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Spectral properties of a torsion-free weakly branch group defined by a three state automaton

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Abstract. We continue the study of a torsion free weakly branch group G without free subgroups defined by a three state automaton which was initiated in [**20**]. We study the spectral properties of G which are related to the amenability question. It is also shown that the conjugacy problem is solvable for G as well as for the Brunner-Sidki-Vierra group.

1. Introduction

The spectra of finitely generated groups play an important role in different topics of mathematics such as random walks, operator algebras, operator K-theory, non commutative geometry etc. It is especially important to study the spectra of torsion free groups. This problem is related to the Kadison-Kaplansky conjecture on idempotents [**31**], to one of the versions of the Atiyah conjecture on L^2- Betti numbers [**27**] and to the Baum-Connes conjecture on the assembly map [**5**].

In the study of this and other kind of questions an important role belongs to the groups generated by finite automata (for a general information about such groups see [**23**]). The spectral properties of some automata groups were studied in [**1**] and [**19**]. It is a question of great importance to continue the study of groups generated by automata and first of all by automata with a small number of states.

In this article we study the spectral properties of a torsion free group G generated by a 3 state automaton (see Figure 1) which was first considered in the paper [**20**]. This group shares several similar properties with a group of Brunner-Sidki-Vieira from [**6**] as was mentioned in [**20**] and there is even deeper relation, as we indicate in Section 6. Among other results in [**20**] the following alternative was proven: Either G is amenable but not subexponentially amenable, or G is a non amenable group without a free subgroup on two generators. Thus the study of amenability properties of G is an actual question because this question is related to modified versions of the Day-Greenleaf-Neumann problem as was indicated in [**20**].

According to a result of Kesten [**25**], the amenability of G is equivalent to the presence of 1 in the spectrum of the Markov operator on the Cayley graph of G (see

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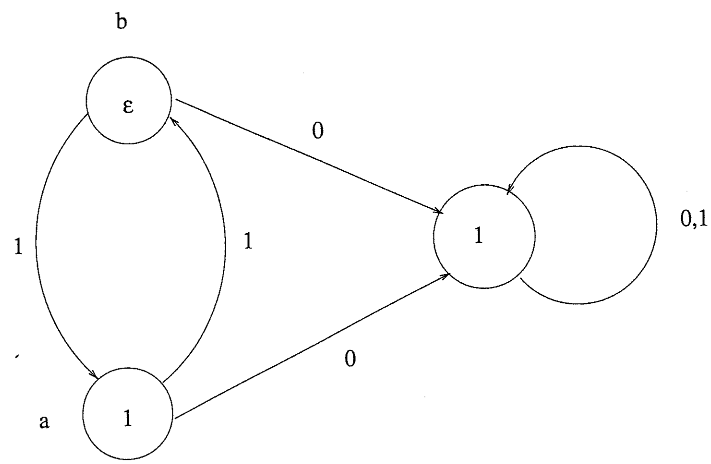

FIGURE 1. The automaton A

also [**24**]). At the moment we are not able to answer the question about amenability but we show that 1 is in the spectrum of the Hecke type operator which naturally arises for any group generated by a finite automaton. Perhaps this should indicate that G is amenable. One of the main properties is that G is a contracting group (this is the first example of a contracting group of exponential growth). It is not known if every contracting group is amenable.

The study of the spectra of groups generated by finite automata was initiated in [**1**] and continued in [**19**] with an application to the strong Atiyah conjecture about L^2 Betti numbers [**22**]. One of the important features of the method developed in [**19**] is the inclusion in the spectral problem of additional parameters and the reduction of it to the study of dynamic properties of a rational mapping of the Euclidean space $\mathbb{R}^d$ for $d \geq 2$.

There are very few examples where such a scheme works and our example is also of this type as we show in this article. But in contrast to the groups considered in [**19**], for the dynamical system associated to G we have a much more complicated case of dynamic behavior.

The Hecke type operator that we study is related to a quasi-regular representation that naturally arises for groups acting on rooted trees, in particular for automata groups. An important question is how the spectrum of this quasi-regular representation is related to the spectrum of the regular representation. In our case the question is of especially high interest as the group is torsion free and the Hecke type operator has many gaps in the spectrum.

To make a comparison of this spectrum we also consider the diagonal action in a direct product of two copies of the tree and see by computer experiments, that the gaps in the spectrum disappear. This shows that the spectrum of the group is much larger than the spectrum of the quasi-regular representation. Perhaps it even contains it and therefore by the Kesten criterion G would be amenable.

To a group generated by a finite automaton one can associate different C^*-algebras. This is related to the idea of self-similarity and our group, its actions, representations and associated C^*-algebras have several self-similarity features. One can consider elements in this C^*-algebras given by the same element of the group ring as the Markov operator. An interesting question is the dependence of the spectrum of this element on the C^*-algebra (in examples considered before there is no dependence, which follows from the amenability property). It will be considered in a forthcoming paper [**21**].

The group we study has many similar properties with the group H from [**6**] as was mentioned above. Here for the groups G and H we establish an important fact, which answers the question from [**6**].

THEOREM 1.1. *The conjugacy problem is solvable in G and in H.*

Let us mention that the conjugacy problem for the group from [**13**] (which is also generated by a finite automaton and historically was the first example of a group of branch type like G) was solved by Y. Leonov [**26**] and A. Rozhkov [**29**].

The last result gives some contribution in the study of algorithmic properties of groups generated by finite automata where the main question is if there is a group generated by a finite automaton with unsolvable conjugacy problem.

1.1. Automata groups. The automata that we use are finite invertible automata with the same input and output alphabet $D = \{0, 1, \dots, d-1\}$ for some $d > 1$. Such an automaton A has a finite set Q of states, transition and exit functions: $\varphi : Q \times D \to Q$, $\omega : Q \times D \to D$ and is characterized by the quadruple $\langle D, Q, \varphi, \omega \rangle$.

The automaton A is invertible if for any $q \in Q$ the function $\omega(q, \cdot) : D \to D$ is a bijection and therefore can be identified with the corresponding element σ_q of the symmetric group S_d on $d = \#D$ symbols, where $\#$ denotes cardinality.

There is a convenient way to describe a finite automaton by a labeled graph $\Gamma(A)$ whose vertices correspond to the elements of Q, two states $q, s \in Q$ are joined by an oriented edge labeled by $i \in D$ if $\varphi(q, i) = s$, and each vertex $q \in Q$ is labeled by the corresponding element σ_q of the symmetric group.

The automata just defined are non-initial automata. To make them initial one has to declare some state $q \in Q$ as the initial state. The initial automaton $A_q = \langle D, Q, \varphi, \omega, q \rangle$ operates on the right on finite and infinite sequences over D in the following way. For each input symbol $x \in D$ it immediately gives the output $y = \omega(q, x)$ and changes his initial state to $\varphi(q, x)$. Joining the output of A_q with the input of another automaton $B_s = \langle D, S, \alpha, \beta, s \rangle$ one gets a map which corresponds to a finite automaton $C_{(q,s)}$ called the composition of A_q and B_s and denoted by $A_q \star B_s$.

This automaton can formally be described as the automaton with the set of states $Q \times S$ and the transition and exit functions Φ, Ω defined as

$$\Phi((x, y), i) = (\varphi(x, i), \alpha(y, \omega(x, i))),$$

$$\Omega((x, y), i) = \beta(y, \omega(x, i)),$$

with the initial state (q, s).

The composition $A \star B$ of two non-initial automata is defined by similar formulas for transition and exit functions only without indicating the initial state.

Two initial automata are called equivalent if they determine the same map on the set of strings. There is an algorithm for minimization of a finite automata which produces an automaton with minimal number of states which is equivalent to the given one [**11**]. The elements of an automaton group are equivalence classes of automata, i.e. automorphisms of trees (see Section 1.2).

An automaton producing the identity map on the set of strings is called trivial. If A is an invertible automaton then for each state q the initial automaton A_q has an inverse automaton A_q^{-1} such that $A_q \star A_q^{-1}$, $A_q^{-1} \star A_q$ are equivalent to a trivial automaton, i.e. an automaton inducing the identity map on the set of sequences.

The inverse automaton can formally be defined as the automaton $\langle D, Q, \widetilde{\varphi}, \widetilde{\omega}, q\rangle$ where $\widetilde{\varphi}(s,i) = \varphi(s, i^{\sigma_s^{-1}})$, $\widetilde{\omega}(s,i) = i^{\sigma_s^{-1}}$ for $s \in Q$. The classes of equivalence of finite invertible automata over the alphabet D constitute a group which is called a finite automata group (it depends on D). Any set of initial automata generate some subgroup of this group.

Now let A be an invertible non-initial automaton as in the beginning. Let $Q = \{q_1, \dots, q_t\}$ be the set of states of Q and let $A_{q_1}, \dots, A_{q_t}$ be the set of initial automata which can be created from A. The group $G(A) = \langle A_{q_1}, \dots, A_{q_t}\rangle$ is called the group determined or generated by A (see [**15**]).

1.2. Action on a tree. The strings over an alphabet $D = \{0, \dots, d-1\}$ are in one-to-one correspondence with vertices of a d-regular rooted tree T_d whose root vertex corresponds to the empty string.

An initial automaton A_q acting on strings over D acts also on T_d as an automorphism. Thus for any group generated by an automata, in particular for a group of the form $G(A)$ there is a canonical action on the corresponding tree. This action is described in more details in [**19, 23**].

$Aut(T)$ is a pro-finite group with a natural topology. Thus for any subgroup $G < Aut(T)$ one can consider the closure $\overline{G}$.

Let now G be any group acting on a regular tree T. The boundary ∂T consisting of geodesic paths joining the root vertex to infinity has a natural topology which makes it homeomorphic to the Cantor set.

