Boundary value problems on domains with non-Lipschitz boundaries and applications in the shape optimization

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Fractals 8, 18 Jun 2025

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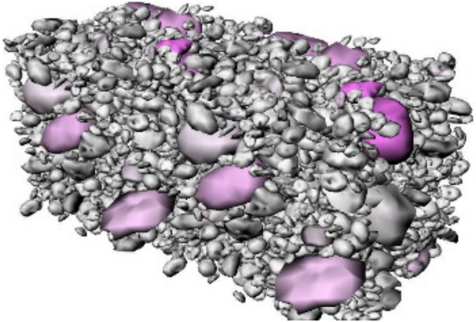
Nature complexity and their models

Porous materials



Traffic noise absorbing wall

“Fractal wall” TM, porous material is the cement-wood (acoustic absorbent),
Patent Ecole Polytechnique-Colas, Canadian and US patent

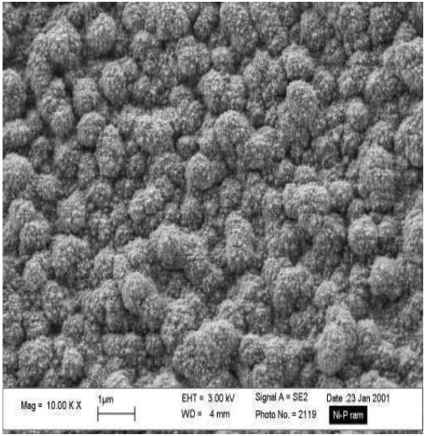
Acoustic anechoic chambers

Test anechoic chamber

Microsoft anechoic chamber -20db noise level, the quietest place on earth

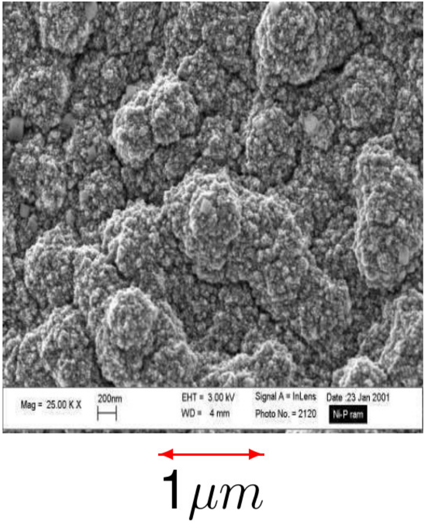
Test semi-anechoic chamber

Irregularity of boundaries

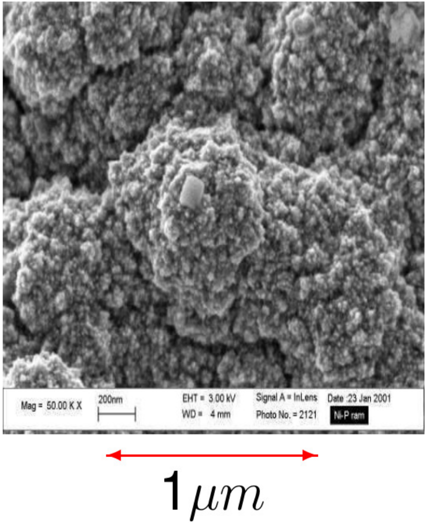


$1\,\mu m$

Irregularity of boundaries

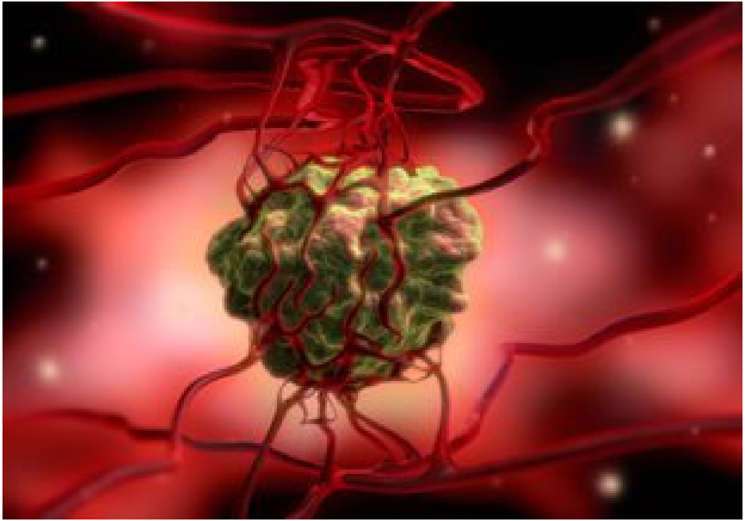


Irregularity of boundaries

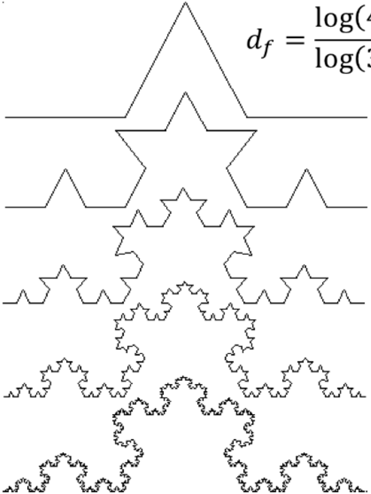


Irregularity of boundaries

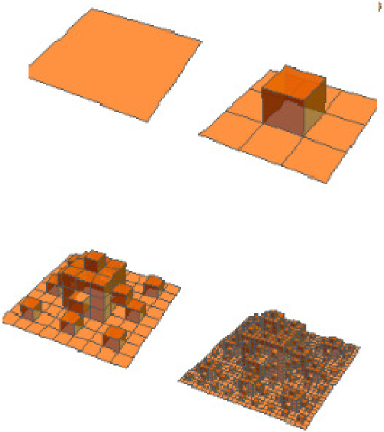
Antigiogenesis of cancerous tumours



Examples of self-similar fractal boundaries



$$d_f = \frac{\log(4)}{\log(3)}$$



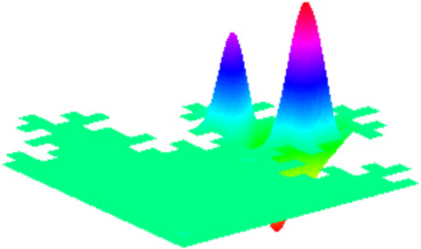
$2 < d = \frac{\log(13)}{\log(3)} \approx 2.33 < 3$ (Wikipedia)

Optimality of fractal models in applications

Physically, models involving fractals are the most

- stable,
- conductive,
- dissipative,
- absorbing structures.

