Chapter Three

Spectrum of the Laplacian

3.1 FOURIER SERIES REVISITED

The Fourier series on I, sine or cosine, gives the expansion of a general function in terms of eigenfunctions, Dirichlet or Neumann, of the Laplacian. By convention, we write

$$-\Delta u = \lambda u \tag{3.1.1}$$

for the eigenfunction equation. We happen to know all solutions to (3.1.1) for all complex λ, without specifying boundary conditions, namely $e^{\alpha x}$ where $-\alpha^2 = \lambda$, although we will eventually arrive at an alternate description using the self-similar structure. Of course, when we impose Dirichlet boundary conditions $u(0) = u(1) = 0$ we obtain the eigenfunctions $\{\sin \pi k x\}$, $k = 1, 2, \ldots$, with eigenvalues $\pi^2 k^2$, while if we impose Neumann boundary conditions $\partial_n u(0) = \partial_n u(1) = 0$, we obtain the eigenfunctions $\{\cos \pi k x\}$, $k = 0, 1, \ldots$, with eigenvalues $\pi^2 k^2$. By coincidence, the Dirichlet and Neumann spectra are identical, except for the Neumann eigenvalue 0. In fact, there are general principles that imply that the two spectra must be close, but the coincidence of eigenvalues must be attributed to coincidence.

We would like to understand these eigenfunctions and eigenvalues in terms of the sequence of graphs Γ_m approximating I. The key observation is that eigenfunctions of the continuous Laplacian on I, when restricted to V_m, are still eigenfunctions of the discrete Laplacian on Γ_m, but the eigenvalues are not the same:

$$\begin{aligned} -\Delta_m u(x) &= 2u(x) - u\left(x + \frac{1}{2^m}\right) - u\left(x - \frac{1}{2^m}\right) \\ &= \left(2 - e^{\alpha/2^m} - e^{-\alpha/2^m}\right) u(x) \\ &= -4 \sinh^2 \frac{\alpha}{2^{m+1}} u(x) \end{aligned} \tag{3.1.2}$$

if $u(x) = e^{\alpha x}$, or

$$-\Delta_m u(x) = 4 \sin^2 \frac{\pi k}{2^{m+1}} u(x) \tag{3.1.3}$$

if $u(x) = \sin \pi k x$ or $\cos \pi k x$. Note that the eigenvalue in (3.1.2) is unchanged if we replace α by $-\alpha$, so it depends only on the eigenvalue λ. We can represent this as

$$-\Delta_m u \big|_{V_m} = \lambda_m u \tag{3.1.4}$$

for a sequence $\{\lambda_m\}$ of discrete eigenvalues depending on λ. We note that

$$\lim_{m \to \infty} 4^m \lambda_m = \lambda, \tag{3.1.5}$$

as would be expected since

(3.1.6) $$\Delta = \lim_{m\to\infty} 4^m \Delta_m.$$

The expression for λ_m as a function of λ is easily read off from (3.1.2):

(3.1.7) $$\lambda_m = 4\sin^2 \frac{\sqrt{\lambda}}{2^{m+1}}.$$

It is a transcendental function, and by coincidence it is easily expressible in terms of well-known functions. What is more important to us is the relationship between λ_m and λ_{m-1}. We compute

$$\begin{aligned}\lambda_{m-1} &= 4\sin^2 2\frac{\sqrt{\lambda}}{2^{m+1}} = 4\left(2\sin\frac{\sqrt{\lambda}}{2^{m+1}}\cos\frac{\sqrt{\lambda}}{2^{m+1}}\right)^2 \\ &= 4\sin^2\frac{\sqrt{\lambda}}{2^{m+1}}\left(4-4\sin^2\frac{\sqrt{\lambda}}{2^{m+1}}\right),\end{aligned}$$

or

(3.1.8) $$\lambda_{m-1} = \lambda_m(4-\lambda_m).$$

We also need to solve for λ_m in terms of λ_{m-1}, which we write as

(3.1.9) $$\lambda_m = 2+\varepsilon_m\sqrt{4-\lambda_{m-1}}, \qquad \varepsilon_m = \pm 1.$$

Note that these relationships are algebraic. So, if we specify λ_{m_0} for some value m_0, then (3.1.8) specifies λ_m for all $m < m_0$. However, for $m > m_0$, we have the free choices of ε_m in (3.1.9) that allow infinitely many continuations. There is one condition we must impose, however. In order for the limit in (3.1.5) to exist, we need $\lambda_m \to 0$ as $m \to \infty$, and this means that all but a finite number of the ε_m must be chosen to be -1. Since

(3.1.10) $$2-\sqrt{4-x} = \frac{1}{4}x + O\left(x^2\right) \quad \text{as } x \to 0,$$

it is easy to see that the limit (3.1.5) exists in that case.