The action of G on T induces the action on ∂T by homeomorphisms and there is a canonical invariant uniform measure μ on ∂T which is the Bernoulli measure on $D^{\mathbb{N}}$ given by the distribution $\left\{\frac{1}{d}, \dots, \frac{1}{d}\right\}$.

There is a canonical way to associate to a dynamical system with an invariant measure a unitary representation. In our case this gives the representation π on $L^2(\partial T, \mu)$ defined as $(\pi(g)f)(x) = f(g^{-1}x)$.

1.3. Automata groups and wreath products. There is a close relation between automata groups and wreath products [**11**]. For a group of the form $G(A)$ it has the following interpretation.

Let $q \in Q$ be a state of A and let $\sigma_q \in S_d$ be the label of this state. For each symbol $i \in D$ denote by $A_{q,i}$ the initial automaton with the initial state $\varphi(q,i)$ (thus $A_{q,i}$ for $i = 0, 1, \dots, d-1$ runs through the set of initial automata A_p for which there is an edge from p to q in the graph of the automaton A.

Let G and F be finitely generated groups such that F is a permutation group of a set D (the case which will be interesting for us is when F is a symmetric group S_d and D is the set $\{0, 1, \dots, d-1\}$). Define the wreath product $G \wr_D F$ of these groups as follows. Elements of $G \wr_D F$ are couples (g, γ) where $g : D \to G$ is a function such that $g(x)$ is different from the identity element id_G of G only for finitely many elements x in D, and where γ is an element of F. The multiplication in $G \wr_D F$ is defined as follows:

$$(g_1, \gamma_1)(g_2, \gamma_2) = (g_3, \gamma_1\gamma_2)$$

where

$$g_3(x) = g_1(x)g_2(x^{\gamma_1}) \quad \text{for } x \in D.$$

We will write the elements of the group $G \wr_D S_d$ as expressions $(a_0, \ldots, a_{d-1})\sigma$, where $a_0, \ldots, a_{d-1} \in G$ and $\sigma \in S_d$.

The group $G = G(A)$ embeds in the wreath product $G \wr_D S_d$ via the map

$$A_q \to (A_{q,0}, \ldots, A_{q,d-1})\sigma_q,$$

where $q \in Q$. This can be seen from the action of $G(A)$ on the tree, described in the previous section (for more on this see Section 3.3 in [**23**]). The expression on the right hand side of the arrow will be called a wreath decomposition of A.

We will write in the text $A_q = (A_{q,0}, \ldots, A_{q,d-1})\sigma_q$ without making special comments. The wreath product leads to the following matrix presentation:

$$A_q = \begin{array}{c} \\ i \end{array} \overset{j = i^{\sigma_q}}{\begin{pmatrix} & \vdots \\ \ldots & A_{q,i} \end{pmatrix}},$$

i.e. the element $A_{q,i}$ is on the intersection of i-th row and $j = i^{\sigma_q}$-th column.

1.4. Branch groups. For a group $G = G(A)$ acting by automorphisms on a corresponding rooted tree T we denote by $St_G(n)$ the subgroup of G consisting of those elements of G which act trivially on the level n in the tree T. Analogously for a vertex $u \in T$ we denote by $St_G(u)$ the subgroup of G consisting of those elements of G which act trivially on u, i.e. fixing u. The group G embeds in the wreath product $G \wr_D S_d$ and let $\phi : G \to G \wr_D S_d$ denote this homomorphism. If an element $g \in G$ belongs to the stabilizer $St_G(1)$ of the first level, we get an embedding $\phi : St_G(1) \to G \times \ldots \times G$ (d times) in the base group of the wreath product. This defines canonical projections $\psi_i : St_G(1) \to G$ $(i = 1, \ldots, d)$ on the i-th coordinate. In future we will often omit the subscript G.

A stabilizer $St_G(n)$ of the n-th level is the intersection of all stabilizers of this level. For any vertex $u \in T$ we can define the projection $\psi_u : St(u) \to G$.

DEFINITION 1.2. A group G is called fractal if for any vertex u, $\psi_u(St_G(u)) = G$ after the identification of the tree T with a subtree T_u with a root u.

A rigid stabilizer of the vertex u is the subgroup $Rist_G(u)$ of automorphisms of G that acts trivially on the complement of the subtree T_u. The rigid stabilizer of the n-th level $Rist_G(n)$ is the group generated by rigid stabilizers of vertices on this level.

A group $G < Aut(T)$ is spherically transitive if its action on each level of the tree T is transitive.

A spherically transitive group $G \leq Aut(T)$ is called a branch group if $Rist_G(n)$ is a subgroup of finite index for every $n \in \mathbb{N}$. A spherically transitive group $G \leq Aut(T)$ is called a weakly branch group if $|Rist_G(n)| = \infty$ for every $n \in \mathbb{N}$.

As long as there is no confusion we will omit the subscript G in $St_G(u)$, $Rist_G(u)$, etc.

DEFINITION 1.3. We say that the group G is regular weakly branch over a subgroup $K \neq \{1\}$ if $\phi(K) \geq K \times \ldots \times K$ (d factors each of which acts on a corresponding subtree Tu, $|u| = 1$).

The embedding $\phi : G \to G \wr_D S_d$, $g \to (g_0, \cdots, g_{d-1})\sigma$ defines a restriction g_i of g in the vertex i of the first level. The iteration of this procedure leads to the notion of restriction g_u of g at any vertex u.

The length of a word w and of the element g is denoted by $|w|$ and $|g|$ respectively.

DEFINITION 1.4. A group G is called contracting if there is $\lambda < 1$ and $C, L \in \mathbb{N}$ such that for any vertex u at the level $l > L$

$$|g_u| \leq \lambda |g| + C.$$

We use the notations $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$ and $\langle X \rangle^Y$ for the normal closure of X in Y.

1.5. Operator recursion. Let us fix an isomorphism between $\mathcal{H}$ and $\mathcal{H} \oplus \mathcal{H}$, where $\mathcal{H}$ is an infinite dimensional Hilbert space, which arises from the splitting of ∂T into two parts E_0 and E_1 corresponding to subtrees T_0, T_1 growing from the first level, the canonical isomorphisms $L^2(\partial T, \mu) \simeq L^2(E_0, \mu_0) \oplus L^2(E_1, \mu_1)$ where μ_i is the restriction of μ on E_i and the isomorphism $L^2(\partial T, \mu) \simeq L^2(E_i, \mu_i)$, $i = 0, 1$ naturally arising from the isomorphism $T \simeq T_i$, $i = 0, 1$.

For simplicity we will denote the generators A_a, A_b and A_c of the group G determined by the automaton A from Figure 1 by a, b and c. It is easy to see that the state A_c produces the identity map and thus $c = 1$. Let $G = \langle a, b \rangle$ be the group determined by the automaton on three states from Figure 1. We will keep this notation until the end of the article.

After fixing the above isomorphism $\mathcal{H} \simeq \mathcal{H} \oplus \mathcal{H}$ the operators $\pi(a)$ and $\pi(b)$, where π is the representation from Section 1.2, which we still denote by a and b, satisfy the following operator recursion:

$$a = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix},$$

which corresponds to the wreath product type relations: $a = (1, b)$ and $b = (1, a)\varepsilon$.

Let π_n be the permutation representation of a group G arising from the action of G on the level n vertices of the associated tree and let $\mathcal{H}_n$ be the Hilbert space of the real functions on the n-th level. Let a_n and b_n be the matrices corresponding to the generators for the representation π_n, $n = 0, 1, \ldots$. Then $a_0 = b_0 = Id$, and

$$a_n = \begin{pmatrix} Id_{n-1} & 0 \\ 0 & b_{n-1} \end{pmatrix}, \quad b_n = \begin{pmatrix} 0 & Id_{n-1} \\ a_{n-1} & 0 \end{pmatrix}, \tag{1.1}$$

where we keep in mind the natural isomorphism $\mathcal{H}_n \simeq \mathcal{H}_{n-1} \oplus \mathcal{H}_{n-1}$.

The Hecke type operator Z associated with the dynamical system $(\partial T, G, \mu)$ is the operator

$$Z = \frac{1}{|S|} \sum_{s \in S} \pi(s)$$

where S is a symmetric set of generators of G. This is a self-adjoint operator whose spectrum is called the spectrum of the dynamical system (it may depend on the system of generators).

Recall that to a group G generated by a finite symmetric set S one can associate a Cayley graph with the vertex set the elements of G and the edges corresponding to pairs (g_1, g_2) such that $g_1^{-1} g_2 \in S$.

More generally, if H is a subgroup of G (not necessarily normal), one can associate to the triple (G, H, S) the Schreier graph with the vertex set the right cosets G/H and the edges corresponding to pairs (gH, sgH) such that $s \in S$ and $g \in G$.

With any connected, locally finite graph $\mathcal{J}$ one can associate the Markov (or the random walk) operator M acting on the Hilbert space $l^2(\mathcal{J}, \deg)$ determined by the weight deg (degree of vertices) as follows:

$$(Mf)(x) = \frac{1}{\deg(x)} \sum_{y \sim x} f(y)$$

where $x, y \in V(\mathcal{J})$ which is the set of the vertices of $\mathcal{J}$ and $\sim$ denotes the incidence relation. The operator M is a self adjoint operator and the spectrum $sp(M)$ (i.e. the set of λ such that the operator $M - \lambda$ is not invertible on $l^2(\mathcal{J})$) is a subset of $[-1, 1]$. The spectral measure $\mu_x(\lambda)$, $x \in V(\mathcal{J})$ is defined by the relation $p^{(n)}(x, x) = \int_{-1}^{1} \lambda^n d\mu_x(\lambda)$ and may depend on x (but in case of the Cayley graph $\mu_x(\lambda)$ does not depend on x).