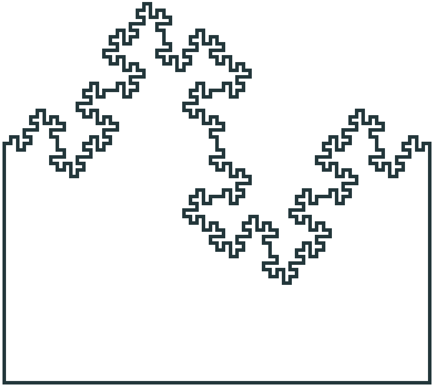
Works of B. Sapoval, M. Filoche, D. Grebenkov, S. Mayboroda, G.David and their co-authors:



Localization (weak) of eigenfunctions

Main difficulties for non-Lipschitz boundary value problems

A fractal and Lipschitz curve boundary $\partial\Omega$ of dimension $d \geq n-1$



No elliptic regularity of solutions: $u \notin H^2(\Omega)$

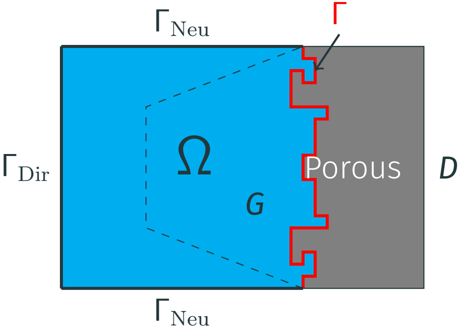
No classical normal derivative $\nabla u \cdot \vec{n}$ $\quad (\nexists \vec{n})$

K. Nyström, 1996, von Koch's snowflake, NTA domains; P. Grisvard, 1974, Lipschitz non-convexe

"Fractal" project: existence of optimal shapes

- **In acoustics** (mixed boundary conditions Dirichlet/Neumann/Robin)
  - F. Magoulès, P.T.K. Ngyuen, P. Omnes, ARP, *Optimal absorbtion of acoustic waves by a boundary*. SIAM J. Control Optim. (2021).
  - M. Hinz, ARP, A. Teplyaev, *Non-Lipschitz uniform domain shape optimization in linear acoustics*. SIAM J. Control Optim. (2021).
- **In architecture** (non-homogeneous Dirichlet and Neumann conditions)
  - M. Hinz, F. Magoulès, ARP, M. Rynkovskaya, A. Teplyaev, *On the existence of optimal shapes in architecture*. Appl. Math. Model., (2021).
- **In heat exchanges** (transmission problem)
  - G. Claret, ARP, *Existence of optimal shapes for heat diffusions across irregular interfaces*, to appear in AMS, CONM book series (2025).
- **In the elliptic framework with Robin type condition**
  - M. Hinz, ARP, A. Teplyaev, *Boundary value problems on Non-Lipschitz uniform domains: Stability, Compactness and the Existence of optimal shapes*. Asymptotic Analysis, (2023).

Minimization of acoustical energy for a fixed frequency and a noise source

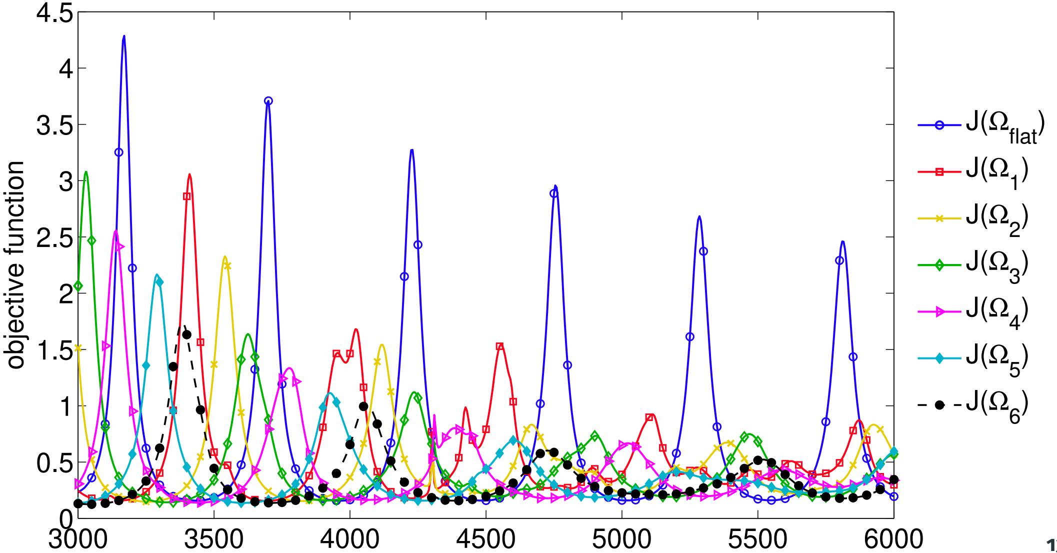


$$\begin{cases} \triangle u + \omega^2 u = f(x) \quad x \in \Omega, \\ u = g(x) \quad \text{on } \Gamma_{Dir}, \qquad \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{Neu}, \\ \frac{\partial u}{\partial n} + \alpha(\omega)\,\mathrm{Tr}\, u = 0 \quad \text{on } \Gamma, \quad \mathrm{Re}(\alpha) > 0 \text{ and } \mathtt{Im}(\alpha) < 0 \end{cases}$$

$$J(\Omega, \mu, u(\Omega, \mu)) := A \int_\Omega |u|^2 dx + B \int_\Omega |\nabla u|^2 dx + C \int_\Gamma |\,\mathrm{Tr}\, u|^2 d\mu$$

$$\min_{\Omega \in U_{ad}(D,G,\ldots)} J(\Omega, \mu, u(\Omega, \mu))$$

The most efficient and easy to construct antinoise wall on a range of frequencies



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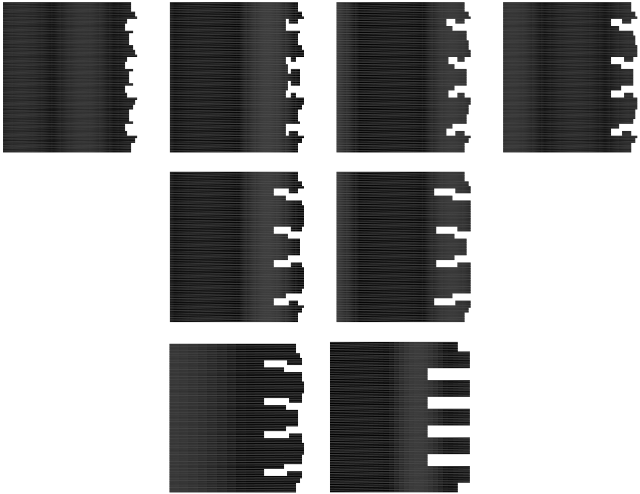