There is one small exception to the above description, namely that if $u\big|_{V_m} \equiv 0$ then we cannot claim that it is an eigenfunction. Does this actually happen? Yes, in fact, in exactly the cases in which we are most interested (for example, $\sin \pi 2^m x$ on V_m). However, any fixed eigenfunction has at most a finite number of zeroes in I, so for m large enough this will not happen. We define the *generation of birth*, m_0, to be the smallest m such that the restriction of u to $V_m \setminus V_0$ is not identically zero. For a generic eigenfunction we have $m_0 = 1$, and the eigenvalue equation on V_1 is the single equation

(3.1.11) $$2u\left(\frac{1}{2}\right) - u(0) - u(1) = \lambda_1 u\left(\frac{1}{2}\right), \qquad u\left(\frac{1}{2}\right) \neq 0.$$

We may regard this equation as giving the value λ_1 in terms of $u\big|_{V_1}$, but it also gives the value of $u(\frac{1}{2})$ in terms of the boundary values $u(0)$, $u(1)$, as

(3.1.12) $$u\left(\frac{1}{2}\right) = \frac{1}{2-\lambda_1}(u(0)+u(1)) \quad \text{provided } \lambda_1 \neq 2.$$

We will therefore refer to 2 as the *forbidden* eigenvalue. Note that (3.1.12) is a variant of the harmonic extension algorithm (1.3.8), which corresponds to the case $\lambda_1 = 0$. What (3.1.12) says is that if we know $u\big|_{V_0}$ and the eigenvalue λ_1, then this determines $u\big|_{V_1}$ provided λ_1 is not forbidden.

The story repeats at all levels. That is, if we know $u\big|_{V_{m-1}}$ and λ_m, then this determines $u\big|_{V_m}$. Indeed, we only have to find the values $u(x)$ for $x \in V_m \setminus V_{m-1}$. Any such x lies between two consecutive points y_0, y_1 in V_{m-1}. The eigenvalue equation at x is

$$2u(x) - u(y_0) - u(y_1) = \lambda_m u(x), \tag{3.1.13}$$

so we obtain

$$u(x) = \frac{1}{2-\lambda_m}(u(y_0)+u(y_1)), \quad \text{provided } \lambda_m \neq 2. \tag{3.1.14}$$

So now we have a recipe for both eigenvalues and eigenfunctions. Choose a value of λ_1 and $\{\varepsilon_m\}$ such that all but a finite number of $\varepsilon_m = -1$; determine λ_m for $m > 1$ by (3.1.9), and λ by (3.1.5). Assume that we never encounter the forbidden eigenvalue. Then u is determined on V_* inductively by (3.1.14) for any choice of boundary values $u(0)$ and $u(1)$, and then by continuity on all of I. We call this recipe *spectral decimation.*

We derived spectral decimation by starting with a knowledge of what the continuous eigenfunctions actually are. Now we want to see that it can stand alone.

Lemma 3.1.1 *Suppose u_{m-1} defined on V_{m-1} satisfies the eigenvalue equation*

$$-\Delta_{m-1}u_{m-1} = \lambda_{m-1}u_{m-1} \quad \text{on } V_{m-1} \setminus V_0. \tag{3.1.15}$$

Let λ_m be related to λ_{m-1} by (3.1.8) or (3.1.9), and assume $\lambda_m \neq 2$. Then extend u_{m-1} to u_m on V_m using (3.1.14). We obtain a function on V_m satisfying the eigenvalue equation

$$-\Delta_m u_m = \lambda_m u_m \quad \text{on } V_m \setminus V_0. \tag{3.1.16}$$

Proof: The fact that the eigenvalue equation (3.1.16) holds on $V_m \setminus V_{m-1}$ follows from the equivalence of (3.1.14) and (3.1.13). What is not obvious, and is somewhat miraculous, is that it also holds on $V_{m-1} \setminus V_0$. To see this we consider three consecutive points y_0, y_1, y_2 in V_{m-1} and fill in to get five consecutive points y_0, x, y_1, z, y_2 in V_m (see Figure 3.1.1).