We also consider an operator L in $l^2(G/P)$ where P is the so-called parabolic subgroup of G, i.e. the group of the type $P = St_G(e)$, $e \in \partial T$. The operator L is given by the same averaged sum of generators as M, considered as an operator for the quasi-regular representation $\lambda_{G/P}$.

For Schreier graphs the degree of all vertices coincide with the number $|S|$ of generators. For graphs with uniformly bounded degree there is a criterion when $1 \in Sp(M)$. Namely 1 is a point in the spectrum of M if and only if the graph $\mathcal{J}$ is amenable i.e. there is a sequence $\{F_n\}_{n=1}^{\infty}$ (called a Følner sequence) of finite subsets of $V(\mathcal{J})$ such that $\bigcup_n F_n = V(\mathcal{J})$ and $\lim_{n \to \infty} \frac{|\partial F_n|}{|F_n|} = 0$ where ∂F_n is the boundary of the set F_n and $|\cdot|$ denotes the cardinality.

This is a theorem of Kesten [**25**] in case of the Cayley graphs and in case of graphs with uniformly bounded degree this was proven in [**10**].

1.6. Some properties of the group *G*. Let as before G be the group given by the automaton from Figure 1 (we will keep this notation until the end of the paper). In this section we recall properties of the group G regarding fractalness, contracting of the action, branchness, torsion, growth, presentation, just non-solvability and classes NF and SG which were obtained in our previous work [**20**].

Definitions of fractal, contracting and weakly branch group were given in Section 1.4. As a reference for growth we give [**24**]. An infinite group is just non-solvable if every proper quotient is solvable. The classes AG, EG, SG and NF are defined as follows. AG is the class of all discrete amenable groups, i.e. those groups G for which there exists a Banach mean on $\ell^\infty(G)$ [**12**]. EG is the class of elementary amenable groups, i.e. the smallest class containing finite and abelian groups which is closed under taking extensions, subgroups, factor groups and direct limits. NF is the class of groups which do not contain a free group of rank 2. The classes EG and NF first appeared in [**9**]. Finally let SG be the class of sub-exponential amenable groups, i.e. the smallest class of groups containing groups of sub-exponential growth and closed under taking extensions and direct limits. This class was introduced in [**24**] (see page 64) where the following problem was stated

PROBLEM 1. *Is it true that $SG = AG$?*

THEOREM 1.5 ([20]). *Let G be the group generated by the automaton from Figure 1. The group G*

a) is fractal;

b) is regular weakly branch over G';

c) is torsion free;

d) has exponential growth;

e) is just non-solvable;

f) has a presentation:

$$G = \langle a, b | \sigma^i([a, a^{b^j}]) = 1, i = 0, 1, \dots, j = 1, 3, \dots \rangle,$$

where

$$\sigma : \begin{cases} a & \rightarrow & b^2 \\ b & \rightarrow & a \end{cases}$$

g) has solvable word problem;

h) is not in the class SG;

i) is in the class NF;

j) is contracting;

k) $\phi(St_G(1)) \geq B \times B$ and $G/G' = \mathbb{Z}^2 = \langle \overline{a}, \overline{b} \rangle$, where $B = \langle b \rangle^G$ and $\overline{x}$ denotes the image of the element x.

As in this paper we will use the result that G is contracting, let us state it precisely:

PROPOSITION 1 ([20]). *The group G is contracting with parameters $\lambda = \frac{2}{3}$, $C = 1$ and $L = 1$.*

For the proof of solvabilty of the conjugacy problem we also need:

LEMMA 1.6 ([20]). *a) Let $g \in St_G(1), g = (g_0, g_1)$. Then*

$$|g_0| + |g_1| \leq |g|. \tag{1.2}$$

and $|g_i| < |g|$ $i = 0, 1$ if

$$g \notin \{a^n, b^{-1}a^n b, ba^n b^{-1}, n \in \mathbb{Z} \setminus 0\} = E.$$

b) Let $g \notin St(1)$, $g = (g_0, g_1)\varepsilon$. Then $|g_0| + |g_1| \leq |g|$ and $|g_i| < |g|$, $i = 0, 1$ if

$$g \notin \{a^n b, b^{-1}a^n, n \in \mathbb{Z}\} = F.$$

1.7. Embedding of G into a finitely presented group. As was observed by L. Bartholdi the presentation from Theorem 1.5 can be simplified. We will use this simplification to embed G into a finitely presented group.

PROPOSITION 2. *The group G has the following presentation*

$$G = \langle a, b | \sigma^k([a, a^b]) = 1, k = 0, 1, \dots \rangle.$$

PROOF. As

$$a^{b^i} \equiv [a^{-b}, b^{-2}] a^{b^{i-2}} \mod [b^2, b^{2a}],$$

$$[b^2, b^{2a}] = [a, a^b]^\sigma$$

and

$$[[a^{-b}, b^{-2}], a] \equiv 1 \mod [a, a^b]$$

we have for any odd i

$$[a, a^{b^i}] = 1 \mod [a, a^b], [a, a^b]^\sigma$$

and hence

$$[a, a^{b^i}]^{\sigma^k} \equiv 1 \mod [a, a^b]^{\sigma^k}, [a, a^b]^{\sigma^{k+1}}.$$

Therefore the presentation f) from Theorem 1.5 can be reduced to the presentation from Proposition 2. □

The following embedding theorem uses the same idea as [**16**].

THEOREM 1.7. *The group G embeds into a finitely presented group*

$$\begin{aligned}\widetilde{G} &= \langle a, b, t | [a^b, a] = 1, a^t = b^2, b^t = a\rangle \\ &= \langle b, t | [b^{tb}, b^t] = 1, b^{t^2} = b^2\rangle\end{aligned}$$

which is an excending HNN-extension

$$\widetilde{G} = \langle G, t | t^{-1} g t = \sigma(g), g \in G\rangle.$$

PROOF. The relators of $\widetilde{G}$ are the relators of G and the relators coming from the action of t by a conjugation on generators of G. Thus

$$\begin{aligned}\widetilde{G} &= \langle a, b, t | \sigma^k[a, a^b] = 1, k = 0, 1, \dots, a^t = \sigma(a), b^t = \sigma(b)\rangle \\ &= \langle a, b, t | [a^b, a] = 1, a^t = b^2, b^t = a, \sigma^k([a, a^b]) = 1, k = 0, 1, \dots\rangle\end{aligned}$$

But the relations $\sigma^k([a, a^b]) = 1$, $k = 1, 2, \dots$ are consequences of the first three relations which lead to the finite presentation given in Theorem 1.7. □

COROLLARY 1. *The group $\widetilde{G}$ does not contain a free group of rank 2 and is not in the class SG.*

PROOF. The following exact sequence holds

$$1 \to Q \to \widetilde{G} \to C \to 1$$

where $Q = \bigcup_{n=1}^{\infty} t^n G t^{-n}$ and C is the cyclic group generated by t.

If the group $\widetilde{G}$ were in SG then because the class SG is closed under taking subgroups the group G would be also in SG as Q is isomorphic to a subgroup of $\widetilde{G}$ and Q contains a copy of G.

Because $G \in NF$ we get $Q \in NF$. As $C \in NF$ this implies $\widetilde{G} \in NF$. □

COROLLARY 2. *The group $\widetilde{G}$ is amenable if and only if G is amenable.*

PROBLEM 2. *Is the group G amenable?*

If the answer to the above question is yes, we obtain first examples of groups which are amenable but which are not in the class SG. This would also answer Problem 1.

If the answer is no, we get the first example of a finitely generated, residually finite non-amenable group without F_2 and an example finitely presented nonamenable group without F_2. Recently an example of a finitely presented nonamenable group without F_2 was constructed by A. Y. Olshanski and M. Sapir [**28**]. If $\widetilde{G}$ happens to be nonamenable we will get a much simpler example of this sort given by a balanced presentation. Let us remind that another candidate for such an example is the R. Thompson group [**7**].

2. Conjugacy problem for G

For an element $x \in G$ let $|x|_a$ denote the exponent sum of a in x. This does not depend on a word representing x (by Lemma 1 in [**20**]).

LEMMA 2.1 ([**20**]). *A pair (x, y) with $x, y \in G$ determines an element of $St_G(1)$ if and only if $|x|_a = |y|_a$.*

PROOF. (compare with the proof of Lemma 2 in [**20**]) As $St_G(1)$ is generated by $a = (1, b)$, $a^b = (b^a, 1)$ and $b^2 = (a, a)$, it is clear that if $(x, y) \in St_G(1)$ then $|x|_a = |y|_a$.

Assume that $x, y \in G$ and $|x|_a = |y|_a$. Using a and b^2 we can reduce (x, y) to the form $(z, 1)$, $|z|_a = 0$. But $z \in B = \langle b \rangle^G$ and $G \geq B \times B$. □

Let $C(g)$ denote the centralizer of the element g.

LEMMA 2.2.

$$C(a) = \langle a \rangle \bmod G',$$
$$C(b) = \langle b \rangle \bmod G'.$$

PROOF. If $C(a) \ni g$ then $g \in St(1)$. Indeed if $g = (x, y)\varepsilon$ then $gag^{-1} = (xbx^{-1}, 1) \neq (1, b) = a$. Moreover $g = (x, y)$ and $y \in C(b)$ because of the relation $a = (1, b)$ and fractalness of G. We use induction on the length of g. We have $|y| \leq |g|$ and by Lemma 1.6 $|y| < |g|$ if $g \notin E$. Then $y = b^i z$ where $z \in G'$.