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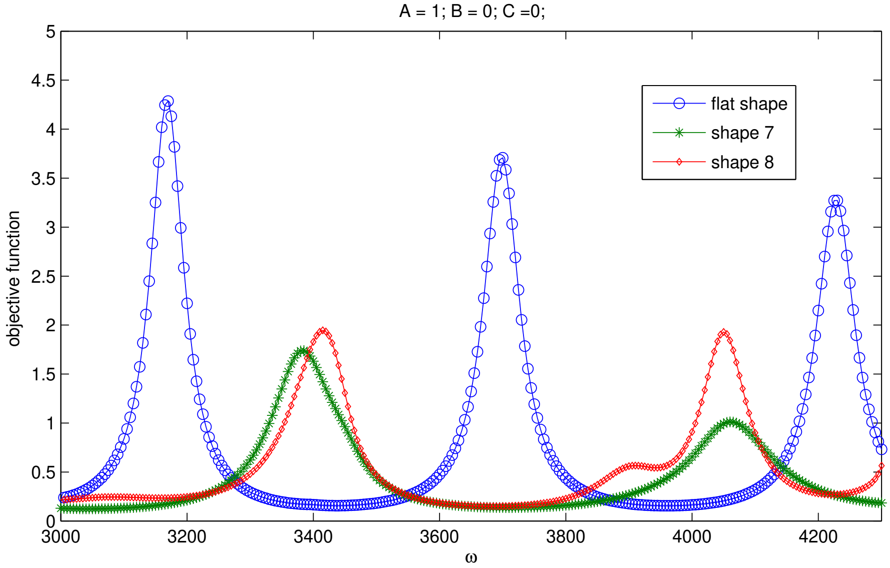

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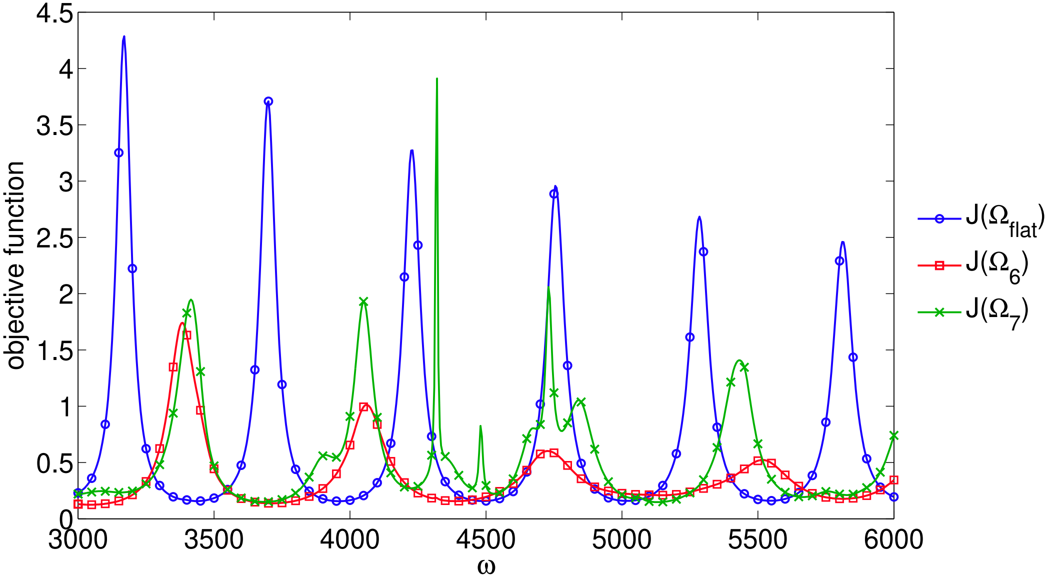

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Sobolev extension domains

Definition

*A domain $\Omega \subset \mathbb{R}^n$ is called a H^1**-extension domain** if there exists a bounded linear extension operator $E_\Omega : H^1(\Omega) \to H^1(\mathbb{R}^n)$:*

$$\forall u \in H^1(\Omega) \quad \exists v = E_\Omega u \in H^1(\mathbb{R}^n) \text{ with } v|_\Omega = u \text{ and } C(\Omega) > 0 :$$

$$\|v\|_{H^1(\mathbb{R}^n)} \leq C\|u\|_{H^1(\Omega)}.$$

Jones [1981]: If Ω is an uniform (or (ε, ∞)-) domain, then it is Sobolev extension domain.

Hajłasz, Koskela and Tuominen [2008]: $\Omega \subset \mathbb{R}^n$ is a H^1-extension domain $\Longleftrightarrow$ Ω is an n-set and $H^1(\Omega) = C^{1,2}(\Omega)$ (space of the fractional sharp maximal functions) with norms' equivalence.

Locally uniform or (ε, δ)-domains ($\varepsilon > 0$, $0 < \delta \leq \infty$)

Definition
An open connected subset Ω of $\mathbb{R}^n$ is an (ε, δ)-domain,

if whenever $x, y \in \Omega$ and $|x - y| < \delta$, ***(thus locally)***

there is a rectifiable arc $\gamma \subset \Omega$ with length $\ell(\gamma)$ joining x to y and satisfying

1. $\ell(\gamma) \leq \frac{|x-y|}{\varepsilon}$ ***(uniformly locally quasiconvex)*** *and*
2. $d(z, \partial\Omega) \geq \varepsilon |x - z| \frac{|y-z|}{|x-y|}$ *for $z \in \gamma$.*

Theorem ($n = 2$, Jones [1981])
A bounded and finitely connected domain Ω is (ε, ∞)-domain $\Longleftrightarrow$ its boundary consists of a finite number of points and quasicircles.

n-sets or measure density condition

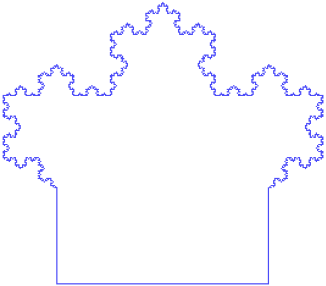
- All (ε, δ)-domains in $\mathbb{R}^n$ are ***n***-sets (***d***-set with $d = n$):

$$\exists c > 0 \quad \forall x \in \overline{\Omega},\ \forall r \in]0, \delta[\cap]0, 1] \quad \lambda(B_r(x) \cap \Omega) \geq C\lambda(B_r(x)) = cr^n,$$

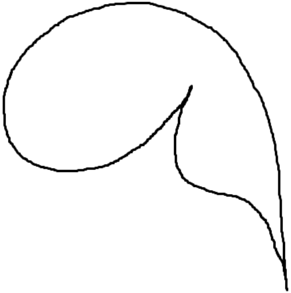
where λ denotes the Lebesgue measure in $\mathbb{R}^n$.

- An ***n***-set Ω cannot be “thin” close to its boundary $\partial\Omega$, since it contain a non trivial ball in its neighborhood.

