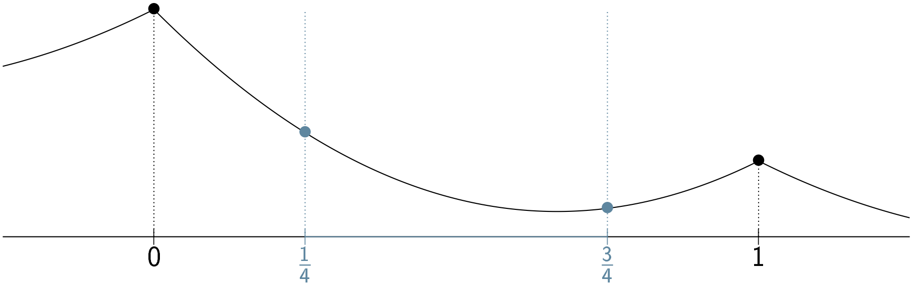

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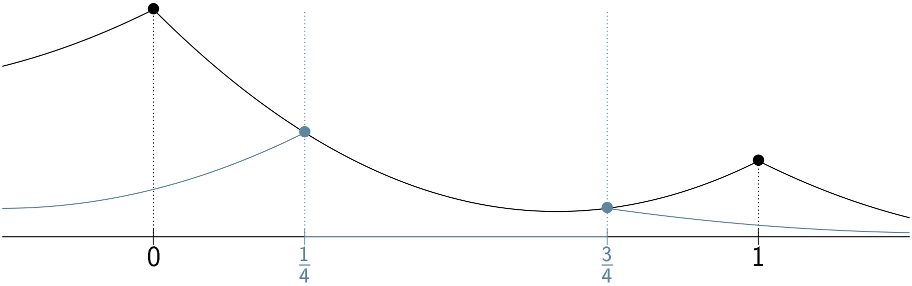

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Did we do things right? – part 2

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Proposition

Assume $\Omega_k \nearrow \Omega$*, and* (E_{Ω_k}) *and* $(\mathrm{E}_{\overline{\Omega}_k^c})$ *are uniformly bounded. If* $(u_k \in V_1(\overline{\Omega}_k^c))_{k\in\mathbb{N}}$ *and* $u \in V_1(\overline{\Omega}^c)$*, then*

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$$\iff \quad \|\mathrm{E}_{\overline{\Omega}^c} u - \mathrm{E}_{\overline{\Omega}_k^c} u_k\|_{H^1(\mathbb{R}^n)} \xrightarrow[k\to\infty]{} 0.$$

Back to the whole space

We built the convergence of the V_1 spaces along the geographic decomposition...

Proposition

Assume $\Omega_k \nearrow \Omega$, *and* (E_{Ω_k}) *and* $(\mathrm{E}_{\overline{\Omega}_k^c})$ *are uniformly bounded. If* $(u_k \in V_1(\mathbb{R}^n \backslash \partial\Omega_k))_k \to u \in V_1(\mathbb{R}^n \backslash \partial\Omega)$ *through* $(\cdot|_{\Omega_k} \oplus \mathcal{E}_k)_k$, *and*

$$u = \mathcal{S}_{\partial\Omega} g - \mathcal{D}_{\partial\Omega} f \quad et \quad u_k = \mathcal{S}_{\partial\Omega_k} g_k - \mathcal{D}_{\partial\Omega_k} f_k,$$

then

$$\mathcal{S}_{\partial\Omega_k} g_k \longrightarrow \mathcal{S}_{\partial\Omega} g \qquad \text{through } (\cdot|_{\Omega_k} \oplus \mathcal{E}_k),$$
$$\mathcal{D}_{\partial\Omega_k} f_k \longrightarrow \mathcal{D}_{\partial\Omega} f \qquad \text{through } (\cdot|_{\Omega_k} \oplus \mathcal{E}_k).$$

... yet **the decomposition in potentials is also preserved!**

The Riemann-Hilbert problem on extension domains

Under the same hypotheses, we can deduce

$$f_k \longrightarrow f \quad \text{through } (\mathrm{Tr}^{\partial\Omega_k} \mathrm{H}_{\partial\Omega_k})$$
$$\Longrightarrow \quad \Phi_{\partial\Omega_k} f_k \longrightarrow \Phi_{\partial\Omega} f \quad \text{through } (\cdot|_{\partial\Omega} \oplus \mathcal{E}_k).$$