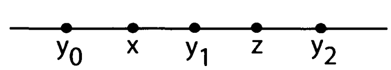

Figure 3.1.1

What we know is that

$$(2-\lambda_{m-1})u_{m-1}(y_1) = u_{m-1}(y_0) + u_{m-1}(y_2), \tag{3.1.17}$$

because this is (3.1.15) at y_1. What we need to show is that

$$(2-\lambda_m)u_{m-1}(y_1) = u_m(x) + u_m(z), \tag{3.1.18}$$

because this is (3.1.16) at y_1. Moreover, $u_m(x)$ and $u_m(z)$ are given by (3.1.14) as

$$u_m(x) = \frac{1}{2-\lambda_m}\left(u_{m-1}(y_0) + u_{m-1}(y_1)\right),$$

$$u_m(z) = \frac{1}{2-\lambda_m}\left(u_{m-1}(y_1) + u_{m-1}(y_2)\right),$$

so adding these yields

$$u_m(x) + u_m(z) = \frac{1}{2-\lambda_m}\left(u_{m-1}(y_0) + 2u_{m-1}(y_1) + u_{m-1}(y_2)\right). \tag{3.1.19}$$

Now we may substitute (3.1.17) into (3.1.19) to eliminate $u_{m-1}(y_0) + u_{m-1}(y_2)$, obtaining $u_m(x) + u_m(y) = \frac{2+2-\lambda_{m-1}}{2-\lambda_m} u_{m-1}(y_1)$. This will be the same as (3.1.18) provided $2 - \lambda_m = \frac{4-\lambda_{m-1}}{2-\lambda_m}$, and a little algebra shows that this is the same as (3.1.8). □

The gist of the argument is that the relationship (3.1.8) between the eigenvalues emerges from the computation. That is, the extension algorithm (3.1.14) is equivalent to the eigenvalue equation at the new points, so this choice is dictated by λ_m, but then the eigenvalue equation at the old points dictates what λ_m must be. If we hadn't known (3.1.8) from the beginning, we could have discovered it in the course of the proof.

There are some technical details needed to complete the story, such as the fact that this recipe produces a continuous function u on V_*, and $-\Delta u = \lambda u$, but these are straightforward and are left to the exercise.

We now come to the description of what happens in the nongeneric cases, when the forbidden eigenvalue may appear and the generation of birth may be greater than 1. We first deal with the case of Dirichlet eigenfunctions, so $u(0) = u(1) = 0$, and $\#(V_m \setminus V_0) = 2^m - 1$. In this case we may regard Δ_m as a symmetric matrix of order $2^m - 1$, so it must have $2^m - 1$ real eigenvalues. When $m = 1$ we see by inspection that $u_1(\frac{1}{2}) = 1$ (the only choice, up to a constant multiple) produces an eigenfunction with eigenvalue $\lambda_1 = 2$, the forbidden value. (A good thing, too, because $u_1(\frac{1}{2})$ is not determined by $u(0)$ and $u(1)$.) So the spectrum of Δ_1 is $\{2\}$. Also, the eigenfunction is the restriction to V_1 of $\pm\sin\pi kx$ for any odd value of k. Passing to Δ_2, we have two choices of λ_2 corresponding to $\lambda_1 = 2$ (namely $\lambda_2 = \varphi_\pm(2)$ for

$$\varphi_\pm(x) = 2 \pm \sqrt{4-x} \tag{3.1.20}$$

by (3.1.9)) that are eigenvalues by Lemma 3.1.1. These eigenvalues have generation of birth $m_0 = 1$. We are still missing one eigenvalue, which must have $m_0 = 2$, so the eigenfunction must have $u(\frac{1}{2}) = 0$. By inspection we see that the choice $u(\frac{1}{4}) = 1$, $u(\frac{3}{4}) = -1$ produces an eigenfunction with $\lambda_2 = 2$. The spectrum of Δ_2, in increasing order, is thus $\{\varphi_-(2), 2, \varphi_+(2)\}$. Moreover, the associated eigenfunction is obtained by restricting $\sin\pi kx$ to V_2, for $k \equiv 1, 2, 3 \mod 4$.