Examples of extension domains



Extension domain



Not an extension domain

Trace operator : boundary measure free framework on extension domains

Definition

*H^1-extension domain Ω is called **H^1-admissible** if its boundary $\partial\Omega$ has positive capacity.*

Proposition

For a H^1-admissible domain Ω of $\mathbb{R}^n$, given $u \in H^1(\Omega)$, let

$$\mathrm{Tr}_i\, u := (E_\Omega u)^{\sim}|_{\partial\Omega}$$

be the restriction of any quasi continuous representative $(E_\Omega u)^{\sim}$ of $E_\Omega u$. Then the ***(interior) trace operator***

$$\mathrm{Tr}_i : H^1(\Omega) \to \mathcal{B}(\partial\Omega)$$

is a well-defined linear surjection.

Consequently, q. e.

$$x \in \partial\Omega \qquad \mathrm{Tr}_i\, u(x) = \lim_{r\to 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega\cap B_r(x)} u(y)dy.$$

Trace theorem ("boundary measure free")

Let $\Omega \subset \mathbb{R}^n$ be an H^1-admissible bounded domain.

$$H^1(\Omega) = H^1_0(\Omega) \oplus V_1(\Omega), \quad V_1(\Omega) = \{u \in H^1(\Omega) | -\Delta u + u = 0 \text{ weakly}\}$$

(i) The space $H^1_0(\Omega) := \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{H^1(\Omega)}}$ is the kernel of Tr_i, that is, $H^1_0(\Omega) = \ker \operatorname{Tr}_i$.

(ii) Endowed with the norm

$$\|f\|_{\mathcal{B}(\partial\Omega)} := \min\{\|v\|_{H^1(\Omega)} \mid v \in H^1(\Omega) \text{ and } \operatorname{Tr}_i v = f\}, \tag{1}$$

the space $\mathcal{B}(\partial\Omega)$ is a Hilbert space.

(iii) $\| \operatorname{Tr}_i \|_{\mathcal{L}(H^1(\Omega),\mathcal{B}(\partial\Omega))} = 1$.
Its restriction $\operatorname{tr}_i|_{V_1(\Omega)} : V_1(\Omega) \to \mathcal{B}(\partial\Omega)$ to $V_1(\Omega)$ is an isometry and onto.

Green formula

Let $\Omega \subset \mathbb{R}^n$ be H^1-admissible.

$$H^1_\Delta(\Omega) := \left\{u \in H^1(\Omega) \,\middle|\, \Delta u \in L^2(\Omega)\right\}$$

Given $u \in H^1_\Delta(\Omega)$, there is a unique element g of $\mathcal{B}'(\partial\Omega)$ such that

$$\langle g, \mathrm{Tr}_i\, v\rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)} = \int_\Omega (\Delta u) v \,\mathrm{d}x + \int_\Omega \nabla u \nabla v \,\mathrm{d}x, \quad v \in H^1(\Omega).$$

We call this element g the *weak interior normal derivative* of u (with respect to Ω) and denote it by $\frac{\partial_i u}{\partial \nu} := g$.

$\frac{\partial_i}{\partial \nu} : H^1_\Delta(\Omega) \to \mathcal{B}'(\partial\Omega)$ is linear and bounded:

$$\left\|\frac{\partial_i u}{\partial \nu}\right\|_{\mathcal{B}'(\partial\Omega)} \leq \|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}.$$

(thanks to multiple works of M. R. Lancia (d-sets, Jonsson measures))

Dirichlet type or harmonic extensions for $-\Delta+1$ on admissible domains

$V_1(\Omega)$ is also the space of weak solutions of the **Dirichlet boundary-value problem**

$$\begin{cases} -\Delta u + u &= 0 \quad \text{in } \Omega \\ u|_{\partial\Omega} &= f \in \mathcal{B}(\partial\Omega) \end{cases}$$

$$E^D : \mathcal{B}(\partial\Omega) \to E^D(\mathcal{B}(\partial\Omega)) = V_1(\Omega) \subset H^1(\Omega)$$
$$f \mapsto u^f = E^D(f),$$

where u^f is the unique weak solution to the Dirichlet boundary problem

Proposition
$E^D : \mathcal{B}(\partial\Omega) \to V_1(\Omega)$ *is an isometry:* $\quad \forall f \in \mathcal{B}(\partial\Omega) \quad \|f\|_{\mathcal{B}(\partial\Omega)} = \|E^D f\|_{H^1(\Omega)}$.
In this sense $E^D = \operatorname{tr}_i^{-1}$.

Neumann problem for $-\Delta + 1$ on admissible domains

Let $\Omega \subset \mathbb{R}^n$ be an H^1-admissible bounded domain.

$$\begin{cases} -\Delta u + u & = 0 \quad \text{in } \Omega \\ \quad \frac{\partial u}{\partial \nu}|_{\partial\Omega} & = g \in \mathcal{B}'(\partial\Omega) \end{cases}$$

$$\forall v \in H^1(\Omega) \quad \exists! u \in H^1(\Omega) \quad \langle u, v \rangle_{H^1(\Omega)} = \langle g, \mathrm{Tr}_i\, v \rangle_{\mathcal{B}'(\partial\Omega), \mathcal{B}(\partial\Omega)}.$$

$V_1(\Omega)$ is the space of the weak solutions of the **Neumann boundary value problem**.

$$E^N : \frac{\partial_i u}{\partial \nu} \in \mathcal{B}'(\partial\Omega) \mapsto u \in V_1(\Omega) \subset H^1(\Omega)$$

where u is the unique weak solution of the Neumann boundary value problem for $-\Delta + 1$, is an isometry, $(E^N)^{-1} = \frac{\partial}{\partial \nu_i}$ on $V_1(\Omega)$

$$\forall g \in \mathcal{B}'(\partial\Omega) \quad \|E^N g\|_{H^1(\Omega)} = \|\,\mathrm{tr}_i^*\, g\|_{(H^1(\Omega))'} = \|g\|_{\mathcal{B}'(\partial\Omega)}.$$

Poincaré-Steklov operator for admissible domains

Theorem
Let Ω be a H^1-admissible bounded domain and $k \in \mathbb{R} \setminus \sigma(-\Delta_D)$. Then the ***Poincaré-Steklov operator***

$$A : \mathcal{B}(\partial\Omega) \to \mathcal{B}'(\partial\Omega)$$
$$\operatorname{Tr} u \mapsto \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}$$

associated with the weak solutions from

$$u \in H^1_\Delta(\Omega) := \left\{v \in H^1(\Omega) \mid \Delta v \in L^2(\Omega)\right\}$$

$$(-\Delta + k)u = 0 \text{ on } \Omega \quad \text{with} \quad \operatorname{Tr} u|_{\partial\Omega} = f \in \mathcal{B}(\partial\Omega), \tag{2}$$

is a linear bounded operator with $\operatorname{Ker} A \neq \{0\}$ *and it coincides with its adjoint.*