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Since the convergence through $(\cdot|_{\partial\Omega} \oplus \mathcal{E}_k)$ can be **strengthened** into an $H^1(\mathbb{R}^n)$ convergence, it also holds

$$\Phi_{\partial\Omega_k} f_k \text{ holomorphic on } \mathbb{C}\backslash\partial\Omega_k \quad \Longrightarrow \quad \Phi_{\partial\Omega} f \text{ holomorphic on } \mathbb{C}\backslash\partial\Omega.$$

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Theorem

Let Ω be a two-sided extension domain of $\mathbb{C}$. Assume there exists a sequence of Lipschitz domains $(\Omega_k)_{k\in\mathbb{N}}$ such that $\Omega_k \nearrow \Omega$, and (E_{Ω_k}) and $(\mathrm{E}_{\overline{\Omega}_k^c})$ be uniformly bounded.
*Then, **we can define a Cauchy integral** $\Phi_{\partial\Omega} : \mathcal{B}(\partial\Omega) \to H^1(\mathbb{C}\backslash\partial\Omega)$ **which solves the Riemann-Hilbert problem**.*

Thank you for your attention!

To find out more:

- G. Claret, A. Rozanova-Pierrat and A. Teplyaev, *Convergence of layer potentials and Riemann-Hilbert problem on extension domains* (2024).
- G. Claret, M. Hinz, A. Rozanova-Pierrat and A. Teplyaev, *Layer potential operators for transmission problems on extension domains* (2024).

A natural model

Many examples of fractal (self-similar) shapes in nature [5]:

Romanesco cabbage

[marcheoutais.com]

Fern

[Gamm Vert]

Lightning bolt

[Vosges Matin]

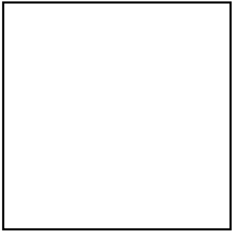
[5] More in: Mandelbrot, *The Fractal Geometry of Nature*, 1982.

Regular case: Lipschitz boundaries

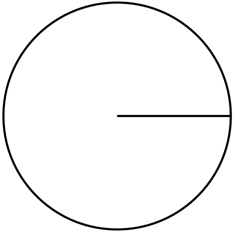
Ω is called *Lipschitz*[6] if:

(i) $\partial\Omega$ is locally the **graph of a Lipschitz continuous function**,

(ii) Ω lies **only on one side** of $\partial\Omega$.



Lipschitz domain



Non-Lipschitz domain

On the boundary: **Lebesgue's measure** $\lambda^{(n-1)}$.

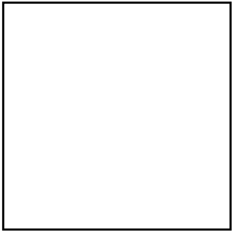
[6] Henrot and Pierre, 2018.

Fractal case: d-sets

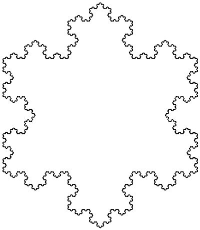
$\partial\Omega$ is a d-set[7], $d \in [n-1, n[$, if there is a **Borel measure** μ:

(i) $\operatorname{supp} \mu = \partial\Omega$,

(ii) $\exists c_1, c_2 > 0,\ \forall x \in \partial\Omega,\ \forall r \in]0,1], \quad c_1 r^d \le \mu(B_r(x)) \le c_2 r^d$.



$(n-1)$-set



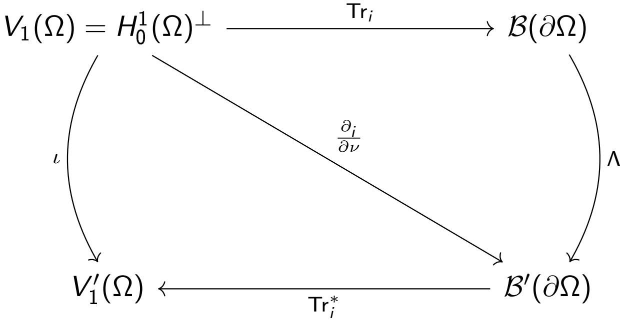
Flocon de Von Koch

$\frac{\ln 4}{\ln 3}$-set

Hausdorff dimension of the boundary d (fixed).