The pattern continues. Having found the spectrum $\{\alpha_1, \ldots, \alpha_{2^{m-1}-1}\}$ and associated eigenfunctions on V_{m-1}, we may use Lemma 3.1.1 to produce $2^m - 2$

eigenvalues, $\{\varphi_\pm(\alpha_j)\}$. (We can check that these are distinct and not the forbidden value.) These have $m_0 < m$. We need one more eigenvalue with $m_0 = m$, and we easily check that we may take $\lambda_m = 2$ and the eigenfunction alternates ± 1 on the points in $V_m \setminus V_{m-1}$. Thus the spectrum of Δ_m in increasing order is

$$\{\varphi_-(\alpha_1), \ldots, \varphi_-(\alpha_{2^{m-1}-1}), 2, \varphi_+(\alpha_{2^{m-1}-1}), \ldots, \varphi_+(\alpha_1)\},$$

because φ_- preserves order and φ_+ reverses order. Moreover, the eigenfunctions are the restriction to V_m of $\sin \pi k x$ for $k \equiv 1, 2, \ldots, 2^m - 1 \mod 2^m$. The relationship between m_0 and the values of $\varepsilon_{m_0+1}, \ldots, \varepsilon_m$, and the value of k, is a bit complicated, so we leave the details to the exercises.

One consequence of the above discussion is the fact that $\sin \pi x$ takes on algebraic values for $x \in V_*$. In fact, the same is true for any rational value of x. Another consequence is that the ground state eigenfunction $\sin \pi x$ is positive and concave down, essentially because the coefficient $\frac{1}{2-\lambda_m}$ in (3.1.14) is greater than $\frac{1}{2}$.

We now briefly discuss the modifications needed to handle the Neumann eigenfunctions. It is not a priori clear what the Neumann conditions should be in the discrete case. The best way to think about it is that the function may be reflected evenly about each boundary point, and then will satisfy the eigenvalue equation at the boundary points as well. This is certainly true of the continuous eigenfunctions $\cos \pi k x$. Also, the analogous statement for Dirichlet eigenfunctions and odd reflection is trivially true. So this means that

$$\begin{cases} 2u(0) - 2u\left(\frac{1}{2^m}\right) = \lambda_m u(0), \\ 2u(1) - 2u\left(1 - \frac{1}{2^m}\right) = \lambda_m u(1) \end{cases} \tag{3.1.21}$$

must be adjoined to the eigenvalue equations, which means that for Δ_m we have a matrix of order $2^m + 1$. When $m = 1$ we see by inspection that the eigenvalues are 0, 2, 4 with corresponding eigenfunctions $(u(0), u(\frac{1}{2}), u(1))$ given by $(1, 1, 1)$, $(1, 0, -1)$, and $(1, -1, 1)$. Passing to $m = 2$ we have no difficulty using Lemma 3.1.1 for $\lambda_1 = 0$ (whence $\lambda_2 = 0$ or 4) or $\lambda_1 = 2$ to produce four eigenfunctions. The eigenvalue $\lambda_1 = 4$ presents only one choice for λ_2, since $\varphi_\pm(4) = 2$, the forbidden eigenvalue. In this case (3.1.14) gives the indeterminate value 0/0, but in fact if we interpret it as 0, then we obtain an eigenfunction with eigenvalue 2. In general, of the $2^{m-1} + 1$ eigenvalues of Δ_{m-1}, all but $\lambda_{m-1} = 4$ bifurcate and generate two values for λ_m, while $\lambda_{m-1} = 4$ extends to only the single eigenvalue $\lambda_m = 2$, again interpreting the right side of (3.1.14) as 0.

In summary, spectral decimation allows us to compute all eigenfunctions of Δ as limits of eigenfunctions of Δ_m, and these eigenfunctions may also be found explicitly. Also, the Dirichlet and Neumann spectrum of Δ is the limit of 4^m times the same spectrum of Δ_m. The orthogonality of Dirichlet or Neumann eigenfunctions corresponding to different eigenvalues follows from the symmetry of the Laplacians; in the discrete case the inner product is just

$$\langle f, g \rangle_m = \sum_{x \in V_m \setminus V_0} f(x) g(x) + \frac{1}{2} \sum_{x \in V_0} f(x) g(x). \tag{3.1.22}$$

In order to give the correct formulas for Fourier coefficients we also need to compute the inner products

$$\langle u, u\rangle = \int_0^1 |u(x)|^2 dx = \lim_{m\to\infty} \frac{1}{2^m}\langle u, u\rangle_m \tag{3.1.23}$$

for the eigenfunctions. This can also be done using spectral decimation to relate the values of $\langle u, u\rangle_{m-1}$ and $\langle u, u\rangle_m$ and then pass to the limit, but we leave the details to the exercises.