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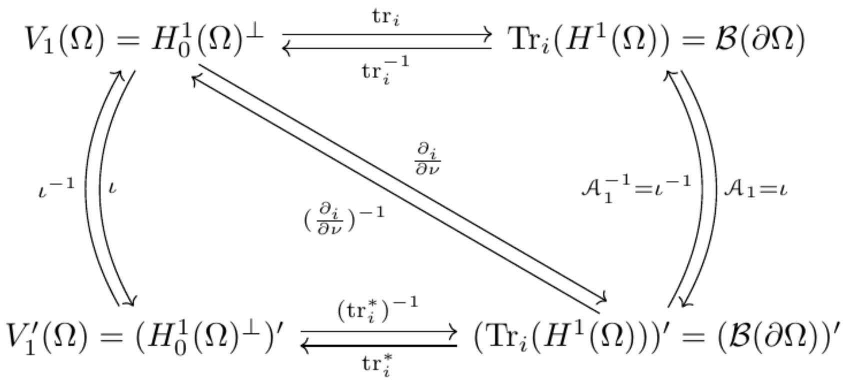
d-sets K. Arfi, A.R.-P. 2019, A.R.-P. 2020, Lipschitz case by W. Arendt, A. F. M. ter Elst 2011, 2015

Different isometries

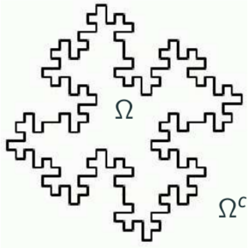


$(E^N)^{-1} = \frac{\partial}{\partial\nu_i}$, $E^D = \mathrm{tr}_i^{-1}$

$$\forall g \in \mathcal{B}'(\partial\Omega) \quad \|g\|_{\mathcal{B}'(\partial\Omega)} = \|\mathrm{tr}_i^* g\|_{(H^1(\Omega))'},$$
$$\forall u \in V_1(\Omega) \quad \|u\|_{H^1(\Omega)} = \|\mathrm{tr}_i u\|_{\mathcal{B}(\partial\Omega)}.$$

in complement to S. N. Chandler Wilde, D. P. Hewett, A. Moiola, 2017-...

Transmission problems: two-sided H^1-admissible domains



$$\begin{cases} (-\Delta+1)u & = 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} & = -f \in \mathcal{B}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} & = g \in \mathcal{B}'(\partial\Omega) \end{cases}$$

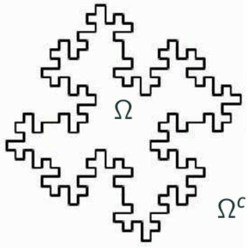
Definition

*$\Omega \subset \mathbb{R}^n$ is a **two-sided H^1-admissible domain** if*

1. *$\Omega \neq \varnothing$ and $\Omega^c = \mathbb{R}^n \setminus \overline{\Omega}$ are H^1-extension domains*
2. $\partial\Omega = \partial(\mathbb{R}^n \setminus \overline{\Omega})$
3. *the Lebesgue measure of $\partial\Omega$ is zero.*

$\Rightarrow \dim_{\mathcal{H}}(\partial\Omega) \geq n-1$, hence its capacity is positive.

Transmission problems: two-sided H^1-admissible domains



$$\begin{cases} (-\Delta+1)u &= 0 \quad \text{on } \mathbb{R}^n \setminus \partial\Omega \\ u_i|_{\partial\Omega} - u_e|_{\partial\Omega} &= -f \in \mathcal{B}(\partial\Omega) \\ \frac{\partial_i u_i}{\partial\nu}|_{\partial\Omega} - \frac{\partial_e u_e}{\partial\nu}|_{\partial\Omega} &= g \in \mathcal{B}'(\partial\Omega) \end{cases}$$

- G. Claret, M. Hinz, AR-P, A. Teplyaev, *Layer potential operators for transmission problems on extension domains* https://hal.science/hal-04505158
- G. Claret, AR-P, A. Teplyaev, *Convergence of layer potentials and Riemann-Hilbert problems on extension domains.* https://hal.science/hal-04762502
- G. Claret, M. Hinz, AR-P *Calderón inverse problem on extension domains.* https://hal.science/hal-05054570v1

Boundary measure framework of admissible domains (Ω, Γ)

Let Ω be a bounded H^1-extension domain of $\mathbb{R}^n$ and $\Gamma \subset \overline{\Omega}$ is such that

- Γ has positive capacity
- Γ is a compact support $\Gamma = \operatorname{supp} \mu$ for a positive Borel measure μ on $\mathbb{R}^n$ satisfying for $d > 0$, $d \in]n-2, n[$ the **upper d-regular** condition: there is a constant $c_d > 0$ such that

$$\mu(B_r(x)) \leq c_d r^d, \quad x \in \Gamma, \quad 0 < r \leq 1. \tag{3}$$

$(\Longrightarrow \dim_H \Gamma \geq d)$

Examples: union of different d-sets, multifractals, . . .
$\Gamma = \partial\Omega$ is a standart case.

Examples, remarks

- d-sets: $\dim_H \Gamma = d > 0$
  $\exists c_1, c_2 > 0,$

$$c_1 r^d \leq \mu(\Gamma \cap \overline{B_r(x)}) \leq c_2 r^d, \quad \text{for } \forall x \in \Gamma,\ 0 < r \leq 1,$$

- Lipschitz and more regular boundaries
- bounded dimension boundaries

$$n - 2 < \dim_H \Gamma < n$$

- **[Jonsson-1979, Biegert-2009]**: for every measurable $E \subset \Gamma$
  $\mathrm{cap}(E) = 0 \Rightarrow \mu(E) = 0$

Definition of the trace operator [A. Jonnson, 2009; M. Biegert, 2009]

Definition
For a Sobolev extension domain Ω of $\mathbb{R}^n$ with $\operatorname{supp}\mu = \partial\Omega$ (for an upper regular Borel measure μ),
the trace operator $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is defined μ-a.e. by

$$x \in \partial\Omega \qquad \mathrm{Tr}\, u(x) = \lim_{r \to 0} \frac{1}{\lambda^n(\Omega \cap B_r(x))} \int_{\Omega \cap B_r(x)} u(y) dy.$$

Properties of $\partial\Omega$, $\partial\Omega = \operatorname{supp}\mu$ are important to caracterize the norm of $B(\partial\Omega) := \mathrm{Tr}(H^1(\Omega))$:

$$H^{\frac{1}{2}}(\partial\Omega), \quad B^{2,2}_{1-\frac{n-d}{2}}(\partial\Omega), \quad B^{2,2}_1(\partial\Omega), \ \ldots$$

d-sets, H. Wallin 1991
A. Jonnson 1997

Trace theorem on boundaries given by upper d-regular measures μ

Let $(\Omega, \partial\Omega)$ be admissible. Then

(i) $\mathrm{Tr} : H^1(\Omega) \to L^2(\partial\Omega, \mu)$ is compact operator and $\exists\, c_{\mathrm{Tr}}(n, \Omega, d, c_d) > 0$, s. t.
$\|\mathrm{Tr}\, f\|_{L^2(\partial\Omega,\mu)} \leq c_{\mathrm{Tr}} \|f\|_{H^1(\Omega)}, \quad f \in H^1(\Omega)$.