[7] Jonsson et Wallin, 1984.

Some important isometries and estimates



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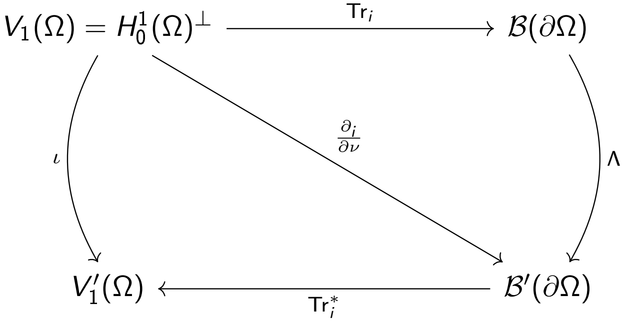

Proposition

Let Ω be a two-sided extension domain. Then, for $v \in V_1(\overline{\Omega}^c)$, it holds

$$\left(\|\mathrm{E}_{\overline{\Omega}^c}\|^2 - 1\right)^{-\frac{1}{2}} \|\mathrm{Tr}_e\, v\|_{\mathcal{B}(\partial\Omega)} \leq \|v\|_{H^1(\overline{\Omega}^c)} \leq \left(\|\mathrm{E}_\Omega\|^2 - 1\right)^{\frac{1}{2}} \|\mathrm{Tr}_e\, v\|_{\mathcal{B}(\partial\Omega)},$$

where the constants are optimal.

Convergence of Hilbert spaces

As we just saw, the notion is weak.

Proposition

Let $(H_k)_{k\in\mathbb{N}}$ and H be separable Hilbert spaces.
For all $(u_k \in H_k)_{k\in\mathbb{N}}$ and $u \in H$ such that $\|u_k\|_{H_k} \to \|u\|_H$, there exists $(\mathcal{T}_k \in \mathcal{L}(H, H_k))_{k\in\mathbb{N}}$ such that $u_k \to u$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$.

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Ex: $H_k = H = \mathbb{R}$.

$$3 \xrightarrow[k\to\infty]{} 3 \qquad \text{through } (\mathrm{id}_{\mathbb{R}})_{k\in\mathbb{N}},$$

but
$$3 \xrightarrow[k\to\infty]{} -3 \qquad \text{through } (-\mathrm{id}_{\mathbb{R}})_{k\in\mathbb{N}},$$

while
$$-3 \xrightarrow[k\to\infty]{} 3 \qquad \text{through } (-\mathrm{id}_{\mathbb{R}})_{k\in\mathbb{N}}.$$

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It is *very* weak.

Proposition

Let $(H_k)_{k\in\mathbb{N}}$ and H be separable Hilbert spaces.
Let $(u_k \in H_k)$ with a finite number of $u_k = 0$, and $u \in H$.
Up to replacing the norms on (H_k) with equivalent norms, there exists $(\mathcal{T}_k \in \mathcal{L}(H, H_k))_{k\in\mathbb{N}}$ such that $u_k \to u$ through $(\mathcal{T}_k)_{k\in\mathbb{N}}$.

The most important part of the statement '$H_k \to H$ through $(\mathcal{T}_k)$' is the sequence $(\mathcal{T}_k)$.

Convergence framework at the boundary: normal derivatives

If $\Omega_k \nearrow \Omega$, which $g_k \in \mathcal{B}'(\partial\Omega_k)$ represents a given $g \in \mathcal{B}'(\partial\Omega)$ best?

First approach: as we did for $\mathcal{B}$.

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$$\begin{array}{ccc}
\mathcal{B}(\partial\Omega_k) & \xleftarrow{\ \mathrm{Tr}^{\partial\Omega_k}\circ \mathrm{H}_{\partial\Omega}\ } & \mathcal{B}(\partial\Omega) \\
\Lambda_{\partial\Omega_k}\big\downarrow & & \big\uparrow \Lambda_{\partial\Omega}^{-1} \\
\mathcal{B}'(\partial\Omega_k) & \leftarrow\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-}\text{-} & \mathcal{B}'(\partial\Omega)
\end{array}$$

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Since $\Omega_k \subset \Omega$, **the representatives are the same** in both cases.