EXERCISES

3.1.1. Verify (3.1.3) using trigonometric double angle formulas.
3.1.2. Show that the extension algorithm (3.1.14) always leads to a uniformly continuous function on V_* provided all but a finite number of ε_m equal -1. Also show $-\Delta u = \lambda u$ if u is extended by continuity to I.
3.1.3. Show that $\sin \pi x$ is algebraic if x is rational.
3.1.4.* Find an algorithm that finds m_0 and the values of ε_m for $m > m_0$ in terms of the integer k (specifically its binary expansion) for the function $\sin \pi kx$.
3.1.5. Show that $\sin \pi x$ is positive and concave down on I, using only (3.1.14).
3.1.6.* Find the relationship between the values $\langle u, u\rangle_{m-1}$ and $\langle u, u\rangle_m$ by using (3.1.14). Use this to give an infinite product formula for $\langle u, u\rangle$.

3.2 SPECTRAL DECIMATION

Our goal in this section is to begin to duplicate the spectral decimation recipe on SG for the eigenfunctions of the standard Laplacian. Here we deal with generic eigenvalues, and in the next section we look at those corresponding to Dirichlet or Neumann boundary conditions. Our approach will be to obtain solutions of the eigenvalue equation

$$-\Delta u = \lambda u \quad \text{on } K \tag{3.2.1}$$

as limits of solutions of the discrete version

$$-\Delta_m u_m = \lambda_m u_m \quad \text{on } V_m \setminus V_0. \tag{3.2.2}$$

As in the case of the interval, we will be lucky that we may take $u_m = u\big|_{V_m}$. We should emphasize that there is no a priori reason this should hold. In fact, it is not true for other Laplacians Δ_μ on SG (or even on I), and there are some very symmetric fractals where nothing like this is true even for the standard Laplacian. But, as always, mathematicans shamelessly exploit good luck!

We look for the analog of Lemma 3.1.1: Given an eigenfunction u_{m-1} on V_{m-1} with eigenvalue λ_{m-1}, how can we extend it to u_m on V_m, so as to be an eigenfunction with eigenvalue λ_m, and what is the relationship between the two eigenvalues? We will essentially go back to the computations in Section 1.3 and redo them, adding in the eigenvalues. So consider an $(m-1)$-cell with boundary points x_0, x_1, x_2 and let y_0, y_1, y_2 denote the points in $V_m \setminus V_{m-1}$ in that cell, with y_i opposite x_i. We know the values $u(x_i)$, and we want to determine the values $u(y_j)$ (for simplicity of notation, we drop the subscripts on u). The λ_m-eigenvalue equation at the points $\{y_i\}$ gives us the system of equations

$$\text{(3.2.3)} \qquad \begin{cases} (4-\lambda_m)u(y_0) = u(y_1)+u(y_2)+u(x_1)+u(x_2), \\ (4-\lambda_m)u(y_1) = u(y_0)+u(y_2)+u(x_0)+u(x_2), \\ (4-\lambda_m)u(y_2) = u(y_1)+u(y_0)+u(x_1)+u(x_0). \end{cases}$$

To solve, we add them and rearrange terms:

$$\text{(3.2.4)} \qquad (2-\lambda_m)(u(y_0)+u(y_1)+u(y_2)) = 2(u(x_0)+u(x_1)+u(x_2)).$$

We see already that 2 should be a forbidden eigenvalue, so we assume $\lambda_m \neq 2$ and obtain

$$\text{(3.2.5)} \qquad u(y_0)+u(y_1)+u(y_2) = \left(\frac{2}{2-\lambda_m}\right)(u(x_0)+u(x_1)+u(x_2)).$$