(ii) $B(\partial\Omega) := \mathrm{Tr}(H^1(\Omega))$ is a Hilbert space, compact and dense in $L^2(\partial\Omega, \mu)$

$$\|\varphi\|_{B(\partial\Omega)} := \min\{\|g\|_{H^1(\Omega)} \mid \varphi = \mathrm{Tr}\ g\}.$$

(iii) the Gelfand triple

$$\boxed{B(\partial\Omega) \hookrightarrow L^2(\partial\Omega, \mu) = (L^2(\partial\Omega, \mu))' \hookrightarrow B'(\partial\Omega), \quad B''(\partial\Omega) = B(\partial\Omega)}.$$

M. Hinz, ARP, A. Teplayev, SIAM SICON, 2021, M. Hinz, F. Magoulès, ARP, M. Rynkovskaya, A. Teplayev, Applied Mathematical Modelling 2021.

Some important corrolaries for admissible (Ω, Γ)

Norm equivalence:

As $\mathrm{Tr}_{\Omega,\Gamma} : H^1(\Omega) \to L^2(\Gamma, \mu)$ is compact, the norm $\|u\|_{H^1(\Omega)}$ on $H^1(\Omega)$ is equivalent to

$$\|u\|_{\mathrm{Tr}} = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x + \int_\Gamma |\mathrm{Tr} u|^2 d\mu \right)^{\frac{1}{2}} .$$

Poincaré inequality:

For Ω an (ε, ∞)-domain, $\exists C_p > 0$ depending only on n, ε, d and c_d

$$\left\| u - \int_\Gamma \mathrm{Tr}_{\Omega,\Gamma} u \, d\mu \right\|_{L^2(\Omega)} \le C_{\mathrm{P}} \|\nabla u\|_{L^2(\Omega, \mathbb{R}^n)}, \quad u \in H^1(\Omega).$$

(M. Hinz, ARP, A. Teplayev, Asymptotic Analysis, 2023)

Mixed boundary Poisson problem

For $\Omega \subset \mathbb{R}^n$ a Sobolev extension domain with a compact boundary $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_R = \mathrm{supp}\mu$ with a d upper regular Borel mesure μ, $d \in (n-2, n)$, and compact Γ_D, Γ_R, s. t. $\mu(\Gamma_D \cap \Gamma_N) = \mu(\Gamma_D \cap \Gamma_R) = \mu(\Gamma_N \cap \Gamma_R) = 0$.

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \qquad (f \in L^2(\Omega)) \\ u = 0 \text{ on } \Gamma_D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_N, \quad \frac{\partial u}{\partial n} + a\mathrm{Tr} u = 0 \text{ on } \Gamma_R, \ (a > 0) \end{cases}$$

$$V(\Omega) := \{u \in H^1(\Omega) |\ Tr_{\Gamma_D} u = 0\}.$$

endowed with

$$\|u\|^2_{V(\Omega)} = \int_\Omega |\nabla u|^2 \, dx + a \int_{\Gamma_R} |Tr_{\partial\Omega} u|^2 \mathrm{d}\mu,$$

$$\forall f \in L^2(\Omega) \ \exists! u \in V(\Omega) : \quad \forall v \in V(\Omega) \quad (u, v)_{V(\Omega)} = (f, v)_{L^2(\Omega)}.$$

$$\exists C(a, C_{Poincaré}(\Omega)) > 0 : \quad \|u\|_{V(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

A. Dekkers, ARP, A. Teplyaev, Calculus of Variations and Partial Differential Equations, 2022 M. Hinz, ARP, A. Teplyaev, 2023

Robin and mixed boundary problems on non-Lipschitz domains

- **D. Daners**, Robin boundary value problems on arbitrary domains. Trans. Amer. Math. Soc. 352(9), 4207–4236 (**2000**).
  $\mu = \mathcal{H}^{n-1}$, if $\mathcal{H}^{n-1}(\Gamma_R) = +\infty$, then $\mathrm{Tr} u|_{\Gamma_R} = 0$ (Dirichlet boundary condition).
- **R. Capitanelli**, Robin boundary condition on scale irregular fractals. Commun. Pure Appl. Anal. 9(5), 1221–1234 (**2010**).
  Von Koch type fractal boundaries in $\mathbb{R}^2$, d-measure.

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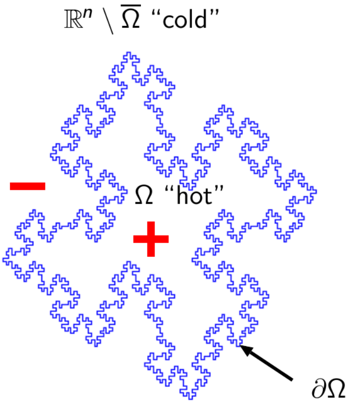
- **R. Capitanelli**, Robin boundary condition on scale irregular fractals. Commun. Pure Appl. Anal. 9(5), 1221–1234 (**2010**).
  Von Koch type fractal boundaries in $\mathbb{R}^2$, d-measure.
- **F. Magoulès, T.P.K. Nguyen, P. Omnes, ARP. SICON, 2021; M. Hinz, ARP, A. Teplyaev, SICON, 2021.** Helmholtz mixed problem for d- and d-upper regular measures
- **A. Dekkers, ARP, A. Teplyaev**, Calc. Var. (2022) $\Gamma = \partial\Omega$ (Westervelt equation);
  M. Hinz, ARP, A. Teplyaev, Asymptotic Anal., (2023) $\Gamma \subset \overline{\Omega}$
  R^n, μ is an upper d-regular Borel measure, $n-2 < d \leq n$.
- **G. David et al.** preprint (2023) (the Robin harmonic measure on d-sets)

"$L^2(\partial\Omega,\mu)$"-transmission heat problem for $0 \le \lambda \le +\infty$