Lemma 2.1 implies that $|x|_a = 0$ as $(x, b^i z) \in G$. Thus $x = b^j t$ for $t \in G'$ and $g = (x, y) = (b^j, b^i) \bmod G' \times G'$. But as $[a, b] = (b, b^{-1}) \bmod G' \times G'$ we get $g = (1, b^{i-j}) \bmod G'$ as $G' \geq G' \times G'$. Thus $g \in \langle a \rangle \bmod G'$ as $a = (1, b)$. So the induction works for the proof of the first relation because the cases when $g \in E$ can be easily verified.

Let $g \in C(b)$. Then either $g \in St(1)$ or $gb \in St(1)$. So we can assume $g \in St(1)$. If $g = (x, y)$ commutes with b then $x = y$ and $x \in C(a)$. Indeed we have in this case $(1, a)\varepsilon = gbg^{-1} = (xy^{-1}, yax^{-1})\varepsilon$.

Let $g = (x, x), x \in C(a)$ and $g \notin E$. Then $x = a^i z, z \in G'$ by inductive assumption. But in this case

$$g = (a^i, a^i) \bmod G' = b^{2i} \bmod G'$$

and lemma is proved because the case $g = (x, x)$, $x \in C(a)$, $g \in E$ can be easily verified. □

Theorem 1.1 for G follows from

PROPOSITION 3. *For any pair $g, h \in G$ solutions of the equation*

$$g^f = h \tag{2.1}$$

in the group G constitute either an empty set or in projection on $\mathbb{Z}^2 = G/G'$ give a set $Z = Z(g, h)$ which is a union of finitely many cosets $w_k C_k$ of subgroups C_k in $\mathbb{Z}^2$.

Given (g, h) the set Z can be constructed effectively by producing elements $w_k \in \mathbb{Z}^2$ and generating sets for subgroups C_k.

Indeed this implies Theorem 1.1 because g and h are conjugated if and only if the set of solutions of (2.1) is nonempty.

PROOF. Proposition 3 will be deduced from the following lemmas.

LEMMA 2.3. *$St_G(1)$ after abelianization can be identified with a subgroup $\mathbb{Z}^3$ in $(\mathbb{Z}^2 \times \mathbb{Z}^2)$ generated by elements $(1, \overline{b}), (\overline{b}, 1), (\overline{a}, \overline{a})$, where $\overline{x}$ is the image of x in $\mathbb{Z}^2 = G/G'$.*

PROOF. This follows from Theorem 1.5 k) and the fact that $St_G(1)$ is generated by $a = (1, b)$, $a^b = (b^a, 1)$ and $b^2 = (a, a)$. $\square$

LEMMA 2.4. *Let $X = \bigcup u_i A_i$, $Y = \bigcup v_j B_j$ be unions of finitely many cosets in $\mathbb{Z}^2$ $(A_i, B_j < G)$. Then there exists a union $Z = \bigcup w_k C_k$ of finitely many cosets in $\mathbb{Z}^2$ $(C_k < \mathbb{Z}^2)$ such that the projection in $\mathbb{Z}^2 = G/G'$ of the set*

$$\{f \in St_G(1) : f = (f_0, f_1), \overline{f_0} \in X, \overline{f_1} \in Y\} \tag{2.2}$$

coincides with the set Z.

Moreover the set Z can be constructed effectively by given data for X, Y. A similar statement also holds in a situation when (2.2) is replaced by

$$\{f \notin St_G(1) : f = (f_0, f_1)\varepsilon, \overline{f_0} \in X, \overline{f_1} \in Y\} \tag{2.3}$$

PROOF. Consider first the set (2.2).

As $\overline{f_0} \in X, \overline{f_1} \in Y$ we have the inclusions $\overline{f_0} \in u_i A_i$, $\overline{f_1} \in v_j B_j$ for some i, j (all such pairs i, j can be effectively found).

Given integer coordinates of u_i, v_j and of generators of A_i, B_j one can express the inclusions $\overline{f_0} \in u_i A_i$, $\overline{f_1} \in v_j B_j$ in the form of relations

$$\begin{cases} (f_0)_a &= (u_i)_a + \sum_\alpha m_\alpha (s_i^\alpha)_a \\ (f_0)_b &= (u_i)_b + \sum_\alpha m_\alpha (s_i^\alpha)_b \\ (f_1)_a &= (v_j)_a + \sum_\beta n_\beta (t_j^\beta)_a \\ (f_1)_b &= (v_j)_b + \sum_\beta n_\beta (t_j^\beta)_b \end{cases} \tag{2.4}$$

where $m_\alpha, n_\beta \in \mathbb{Z}$, $\{s_i^\alpha\}$ is a generating set of A_i, $\{t_j^\beta\}$ is a generating set of B_j. The vector (f_0, f_1) determines an element of G if and only if $|f_0|_a = |f_1|_a$. Thus

$$(u_i)_a + \sum_\alpha m_\alpha (s_i^\alpha)_a = (v_j)_b + \sum_\beta n_\beta (t_j^\beta)_b. \tag{2.5}$$

Keeping in (2.4) parameters $m_\alpha, n_\beta \in \mathbb{R}$ satisfying the condition (2.5) we get some lattice in $\mathbb{R}^4$ defined by equations with integer coefficients which is a union of finitely many cosets wC for some $w \in \mathbb{Z}^4, C < \mathbb{Z}^4$ and can be determined effectively. Projection of the coset wC in $\mathbb{Z}^4$ is again a coset. The image of $St(1)$ is a subgroup A of index 2 in the abelianization $\mathbb{Z}^2$ of G. The abelianization of $St(1)$ (which is $\mathbb{Z}^3$) projects on $\mathbb{Z}^2$ in a canonical way and the set Z from the statement of Lemma 2.3 is the projection of the union of the above cosets, and thus can be determined effectively. This finishes the proof of the first part of the statement.

The second part easy reduces to the first. Indeed if $f \notin St_G(1)$ then $bf = (1, a)\varepsilon(f_0, f_1)\varepsilon = (f_1, af_0)$ and $\overline{f_1} \in Y, \overline{af_0} \in aX$. But aX is a union of cosets the data for which can be constructed with data for X effectively. $\square$

LEMMA 2.5. *Let $X = \bigcup u_i A_i$ be a union of finitely many cosets in $\mathbb{Z}^2$ $(A_i < G)$ and $\xi, \eta \in G$. Then there exists a union $Z = \bigcup w_k C_k$ of finitely many cosets in $\mathbb{Z}^2$ $(C_k < \mathbb{Z}^2)$ such that the projection in $\mathbb{Z}^2 = G/G'$ of the set*

$$\{f \in St_G(1) : f = (f_0, \xi f_0 \eta), \overline{f_0} \in X\} \tag{2.6}$$

coincides with the set Z.

Moreover the set Z can be constructed effectively by given data for X. A similar statement also holds in a situation when (2.6) is replaced by

$$\{f \notin St_G(1) : f = (f_0, \xi f_0 \eta)\varepsilon, \overline{f_0} \in X\} \tag{2.7}$$

PROOF. Consider first the set (2.6).

As $\overline{f_0} \in X$ we have the inclusions $\overline{f_0} \in u_i A_i$ for some i (all such i can be effectively found).

Given integer coordinates of u_i and of generators of A_i one can express the inclusion $\overline{f_0} \in u_i A_i$ in the form of relations

$$\begin{cases} (f_0)_a &= (u_i)_a + \sum_\alpha m_\alpha (s_i^\alpha)_a \\ (f_0)_b &= (u_i)_b + \sum_\alpha m_\alpha (s_i^\alpha)_b \end{cases} \tag{2.8}$$

where $m_\alpha \in \mathbb{Z}, \{s_i^\alpha\}$ is a generating set of A_i. The vector (f_0, f_1) determines an element of G if and only if $|f_0|_a = |f_1|_a$. Thus for $f_1 = \xi f_0 \eta$, we have $(f_0, f_1) \in G$ if and only if

$$|\xi|_a + |\eta|_a = 0 \text{ and } |\xi|_b + |\eta|_b = 0. \tag{2.9}$$

Satisfying the condition (2.9) we get some lattice in $\mathbb{R}^4$ defined by equations with integer coefficients which is a union of finitely many cosets wC for some $w \in \mathbb{Z}^4, C < \mathbb{Z}^4$ and can be determined effectively. Projection of the coset wC in $\mathbb{Z}^4$ is again a coset. The image of $St(1)$ is a subgroup A of index 2 in the abelianization $\mathbb{Z}^2$ of G. The abelianization of $St(1)$ (which is $\mathbb{Z}^3$) projects on $\mathbb{Z}^2$ in a canonical way and the set Z from the statement of Lemma 2.3 is the projection of the union of the above cosets, and thus can be determined effectively. This finishes the proof of the first part of the statement.

The second part easy reduces to the first. □

We also use the following lemma.

LEMMA 2.6. *Let $f, g, h \in G$ and assume that the relation (2.1) holds (in particular $g, h \in St(1)$ or $g, h \notin St(1)$).*

1. *Let $g, h \in St(1)$, $g = (g_0, g_1)$, $h = (h_0, h_1)$.*
   a) If $f \in St(1)$, $f = (f_0, f_1)$ then (2.1) is equivalent to

$$\begin{cases} g_0^{f_0} &= h_0 \\ g_1^{f_1} &= h_1 \end{cases} \tag{2.10}$$

   b) If $f \notin St(1)$, $f = (f_0, f_1)\varepsilon$ then (2.1) is equivalent to

$$\begin{cases} g_0^{f_0} &= h_1 \\ g_1^{f_1} &= h_0 \end{cases} \tag{2.11}$$

2. *Let $g, h \notin St(1)$, $g = (g_0, g_1)\varepsilon$, $h = (h_0, h_1)\varepsilon$.*
   a) If $f \in St(1)$, $f = (f_0, f_1)$ then (2.1) is equivalent to

$$\begin{cases} (g_0 g_1)^{f_0} &= h_0 h_1 \\ f_1 &= g_1 f_0 h_1^{-1} \end{cases} \tag{2.12}$$

b) If $f \notin St(1)$, $f = (f_0, f_1)\varepsilon$ then (2.1) is equivalent to

$$\left\{ \begin{array}{ccc} (g_0 g_1)^{f_0} & = & h_1 h_0 \\ f_1 & = & g_0^{-1} f_0 h_1 \end{array} \right. \tag{2.13}$$

We use the induction on the number $|g| + |h| \geq 2$. If $|g| = |h| = 1$ then as $a \in St(1)$ and $b \notin St(1)$, $g = h$ and the set of solutions of (2.1) is $C(a)$ or $C(b)$ and we can apply Lemma 2.2. Thus we obtain the base of induction.