We add $u(y_i)$ to both sides of the corresponding equation in (3.2.3) and then use (3.2.5) to simplify the right side:

$$\text{(3.2.6)} \qquad \begin{cases} (5-\lambda_m)u(y_0) = \left(\frac{2}{2-\lambda_m}\right)(u(x_0)+u(x_1)+u(x_2))+u(x_1)+u(x_2), \\ (5-\lambda_m)u(y_1) = \left(\frac{2}{2-\lambda_m}\right)(u(x_0)+u(x_1)+u(x_2))+u(x_0)+u(x_2), \\ (5-\lambda_m)u(y_2) = \left(\frac{2}{2-\lambda_m}\right)(u(x_0)+u(x_1)+u(x_2))+u(x_1)+u(x_0). \end{cases}$$

Now we see that 5 should also be a forbidden eigenvalue, so we assume $\lambda_m \neq 5$ and obtain

$$\text{(3.2.7)} \qquad u(y_0) = \frac{(4-\lambda_m)(u(x_1)+u(x_2))+2u(x_0)}{(2-\lambda_m)(5-\lambda_m)}, \text{ etc.}$$

This is the analog of (3.1.14). Note that it is a variant of the "$\frac{1}{5}-\frac{2}{5}$ rule", because the coefficients $\frac{4-\lambda_m}{(2-\lambda_m)(5-\lambda_m)}$ and $\frac{2}{(2-\lambda_m)(5-\lambda_m)}$ corresponding to the adjacent and opposite vertices reduce to $\frac{2}{5}$ and $\frac{1}{5}$ when $\lambda_m = 0$.

Next, we go back to the points in $V_{m-1} \setminus V_0$ and compare the λ_{m-1}-eigenvalue equation on V_{m-1}, which we know holds, and the λ_m-eigenvalue equation on V_m, which we want to hold. So consider a point x_0 in $V_{m-1} \setminus V_0$. In addition to the $(m-1)$-cell considered above, there is another $(m-1)$-cell, with boundary points x_0', x_1', x_2' and interior points y_0', y_1', y_2', and $x_0' = x_0$. Then the λ_{m-1}-eigenvalue equation says

$$\text{(3.2.8)} \qquad (4-\lambda_{m-1})u(x_0) = u(x_1)+u(x_2)+u\left(x_1'\right)+u\left(x_2'\right)$$

and the λ_m-eigenvalue equation says

$$\text{(3.2.9)} \qquad (4-\lambda_m)u(x_0) = u(y_1)+u(y_2)+u\left(y_1'\right)+u\left(y_2'\right).$$

Now each term on the right side of (3.2.9) is given by (3.2.7), and the sum is

$$\frac{(6-\lambda_m)\left(u(x_1)+u(x_2)+u\left(x_1'\right)+u\left(x_2'\right)\right)+4(4-\lambda_m)u(x_0)}{(2-\lambda_m)(5-\lambda_m)}. \tag{3.2.10}$$

When we substitute (3.2.8) in (3.2.10) we obtain

$$\left(\frac{(6-\lambda_m)(4-\lambda_{m-1})+4(4-\lambda_m)}{(2-\lambda_m)(6-\lambda_m)}\right)u(x_0). \tag{3.2.11}$$

So (3.2.9) will be valid provided

$$4-\lambda_m=\frac{(6-\lambda_m)(4-\lambda_{m-1})+4(4-\lambda_m)}{(2-\lambda_m)(5-\lambda_m)} \tag{3.2.12}$$

(this condition is also clearly necessary, unless $u(x_0)=0$ for all $x_0\in V_{m-1}\setminus V_0$).

It is possible to simplify (3.2.12), but first we must record one more forbidden eigenvalue, $\lambda_m\neq 6$ (if $\lambda_m=6$ then (3.2.12) is true, regardless of the value of λ_{m-1}). Then (3.2.12) becomes

$$((2-\lambda_m)(5-\lambda_m)-4)(4-\lambda_m)=(6-\lambda_m)(4-\lambda_{m-1}),$$

and since $(2-\lambda_m)(5-\lambda_m)-4=(6-\lambda_m)(1-\lambda_m)$, this becomes

$$(6-\lambda_m)(1-\lambda_m)(4-\lambda_m)=(6-\lambda_m)(4-\lambda_{m-1}).$$

Since $\lambda_m\neq 6$ we may cancel the factor $6-\lambda_m$ to obtain $(1-\lambda_m)(4-\lambda_m)=4-\lambda_{m-1}$, and finally

$$\lambda_{m-1}=\lambda_m(5-\lambda_m). \tag{3.2.13}$$

This is the analog of (3.1.8). We can also solve

$$\lambda_m=\frac{5+\varepsilon_m\sqrt{25-4\lambda_{m-1}}}{2}\quad\text{for } \varepsilon_m=\pm 1. \tag{3.2.14}$$