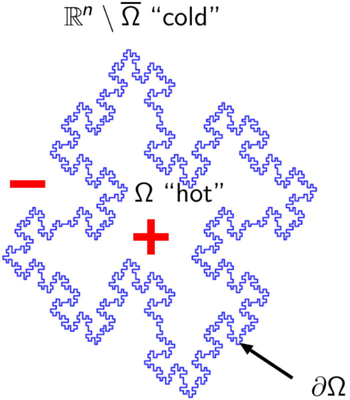


μ is a d-upper regular measure on $\partial\Omega$

$$\begin{cases} \partial_t u^{\pm} - D_{\pm}\Delta u^{\pm} = 0, \ x \in \mathbb{R}^n, \ t > 0, \\ u^+|_{t=0} = 1_{\Omega}(x), \quad u^-|_{t=0} = 0, \\ D_- \frac{\partial u^-}{\partial n}|_{\partial\Omega} = -\lambda(u^+ - u^-)|_{\partial\Omega}, \\ D_+ \frac{\partial u^+}{\partial n}|_{\partial\Omega} = D_- \frac{\partial u^-}{\partial n}|_{\partial\Omega}, \end{cases}$$

$$N(t) = \int_{\mathbb{R}^n\setminus\Omega} u(x,t)\mathrm{d}x = \int_{\Omega}(1 - u(x,t))\mathrm{d}x \quad t \to +0$$

"$L^2(\partial\Omega,\mu)$"-transmission heat problem for $0\le\lambda\le+\infty$



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- G. Claret, ARP, *Existence of optimal shapes for heat diffusions across irregular interfaces*, to appear 2025, `https://hal.science/hal-04505158`
- ARP, A. Teplyaev, S. Winter, M. Zaehle *Fractal curvatures and short time asymptotics of heat diffusion.* to appear 2025, `https://hal.science/hal-04927310`

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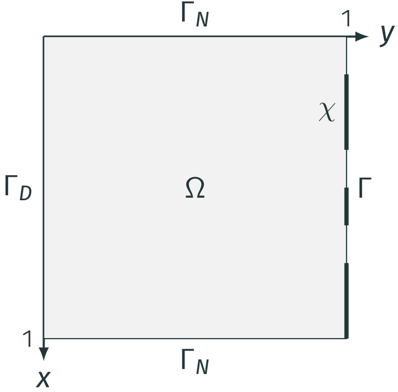
Applications to parametric optimization of wave absorbtion

Parametric optimization

$$\begin{cases} \triangle u + \omega^2 u = f(x) \quad x \in \Omega, \\ u = 0 \quad \text{on } \Gamma_{Dir}, \qquad \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_{Neu}, \\ \dfrac{\partial u}{\partial n} + \chi(x)\alpha(\omega)\,\mathrm{Tr}\, u = 0 \quad \text{on } \Gamma, \quad \mathrm{Re}(\alpha) > 0 \text{ and } \mathtt{Im}(\alpha) < 0 \end{cases}$$

- F. Magoulès, M. Menoux, ARP, *Frequency range non-Lipschitz parametric optimization of a noise absorption*. SIAM J. Control Optim. (2025).
- N. Alami, E. Lucyszyn, R. Pain Dit Hermier, ARP, *Parametric shape optimization for the convected Helmholtz equation with a generalized Myers boundary condition.* https://hal.science/hal-05007470

Distribution of pourous medium (M. Menoux, F. Magoulès, ARP, SIAM SICON 2025)



We fix the percentage rate of the absorbent material on Γ

$$0 < \int_\Gamma \chi d\mu = \beta < \mu(\Gamma) = \int_\Gamma 1 d\mu = 1.$$

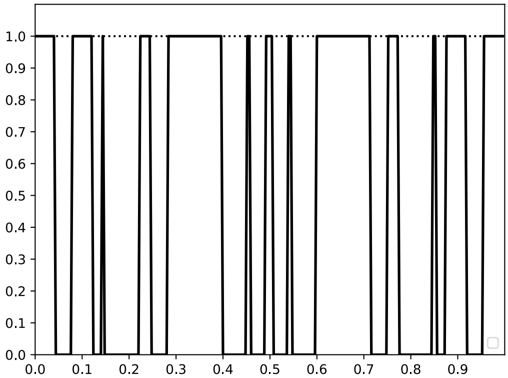
$$\min_{\chi \in U_{ad}(\beta)} J(\omega, \chi), \qquad \min_{\chi \in U_{ad}(\beta)} \int_I J(\omega, \chi) d\omega,$$

$$J(\omega, \chi) = \int_\Omega |u(\omega, \chi)|^2 dx$$

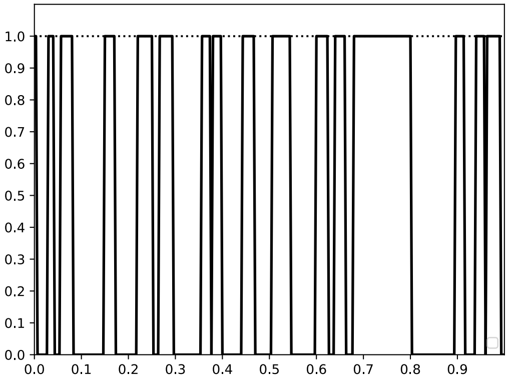
$$U_{ad}(\beta) = \{\chi \in L^\infty(\Gamma, \mu) | \; \mu\text{-a.e. on } \Gamma \quad \chi(\sigma) \in \{0, 1\}, \quad 0 < \beta = \int_\Gamma \chi d\mu < 1\}.$$

$$U^*_{ad}(\beta) = \{\chi \in L^\infty(\Gamma, \mu) | \; \mu\text{-a.e. on } \Gamma \quad \chi(\sigma) \in [0, 1], \quad 0 < \int_\Gamma \chi d\mu = \beta < \mu(\Gamma) = 1\}.$$

"Almost" optimal distribution on a frequency range, $\beta = 0.5$
(M. Menoux, F. Magoulès, AR-P, SIAM SICON 2025)

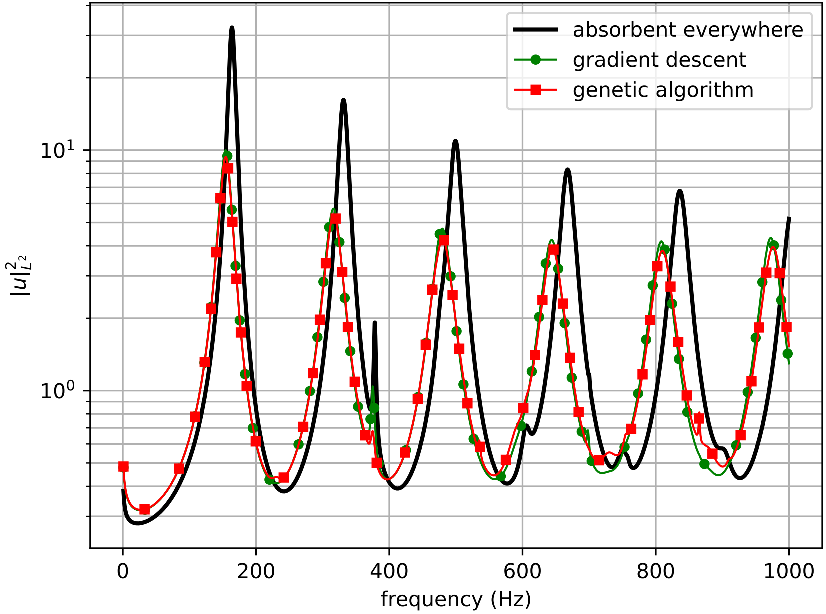


Descent gradient method



Genetic method

Energy as a function of frequency (M. Menoux, F. Magoulès, AR-P, SIAM SICON 2025)