If $g, h \notin X$ or $g, h \notin Y$, depending on case 1 or 2, we can use inductive assumption for each of the equations entering any of the system. The projection of the set of solutions is a union of finitely many cosets a data of which is effectively constructed. The intersection of such sets has again a similar form and can be effectively computed.

By Lemma 2.4, Lemma 2.5 and Lemma 2.6 the set of solutions of (2.1) after projection in $\mathbb{Z}^2$ again is a union of finitely many cosets a data of which can be effectively constructed. This finishes the proof of Proposition 3. □

3. Approximation method

We have three different operators M, L and Z. The operator M is the Markov operator on $l^2(G)$, L is the Markov operator on $l^2(G/P)$ and Z is the Hecke type operator in $L^2(\partial T, \mu)$. First, we are going to provide some approximation method for the computation of the spectrum.

Let $G_n = G/St_G(n)$ be a finite group acting transitively on the set V_n, $|V_n| = 2^n$, of the vertices of the n-th level. Let P be a parabolic subgroup determined by a path $e \in \partial T$, i.e. $P = St_G(e)$, and let e_n be a vertex of the path e belonging to the n-th level. Then V_n can be identified with $G/St_G(e_n) \simeq G_n/St_{G_n}(e_n)$.

Let us denote $P_n = St_G(e_n)$ and $\overline{P}_n = St_{G_n}(e_n)$ the image of P_n in G_n under the canonical factorization $G \to G_n$. Consider the Schreier graphs $\mathcal{J} = C(G, P)$, $\mathcal{J}_n = C(G, P_n)$ and $\overline{\mathcal{J}_n} = C(G_n, \overline{P}_n)$ constructed with respect to the system of generators a, b for G and their images in G_n. These graphs are marked with a distinguished point corresponding to the cosets P, P_n and $\overline{P}_n$. Then $\mathcal{J}_n = C(G, P_n)$ and $\overline{\mathcal{J}_n} = C(G_n, \overline{P}_n)$ are finite, canonically isomorphic graphs and $\mathcal{J}_n$ converges to $\mathcal{J}$ in the sense of [**19**] because $P = \bigcap_{n=1}^{\infty} P_n$.

Let L_n be the Markov operator on $\mathcal{J}_n$. Then $\{L_n\}_{n=1}^{\infty}$ is a sequence of finite dimensional operators approximating the operator L, i.e. the Kesten spectral measures of L_n converge to the Kesten spectral measure of L and therefore we have convergence of spectra as well.

Let now $\xi_n = \{E_w; w \in \{0,1\}^n\}$ be a partition of the boundary ∂T on 2^n atoms, where E_w is an open and closed subset consisting of paths starting with the string w.

Let $\mathcal{H} = L^2(\partial T, \mu)$ and $\mathcal{H}_n$ be a subspace of $\mathcal{H}$ spanned by characteristic functions of atoms of the partition ξ_n. Then $\dim \mathcal{H}_n = 2^n$ and $\mathcal{H}_n$ is invariant with respect to the unitary representation π, determined by the dynamical system $(G, \partial T, \mu)$ because any automorphism of a tree permutes the atoms of the partition ξ_n.

Let $\pi_n = \pi|_{\mathcal{H}_n}$. It is easy to see that this is a permutation-like representation of a group G arising from the action of G on the set of 2^n atoms of ξ_n. This is not a faithful representation and the factorization by the kernel gives a faithful permutation-like representation $\overline{\pi_n}$ of G_n on V_n. From these considerations we conclude that π_n is isomorphic to the quasi-regular representation λ_{G/P_n} and $\overline{\pi_n}$ is isomorphic to $\lambda_{G_n/\overline{P}_n}$.

The Hecke type operator for the representation $\lambda_{G/P}$ (the average sum of operators, corresponding to generators) is equivalent to the Markov operator L on the Schreier graph $\mathcal{J}$ and similarly the operators λ_{G/P_n}, $\lambda_{\overline{G}_n/P}$ are equivalent to the operator L_n. Let λ_G be the regular representation and let $sp(\lambda_{G/P})$ denote the spectrum of the corresponding Markov operator.

PROPOSITION 4 ([**1**]). *Let G be amenable. Then*

$$sp(\lambda_{G/P}) = sp(\pi) = \overline{\bigcup_{n\geq 0} sp(\pi_n)} \subseteq sp(\lambda_G).$$

We do not know if the group studied in this paper is amenable. However we are able to use

PROPOSITION 5 ([**1**]). *If either P or G/P is amenable then*

$$sp(\lambda_{G/P}) = sp(\pi) = \overline{\bigcup_{n\geq 0} sp(\pi_n)}.$$

4. Spectrum of the automaton

There are two natural ways to associate to an automaton A the spectrum $sp(A)$. The first one is $sp(\pi)$ where π is the unitary representation of $G(A)$ described in Section 3 and the second one is $sp(\lambda_{G(A)/P})$ for the corresponding quasi-regular representation. In case where the action of G on the corresponding tree T is contracting these two spectra coincide (as sets), because in this situation the graph $\mathcal{S}(G,P)$ has polynomial growth (see [**1**]) and thus G/P is amenable. As follows from Theorem 1.5 j) we are exactly in this situation, as G/P has polynomial growth. There is also no dependence on the path e determining the parabolic subgroup as for any $\xi, \nu \in \partial T$ the corresponding Schreier graphs are locally isomorphic and therefore they have the same spectra.

For the study of $sp(A)$ of the automaton from Figure 1 let us introduce two parameter matrix and its determinant:

$$\begin{aligned} Q_n(\mu,\lambda) &= a_n + a_n^{-1} + \mu(b_n + b_n^{-1}) - \lambda, \\ \Psi_n(\mu,\lambda) &= \det(a_n + a_n^{-1} + \mu(b_n + b_n^{-1}) - \lambda). \end{aligned}$$

For $n = 0, 1$

$$\begin{aligned} \Psi_0 &= 2 + 2\mu - \lambda, \\ \Psi_1 &= (2 + 2\mu - \lambda)(2 - 2\mu - \lambda). \end{aligned}$$

Let also

$$Q(\mu,\lambda) = \pi(a) + \pi(a^{-1}) + \mu(\pi(b) + \pi(b^{-1})) - \lambda.$$

THEOREM 4.1. *The spectrum Σ of $Q(\mu,\lambda)$, i.e. the set of pairs (μ,λ) (including multiplicities) for which the operator $Q(\mu,\lambda)$ is not invertible, is invariant with respect to the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$F : \begin{cases} \lambda & \to & -2 - \frac{\lambda(2-\lambda)}{\mu^2} \\ \mu & \to & \frac{\lambda-2}{\mu^2} \end{cases},$$

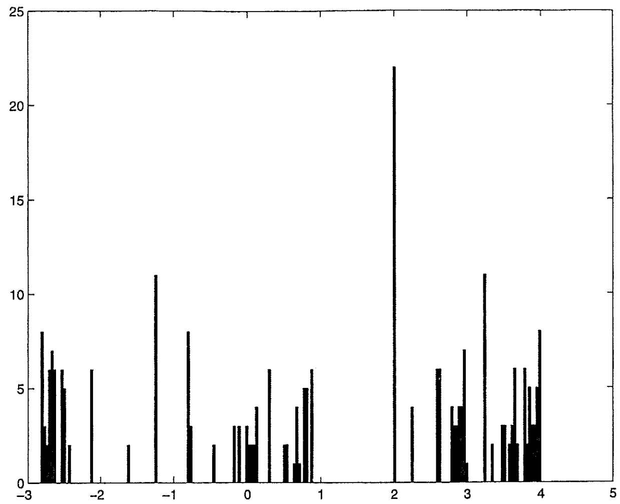

FIGURE 2. The histogram of the spectrum of the automaton A

i.e. $F^{-1}(\Sigma) = \Sigma$.

LEMMA 4.2. *If* $n \geq 1$ *the following recursion holds:*

$$\Psi_{n+1}(\mu, \lambda) = \mu^{2^{n+1}} \Psi_n \left(F(\lambda, \mu)\right).$$

PROOF. First of all let us recall a well known fact (see for instance [**19**])

LEMMA 4.3. *Let* A, B, C *and* D *be* n *by* n *matrices with complex coefficients such that* $AC = CA$. *Then*

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - CB).$$

Applying the above formula we get

$$\begin{aligned}
\Psi_{n+1}(\mu, \lambda) &= \det(a_{n+1} + a_{n+1}^{-1} + \mu(b_{n+1} + b_{n+1}^{-1}) - \lambda) \\
&= \det \begin{pmatrix} 2 - \lambda & \mu(1 + a_n^{-1}) \\ \mu(1 + a_n) & b_n + b_n^{-1} - \lambda \end{pmatrix} \\
&= \det((2 - \lambda)(b_n + b_n^{-1} - \lambda) - \mu^2(1 + a_n)(1 + a_n^{-1})). \\
&= \det(-\mu^2(a_n + a_n^{-1}) + (2 - \lambda)(b_n + b_n^{-1}) - (2\mu^2 + \lambda(2 - \lambda))) \\
&= \mu^{2^{n+1}} \Psi_n \left(\frac{\lambda - 2}{\mu^2}, -2 - \frac{\lambda(2 - \lambda)}{\mu^2} \right) = \mu^{2^{n+1}} \Psi_n \left(F(\lambda, \mu)\right).
\end{aligned}$$

Thus the solutions of the equation $\Psi_{n+1}(\mu, \lambda) = 0$ are the preimages of the solutions of the equation $\Psi_n(\mu, \lambda) = 0$. Using Proposition 5 we obtain the statement of Theorem 4.1. □

Let

$$\begin{aligned}
\Phi_0(\mu, \lambda) &= 2 + 2\mu - \lambda, \\
\Phi_1(\mu, \lambda) &= 2 - 2\mu - \lambda, \\
\Phi_2(\mu, \lambda) &= 4\mu^2 + 4 - \lambda^2,
\end{aligned}$$

and in general

$$\Psi_n(\mu, \lambda) = \Phi_0 \Phi_1 \dots \Phi_n.$$

It is easy to see that if $n \geq 1$ the recursion

$$\Phi_{n+1}(\mu, \lambda) = \mu^{2^n} \Phi_n \left(F(\mu, \lambda)\right)$$

holds, which is similar to the recursion for Ψ_n.