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Westervelt problem and known results (PhD of A. Dekkers)

$$\begin{cases} \partial_t^2 u - c^2 \Delta u - \nu \Delta \partial_t u = \alpha u \partial_t^2 u + \alpha (\partial_t u)^2 + f \quad on \quad]0,T] \times \Omega, \\ u = 0 \quad \text{on } \Gamma_D \times [0,T], \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_N \times [0,T], \\ \frac{\partial u}{\partial n} + au = 0 \quad \text{on } \Gamma_R \times [0,T], \\ u(0) = u_0, \quad \partial_t u(0) = u_1. \end{cases}$$

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Bounded domain with C^2 boundary:

- B. Kaltenbacher, I. Lasiecka, 2009, 2012 ($\partial\Omega = \Gamma_D$ non homogeneous) 2011 (Robin or Neumann non homogeneous) $n \leq 3$;
- S. Meyer, M. Wilke, 2013 (Dirichlet non homogeneous case, all n, $W^{k,p}$).

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In the Non-Lipschitz case, no access to

- the H^2-regularity (thus high energy a priori estimates)
- Nyström: $w \in H_0^1(\Omega),\ -\Delta w = f \in L^2(\Omega) \quad \|\nabla w\|_{L^6(\Omega)} \not\lesssim C\|\Delta w\|_{L^2(\Omega)}$

Westervelt problem and known results (PhD of A. Dekkers)

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Domain Ω	Linear equation	Nonlinear equation
$\partial\Omega = \Gamma_D$ in $\mathbb{R}^2$	arbitrary	NTA or limit of NTA domains
$\partial\Omega = \Gamma_D$ in $\mathbb{R}^3$	arbitrary	arbitrary
$\Gamma_R \neq \varnothing$ in $\mathbb{R}^2$ or $\mathbb{R}^3$	Sobolev admissible	Sobolev admissible

Estimate of $\|u\|_{L^\infty(\Omega)}$

Theorem
Let Ω be a bounded domain and $f \in L^p(\Omega)$ $p \geq 2$, then for u weak solution of the Poisson problem

$$\|u\|_{L^\infty(\Omega)} \leq C\|f\|_{L^p(\Omega)} = C\|\Delta u\|_{L^p(\Omega)}.$$

1. *If $\partial\Omega = \Gamma_{Dir}$*
   - $\Omega \subset \mathbb{R}^2$ *NTA domains (Nyström (1994)),*
   - $\Omega \subset \mathbb{R}^3$ *arbitrary domain (Xie (1991)).*
2. *If $\partial\Omega = \Gamma_{Rob}$ and $\Omega \subset \mathbb{R}^n$*
   - *Daners (2000): $p > n$ for $n-1$-dimensional boundaries, $C = \tilde{C}\max\left(1, \frac{1}{a}\right)$*
   - *A. Dekkers, ARP: $p \geq 2$ for Sobolev admissible domains;*
3. *If $\partial\Omega = \Gamma_{Rob} \cup \Gamma_{Dir} \cup \Gamma_{Neu}$, $\Omega \subset \mathbb{R}^n$*
   - *A. Dekkers, ARP, A. Teplyaev, 2022: $p \geq 2$, if Ω is (ε, ∞)-domain, then $C = C(\varepsilon, n, C_P)$, but not on a.*

Mixed problem for the Westervelt equation, $\nu > 0$, $p = 2$

Theorem
Let Ω be bounded Sobolev admissible domain of $\mathbb{R}^2$ or $\mathbb{R}^3$.
For all $\phi \in L^2(\mathbb{R}^+; V(\Omega))$ with $u(0) = u_0 \in \mathcal{D}(-\Delta)$ and $\partial_t u(0) = u_1 \in V(\Omega)$, $f \in L^2(\mathbb{R}^+; L^2(\Omega))$,

$$\|f\|_{L^2(\mathbb{R}^+;L^2(\Omega))} + \|u_0\|_{\mathcal{D}(-\Delta)} + \|u_1\|_{V(\Omega)} \leq \frac{\nu}{C_2} r, \tag{4}$$

$$\int_0^{+\infty} (\partial_t^2 u, \phi)_{L^2(\Omega)} + c^2 (u, \phi)_{V(\Omega)} + \nu(\partial_t u, \phi)_{V(\Omega)} ds - \int_0^{+\infty} \alpha(u\partial_t^2 u + (\partial_t u)^2 + f, \phi)_{L^2(\Omega)} ds = 0,$$

$$\exists!\, u \in X^2 := H^1(\mathbb{R}^+; \mathcal{D}(-\Delta)) \cap H^2(\mathbb{R}^+; L^2(\Omega)) :$$

$$\exists r_* > 0 : \quad \forall r \in [0, r_*[\quad (4) \Rightarrow \quad \|u\|_{X^2} \leq 2r.$$

Application of M.F. Sukhinin's Theorem $Lu + \Phi(u) = F$

Methods for evolutive in time problems

- Galerkin method based on the spectral problem of $-\Delta$
- To work in the Hilbert space of the weak solutions of the Poisson problem:

$$\begin{aligned}\mathcal{D}(-\Delta_R) = \{u \in H^1(\Omega)| \quad & -\Delta u \in L^2(\Omega) : \\ & \exists f \in L^2(\Omega) \quad \forall v \in V(\Omega) \quad (u,v)_{V(\Omega)} = (f,v)_{L^2(\Omega)}\} \\ & \|u\|_{\mathcal{D}(-\Delta_R)} = \|\Delta_R u\|_{L^2(\Omega)} = \|f\|_{L^2(\Omega)}\end{aligned}$$

- Fix point type theorems of functional analysis
- Approximation by the solutions on regular boundaries

(with converging (extension) sequence of initial conditions; $\rightharpoonup H^1(\mathbb{R}^n)$)

  - $\Omega_m \to \Omega$ in the sense of Hausdorff and caracteristic functions in D;
  - Mosco convergence; $VF_m(v_m, \phi) \to VF(u, \phi)\ \forall \phi \in H(D)$
  - uniform on m linear bounded extension $E : H^1(\Omega_m) \to H^1(D)$
  - $(Ev_m)_{m\in\mathbb{N}}$ is uniformly bounded on m
  - $\forall t \geq 0 \quad Ev_{m_k}|_\Omega \to u$ in $H^1(\Omega)$

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Conclusion

Solving PDEs on domains with Non-Lipschitz boundaries.

Approximation of solutions on domains with a d-set voundary.

Rough boundaries are the energy minimizers.

Thank you very much for your attention!