From Theorem 4.1 we see that the spectral properties of the automaton A (or a group G) depend very much on dynamical properties of the map F. The dynamic of the rational map of the plane in the general case is very complicated and probably this is also the case for F. We hope in future to get more detailed picture for iterations of F. Here we provide just first observations.

The essential difference from the examples considered in [**1**] and [**19**] is that the curves $F^{-1}(q)$ are not union of lines and hyperbolas and the degree of them is increasing. This makes the study of F (and the study of the spectrum) more complicated.

We know that

$$\mathit{spectrum} = \overline{\bigcup_{n=0}^{\infty} F^{-n}(q)}$$

where q is the curve $2\mu + 2 - \lambda = 0$. The only fixed points of F are solutions of

$$\mu^4 - \mu^3 + 2\mu - 4 = 0,$$

$$\lambda = \mu^3 + 2.$$

Thus the only real fixed points of F are

$$\begin{aligned} (\mu_0, \lambda_0) &= (\sqrt{2}, 2\sqrt{2} + 2), \\ (\mu_1, \lambda_1) &= (-\sqrt{2}, -2\sqrt{2} + 2). \end{aligned}$$

We have the following decomposition into irreducible polynomials:

$$
\begin{aligned}
\Psi_1(\mu,\lambda) &= 2\mu+2-\lambda,\\
\Psi_2(\mu,\lambda) &= -(2\mu-2+\lambda)(2\mu+2-\lambda),\\
\Psi_3(\mu,\lambda) &= (2\mu+2-\lambda)(2\mu-2+\lambda)(4\mu^2-\lambda^2+4)\\
\Psi_4(\mu,\lambda) &= -(-2+\lambda)(2\mu+2-\lambda)(2\mu-2+\lambda)\\
&\quad (2\lambda^2-8-\lambda^3+4\lambda+4\mu^2\lambda)(4\mu^2-\lambda^2+4)\\
\Psi_5(\mu,\lambda) &= (-2+\lambda)^2(2\mu+2-\lambda)(2\mu-2+\lambda)(4\mu^2+2\lambda-\lambda^2)\\
&\quad (4\mu^2+4-\lambda^2)(4\mu^2\lambda-8+4\lambda+2\lambda^2-\lambda^3)\\
&\quad (16\mu^4\lambda-8\mu^2\lambda^3+16\mu^2\lambda^2+8\mu^2\lambda-16\mu^2\\
&\quad +\lambda^5-4\lambda^4+16\lambda^2-16\lambda)\\
\Psi_6(\mu,\lambda) &= -(-2+\lambda)^3(2\mu-2+\lambda)(2\mu+2-\lambda)\\
&\quad (2\lambda^2-8-\lambda^3+4\lambda+4\mu^2\lambda)(4\mu^2-\lambda^2+4)\\
&\quad (16\mu^4\lambda-16\mu^2+8\mu^2\lambda-16\lambda+16\mu^2\lambda^2+16\lambda^2-8\mu^2\lambda^3\\
&\quad -4\lambda^4+\lambda^5)(8\mu^4-16+12\mu^2\lambda+16\lambda-6\mu^2\lambda^2-4\lambda^3+\lambda^4)\\
&\quad (-8\lambda^8+16\lambda^7-384\mu^2\lambda-160\mu^6\lambda^3-320\mu^2\lambda^2+72\mu^4\lambda^5\\
&\quad -14\mu^2\lambda^7-512\lambda^2+256\mu^2+256\lambda-608\mu^4\lambda+608\mu^4\lambda^2\\
&\quad +128\lambda^4+256\lambda^3+128\mu^8\lambda+736\mu^2\lambda^3+320\mu^6\lambda^2+96\mu^6\lambda\\
&\quad -272\mu^2\lambda^4-160\lambda^5+136\mu^4\lambda^3-288\mu^4\lambda^4+32\lambda^6+\lambda^9\\
&\quad -104\mu^2\lambda^5+84\mu^2\lambda^6-192\mu^6)(4\mu^2+2\lambda-\lambda^2)^2.
\end{aligned}
$$

The corresponding curves are drawn in Figure 3. The equation $\Psi_n(1,\lambda)=0$ gives the spectrum of the n-th approximation of the spectrum of the Hecke type operator for Z (or in our case the Markov operator M). Therefore $sp(Z)$ is the set of points from the intersection of curves from Figure 3 with the line $\mu=1$. The histogram for the 7-th approximation is given in Figure 2.

5. Double action of the automaton A

Let us consider the diagonal action of the group $G(A)$ on the product of two trees $T\times T$. Such an action can be described by an automaton over the alphabet $D\times D$ whose diagram is presented in Figure 4.

Let V_n denote the vertices on the n-th level of T. We remark that

LEMMA 5.1. *The unitary representation corresponding to the diagonal action on $V_n\times V_n$ (here we mean the pairs (u,v) with $|u|=|v|$) contains the unitary representation corresponding to the action on V_n. Thus the spectrum of the first includes the spectrum of the second.*

There is the following operator recursion:

$$
(a,a)=\begin{pmatrix} (1,1) & 0 & 0 & 0\\ 0 & (1,b) & 0 & 0\\ 0 & 0 & (b,1) & 0\\ 0 & 0 & 0 & (b,b)\end{pmatrix},
$$

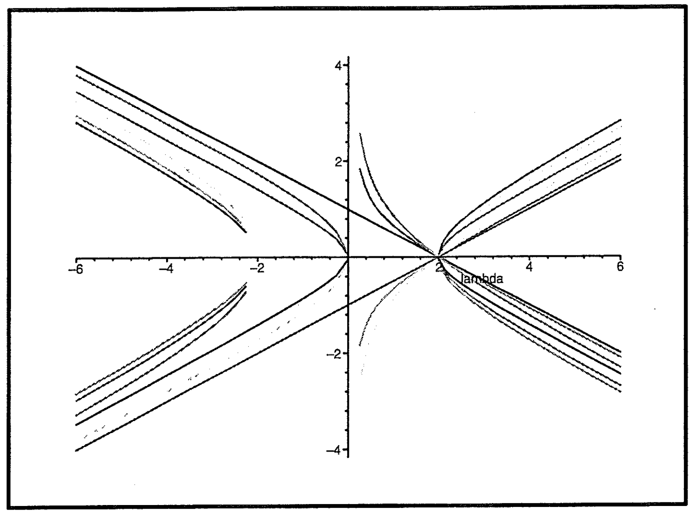

FIGURE 3. Zeros of $\Psi_n(\mu, \lambda)$

$$(a,b) = \begin{pmatrix} 0 & (1,1) & 0 & 0 \\ (1,a) & 0 & 0 & 0 \\ 0 & 0 & 0 & (b,1) \\ 0 & 0 & (b,a) & 0 \end{pmatrix},$$

$$(a,id) = \begin{pmatrix} (1,1) & 0 & 0 & 0 \\ 0 & (1,1) & 0 & 0 \\ 0 & 0 & (b,1) & 0 \\ 0 & 0 & 0 & (b,1) \end{pmatrix},$$

$$(b,a) = \begin{pmatrix} 0 & 0 & (1,1) & 0 \\ 0 & 0 & 0 & (1,b) \\ (a,1) & 0 & 0 & 0 \\ 0 & (a,b) & 0 & 0 \end{pmatrix},$$

$$(b,b) = \begin{pmatrix} 0 & 0 & 0 & (1,1) \\ 0 & 0 & (1,a) & 0 \\ 0 & (a,1) & 0 & 0 \\ (a,a) & 0 & 0 & 0 \end{pmatrix},$$

$$(b,1) = \begin{pmatrix} 0 & 0 & (1,1) & 0 \\ 0 & 0 & 0 & (1,1) \\ (a,1) & 0 & 0 & 0 \\ 0 & (a,1) & 0 & 0 \end{pmatrix},$$

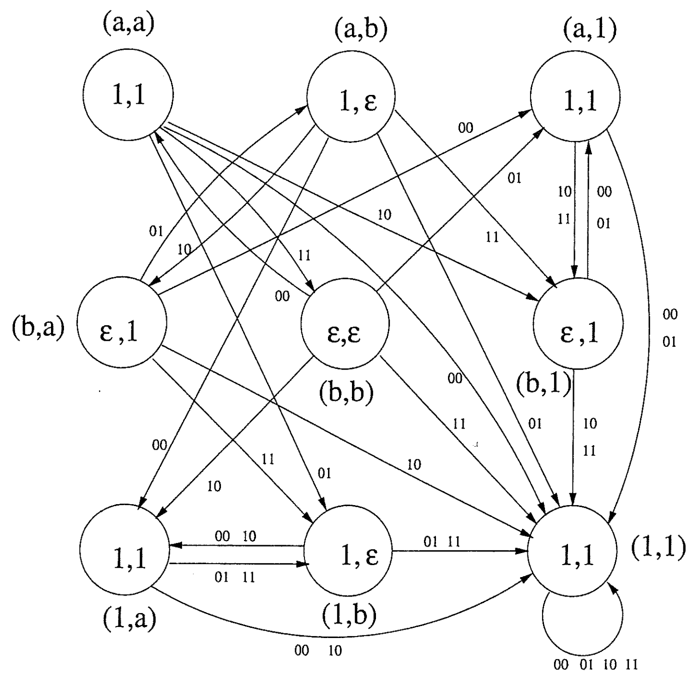

FIGURE 4. The double action of the automaton A

$$(1,a) = \begin{pmatrix} (1,1) & 0 & 0 & 0 \\ 0 & (1,b) & 0 & 0 \\ 0 & 0 & (1,1) & 0 \\ 0 & 0 & 0 & (1,b) \end{pmatrix},$$

$$(1,b) = \begin{pmatrix} 0 & (1,1) & 0 & 0 \\ (1,a) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1,1) \\ 0 & 0 & (1,a) & 0 \end{pmatrix},$$

$$(1,1) = \begin{pmatrix} (1,1) & 0 & 0 & 0 \\ 0 & (1,1) & 0 & 0 \\ 0 & 0 & (1,1) & 0 \\ 0 & 0 & 0 & (1,1) \end{pmatrix}.$$

The computer experiment, which can be used for computation of the spectrum, for the 7-th approximation gives the picture of the histogram (see Figure 5) which shows the disappearance of all gaps in the spectrum. Perhaps this means that the spectrum of the Laplace operator on the group G has no gaps, but further investigation in this direction is necessary.

6. The Brunner-Sidki-Vieira group

Let H be the group generated by the automaton B from Figure 6. Then $H = \langle \mu, \tau \rangle$ where $\tau = (1,\tau)\varepsilon$ and $\mu = (1,\mu^{-1})\varepsilon$. The generator τ is the so called adding machine [**23**].

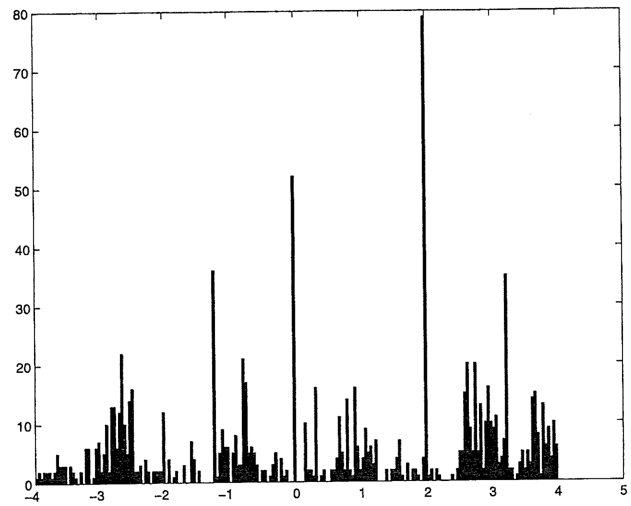

FIGURE 5. The histogram of the spectrum of the double action of the automaton A

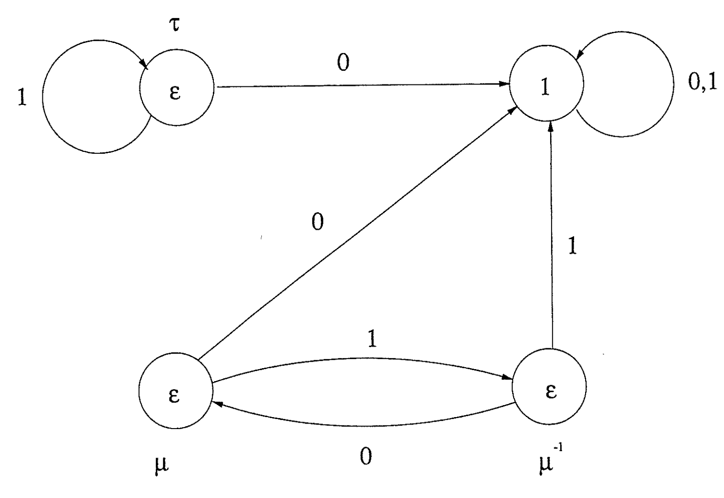

FIGURE 6. The automaton generating the Brunner-Sidki-Vieira group

This group was investigated in [**6**] where it was proven that H has the following presentation in generators $a = \tau$, $b = \tau\mu^{-1}$

$$H = \langle a, b | \xi^k(r), \xi^k(r'), k \geq 0 \rangle,$$

where $r = [b, b^a]$, $r' = [b, b^{a^3}]$ and ξ is a substitution defined by $\xi(a) = a^2$, $\xi(b) = a^2 b^{-1} a^2$.

PROPOSITION 6. *H embeds into a finitely presented group $\widetilde{H}$ which is an ascending HNN extension of H. Explicitly*

$$\widetilde{H} = \langle a, b, t | a^t = a^2, b^t = a^2 b^{-1} a^2, [b, b^a] = 1, [b, b^{a^3}] = 1 \rangle.$$

PROOF. The substitution ξ induces an endomorphism of H. Moreover we have

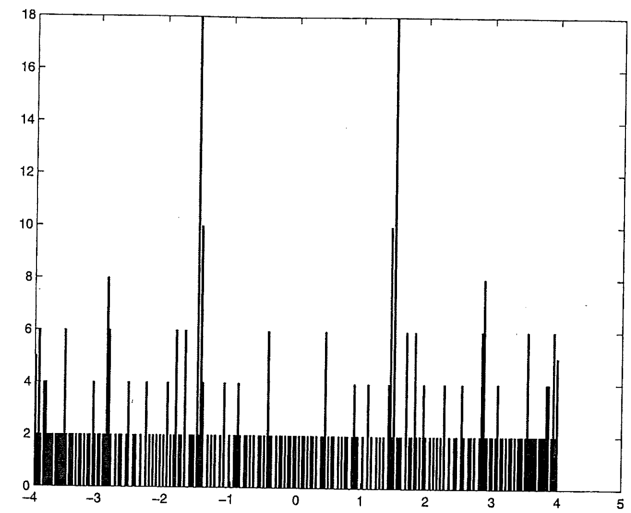

FIGURE 7. The histogram of the spectrum of the Brunner-Sidki-Vieira group

LEMMA 6.1. *For the canonical projection $\pi_2 : H \longrightarrow H$ on the second coordinate*

$$\pi_2(\xi(g)) = g$$

for every $g \in H$.

PROOF. As the group H is generated by a and b this follows from

$$\begin{aligned} \pi_2(\xi(a)) &= \pi_2(a^2) = \pi_2(\tau^2) = \pi_2((\tau,\tau)) = \tau = a, \\ \pi_2(\xi(b)) &= \pi_2(a^2b^{-1}a^2) = \pi_2(\tau^2(\tau\mu^{-1})^{-1}\tau^2) = \pi_2((\tau^2,\tau\mu^{-1})) = \tau\mu^{-1} = b. \end{aligned}$$

□

Thus ξ is injective. For the ascending HNN extension

$$\widetilde{H} = \langle H, t; t^{-1}ht = \xi(h), h \in H\rangle$$

the relations in the above presentation can be easily reduced (similarly as in the proof of Theorem 1.7) to the form as in Proposition 6. □

PROPOSITION 7. *The group H is contracting with parameters $\lambda = \frac{1}{2}, C = \frac{1}{2}, L = 1$ and for any $h \in H$, $h = (h_1, h_2)\varepsilon^i$, where $i \in \{0, 1\}$ we have*

$$|h| \geq |h_1| + |h_2|. \tag{6.1}$$

PROOF. Because for the generators we have $\tau = (1,\tau)\varepsilon$ and $\mu = (1,\mu^{-1})\varepsilon$ the inequality (6.1) is clear. All words of length two, up to taking the inverses, are τ^2, $\tau\mu$, $\mu\tau$, μ^2, $\tau^{-1}\mu$, $\tau\mu^{-1}$. For the elements corresponding to these words we have

$$\begin{aligned} \tau^2 &= (\tau,\tau) \\ \tau\mu &= (\mu^{-1},\tau) \\ \mu\tau &= (\tau,\mu^{-1}) \\ \mu^2 &= (\mu^{-1},\mu^{-1}) \\ \tau^{-1}\mu &= ((\tau^{-1},\mu^{-1}),(1,1)) \\ \tau\mu^{-1} &= ((1,1),(\mu^{-1},\tau)) \end{aligned} \tag{6.2}$$

and we have reduction of the length on the second level by $\frac{1}{2}$. Therefore the statement about the contractness holds for all words of length 2. Up to adding or removing one letter, every non trivial word can be written using a product of words from (6.2). Thus for $C = \frac{1}{2}$ we get the contraction with coefficient $\lambda = \frac{1}{2}$. □

By Proposition 3.13 from [**2**] this implies

COROLLARY 3. *The Schreier graphs $\mathcal{S}(H, P)$ where P is the parabolic subgroup, have polynomial growth. In particular the H space H/P is amenable and for the operators corresponding to the automaton B we have $sp(Z) = sp(L) = sp(M)$ and $1 \in sp(Z)$.*

A similar statement holds for the group G. The Schreier graph of G is closely related to the mapping $z \to z^2 - 1$ ([**3**]).

Unfortunately it seems that the study of spectral properties of the automaton given by Figure 6 is rather a harder problem than for the automaton given by Figure 1. At the moment we do not know if there is a map for which the analogue of Theorem 4.1 holds.

The following two propositions have the proofs similar to the one given in [**19**] so we will omit them. We heard from S. Sidki that Proposition 8 was also proven by Edmeia F. da Silva.

PROPOSITION 8. *The group H does not contain F_2.*

PROPOSITION 9. *The group H is not in the class SG.*

As this finitely presented group $\widetilde{H}$ is an ascending HNN extension of H, these groups are simultaneously either amenable or not, like in Corollary 2.

The histogram of the 7-th approximation of the spectrum of the operator Z for the group H is given in Figure 7.

The next statement gives a closer relation between G and H.

PROPOSITION 10. *The group H embeds in $\overline{G}$ which is a closure of G in $Aut(T)$.*

PROOF. Consider the element $\rho \in G$

$$\rho = a^{-1}b = (1, b^{-1}a)\varepsilon = (1, \rho^{-1})\varepsilon.$$

Thus ρ is the initial automaton which coincides with the automaton μ and so $\rho = \mu$ and $\mu \in G$. Also consider the element $\nu \in \overline{G}$ given as an infinite product

$$\nu = ab(c, 1)(c, 1, 1, 1)(c, 1, 1, 1, 1, 1, 1, 1)\dots,$$

where $c = [a, b]$. For this element one can check the relation $\nu = (1, \nu)\varepsilon$ and so as an automorphism of a binary tree ν coincides with the automorphism τ. □

7. Conjugacy problem for H

In this section we solve the conjugacy problem for $H = \langle \tau, \mu \rangle$ in a similar way we did this for G.

LEMMA 7.1. *The equation in f*

$$(\tau^{\varepsilon})^f = \mu^{\eta}$$

where $\varepsilon, \eta = \pm 1$ are fixed has no solution.

PROOF. We will prove this for $\varepsilon, \eta = 1$ as other cases are similar. Suppose that the equation

$$(\tau)^f = \mu \tag{7.1}$$

has a solution and let us take f to have the minimal length. Clearly this length has to be at least 2. Now $f = ((f^{(1)}, f^{(2)})\varepsilon^i, (f^{(3)}, f^{(4)})\varepsilon^j)\varepsilon^k$, where $i, j, k \in \{\pm 1\}$. Thus we get

$$(\tau)^{f^{(l)}} = \mu$$

for some $l \in \{1, \ldots, 4\}$ which gives a desired contradiction because H is contracting. □

LEMMA 7.2. *For the centralizers $C(\tau)$ and $C(\mu)$ of the elements τ and μ we have:*

$$C(\tau) = \langle\tau\rangle,$$
$$C(\mu) = \langle\mu\rangle.$$

PROOF. We will prove it for $C(\tau)$ as the proof for $C(\mu)$ is similar. Suppose that $C(\tau) \neq \langle\tau\rangle$ and let $z \in C(\tau) \setminus \langle\tau\rangle$ be of minimal length. We have $z = (x, y)$ or $z = (x, y)\varepsilon$ for some $x, y \in H$.

In the case $z = (x, y)$ because $z \in C(\tau)$ we get $x = y$ and $x \in C(\tau)$. By (6.1) we have $|z| \geq |x| + |x|$ which implies $|x| < |z|$. By miminality of $|z|$ we have $x = \tau^k$. But then $z = (\tau^k, \tau^k) = \tau^{2k}$ which gives the desired contradiction.

In the case $z = (x, y)\varepsilon$ because $z \in C(\tau)$ we get $\tau x = y$ and $x \in C(\tau)$. By (6.1) we have $|z| \geq |x| + |\tau x|$ which implies $|x| < |z|$, unless $x = \tau^{-1}$ which implies $z = \tau^{-1}$. By minimality of $|z|$ we have $x = \tau^k$. But then $z = (\tau^k, \tau^{k+1}) = \tau^{2k+1}$ which gives the desired contradiction. □

Theorem 1.1 for H follows from

PROPOSITION 11. *For any pair $g, h \in H$ solutions of the equation*

$$g^f = h \tag{7.2}$$

in the group H constitute either an empty set or in projection on $\mathbb{Z}^2 = H/H'$ give a set $Z = Z(g, h)$ which is a union of finitely many cosets $w_k C_k$ of subgroups C_k in $\mathbb{Z}^2$.

Given (g, h), the set Z can be constructed effectively by producing elements $w_k \in \mathbb{Z}^2$ and generating sets for subgroups C_k.

Indeed this implies Theorem 1.1 because g and h are conjugated if and only if the set of solutions of (7.2) is nonempty.

PROOF. We use here the fact that $H/H' \simeq \mathbb{Z}^2$ which can be found in [**6**]. Proposition 11 will be deduced from the following lemmas.

LEMMA 7.3. *We have*

$$St_H(1) = \langle\tau^2, \mu^2, \tau\mu\rangle.$$

PROOF. $St_H(1)$ consists of elements which are products of τ^2, μ^2, $\tau^{\pm 1}\mu^{\pm 1}$, τ^{-2}, μ^{-2}, $\mu^{\pm 1}\tau^{\pm 1}$. All these elements belong to $\langle\tau^2, \mu^2, \tau\mu\rangle$. □

LEMMA 7.4. *$St_H(1)$ after abelianization can be identified with a subgroup $\mathbb{Z}^3$ in $(\mathbb{Z}^2 \times \mathbb{Z}^2)$ generated by elements $(\overline{\tau}, \overline{\tau}), (\overline{\mu^{-1}}, \overline{\mu^{-1}}), (\overline{\mu^{-1}}, \overline{\tau})$, where $\overline{x}$ is the image of x in H/H'.*

PROOF. This follows from the fact that $St_H(1)$ is generated by $\tau^2 = (\tau, \tau)$, $\mu^2 = (\mu^{-1}, \mu^{-1})$, $\tau\mu = (\mu^{-1}, \tau)$ and the vectors $(\overline{\tau}, \overline{\tau}), (\overline{\mu^{-1}}, \overline{\mu^{-1}}), (\overline{\mu^{-1}}, \overline{\tau})$ in $\mathbb{Z}^2 \oplus \mathbb{Z}^2$ are linearly independent. □

LEMMA 7.5. *Let $X = \bigcup u_i A_i$, $Y = \bigcup v_j B_j$ be unions of finitely many cosets in $\mathbb{Z}^2$ $(A_i, B_j < H)$. Then there exists a union $Z = \bigcup w_k C_k$ of finitely many cosets in $\mathbb{Z}^2$. $(C_k < \mathbb{Z}^2)$ such that the projection in $\mathbb{Z}^2 = H/H'$ of the set*

$$\{f \in St_G(1) : f = (f_0, f_1), \overline{f_0} \in X, \overline{f_1} \in Y\} \tag{7.3}$$

coincides with the set Z.

Moreover the set Z can be constructed effectively by given data for X, Y. A similar statement also holds in a situation when (7.3) is replaced by

$$\{f \notin St_G(1) : f = (f_0, f_1)\varepsilon, \overline{f_0} \in X, \overline{f_1} \in Y\} \tag{7.4}$$

PROOF. We have the following commutative diagram of maps and inclusions:

$$\begin{array}{ccccc}
 & H & \xrightarrow{\rho} & H_{ab} & \simeq \mathbb{Z}^2 \\
 & \triangledown & & \triangledown & \\
 & St_H(1) & \xrightarrow{\rho} & \rho(St_H(1)) & \simeq \mathbb{Z}^2 \\
 & \phi \downarrow & & \uparrow \xi & \\
H \times H > & \phi(St_H(1)) & \xrightarrow{\rho \times \rho} & B & \simeq \mathbb{Z}^3 < \mathbb{Z}^2 \times \mathbb{Z}^2 \simeq H_{ab} \times H_{ab}
\end{array}$$

In the above diagram ρ is a homomorphism of abelianization of H, ϕ is the canonical embedding described in Section 1.4 and B is the image of $\phi(St_H(1))$ under $\rho \times \rho$. The existence of the homomorphism ξ follows from the fact that $\phi(H) > \phi(H') > H' \times H'$ which is proved in [**6**]. The fact that $B \simeq \mathbb{Z}^3$ follows from Lemma 7.4.

LEMMA 7.6. *Let $x, y \in H$. Then $(x, y) \in \phi(St_H(1))$ if and only if $(\overline{x}, \overline{y}) \in B$.*

PROOF. It is obvious that if $(x, y) \in \phi(St_H(1))$ then $(\overline{x}, \overline{y}) \in B$. In order to prove the converse we have to show that if $(x, y) \in Ker(\rho \times \rho)$ then $(x, y) \in H$. The last follows from the fact that $\phi(H) > \phi(H') > H' \times H'$. □

Now the set (7.3) can described as

$$\xi((X \times Y) \cap B)$$

and therefore they are unions of finitely many costes the data of which can be determined effectively. The second part of the statement can be proved by reduction to the first part in a similar manner it was done in Lemma 2.4. This finishes the proof of Lemma 7.5. □

We use the induction on the number $|g| + |h| \geq 2$. If $|g| = |h| = 1$ then it is clear by Lemma 7.1 and Lemma 7.2.

So assume $|g| + |h| > 3$ and use Lemma 2.6. As $|g| > 1$ or $|h| > 1$ we can use inductive assumption for each of the equations entering any of the system. The projection of the set of solutions is a union of finitely many cosets a data of which

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is effectively constructed. The intersection of such sets has again a similar form and can be effectively computed.

For the group H also holds the analogue of Lemma 2.5. Therefore by Lemma 7.5 and Lemma 2.6 the set of solutions of (7.2) after projection in $\mathbb{Z}^2$ again is a union of finitely many cosets a data of which can be effectively constructed.

This finishes the proof of Proposition 11. □

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