Lemma 3.2.1 *Suppose $\lambda_m\neq 2, 5,$ or 6, and λ_{m-1} is given by (3.2.13). (a) If $u\big|_{V_{m-1}}$ is a λ_{m-1}-eigenfunction of Δ_{m-1} and is extended to V_m by (3.2.7), then it is a λ_m-eigenfunction of Δ_m. (b) Conversely, if $u\big|_{V_m}$ is a λ_m-eigenfunction of Δ_m, then $u\big|_{V_{m-1}}$ is a λ_{m-1}-eigenfunction of Δ_{m-1}.*

Proof: Part (a) follows from the above discussion. For part (b), since the λ_m-eigenvalue equation holds, we have (3.2.7). Then the reasoning from (3.2.8) to (3.2.9) may be reversed if (3.2.12) holds, so (3.2.8) follows from (3.2.9) and (3.2.12). □

Of course, if $m=1$ there is no equation on V_0, so (a) above is valid without any assumption on $u\big|_{V_0}$.

Next we want to take the limit as $m\to\infty$. We assume that we have an infinite sequence $\{\lambda_m\}_{m\geq m_0}$ related by (3.1.13) or (3.1.14), with all but a finite number of $\varepsilon_m=-1$. Then we may define

$$\lambda=\frac{3}{2}\lim_{m\to\infty}5^m\lambda_m. \tag{3.2.15}$$

It is easy to see that the limit exists since

(3.2.16) $$\frac{5-\sqrt{25-4x}}{2} = \frac{1}{5}x + O\left(x^2\right) \quad \text{as } x \to 0.$$

Now suppose we start with a λ_{m_0}-eigenfunction u of Δ_{m_0} on V_{m_0}, and extend u to V_* by successively using (3.2.7), assuming that none of the λ_m is a forbidden eigenvalue (2, 5, 6). (If $m_0 = 1$ we can choose any values of $u\big|_{V_0}$ and extend to V_1 by (3.2.7).) Since (3.2.16) implies $\lambda_m = O(\frac{1}{5^m})$ as $m \to \infty$, it is easy to see that u is uniformly continuous on V_* and so extends to a continuous function on K. Moreover, it satisfies the λ-eigenvalue equation for Δ. This is the generic case of the spectral decimation recipe.

It is clear that this recipe constructs many eigenfunctions of Δ, but does it construct them all? It would be nice to have a direct argument to show that any eigenfunction of Δ must restrict to an eigenfuncton of Δ_m for m large enough, but it seems unlikely that such an argument exists (for I, we were able to use known properties of exponentials, sines, and cosines). We will eventually answer this question in the next section.

First we consider "small" eigenvalues. Suppose we start with $|\lambda_1| < 2$ and choose all $\varepsilon_m = -1$. (Here we allow complex eigenvalues and eigenvectors.) It is easy to see that $|\lambda_m| < 2$ for all m, so we never encounter a forbidden eigenvalue. Therefore, the recipe produces a three-dimensional space of eigenfunctions, with arbitrary boundary values. But the dimension of the space of eigenfunctions for a fixed eigenvalue is at most three, unless the eigenvalue is a Dirichlet eigenvalue (a four-dimensional space must contain a nonzero function vanishing on V_0). In the next section we will see that these eigenvalues are never Dirichlet eigenvalues. So we have constructed all eigenfunctions for these eigenvalues.

What exactly is this set of eigenvalues? It is difficult to give an exact description, but we can give a qualitative description. Define

(3.2.17) $$\varphi_\pm(x) = \frac{5 \pm \sqrt{25-4x}}{2}$$

and

(3.2.18) $$\Phi(z) = \frac{3}{2}\lim_{m\to\infty} 5^m \varphi_-^{(m)}(z) \quad (m\text{-fold iterate}).$$

Using (3.2.16) it is not difficult to show that Φ is an analytic function in the complement of $[\frac{25}{4}, \infty)$ with $\Phi(0) = 0$, $\Phi'(0) \neq 0$. Since the set of small eigenvalues is $\Phi(B_2)$, for B_2 the open ball of radius 2 about 0, we know that it forms a bounded open neighborhood of 0. In particular, it contains B_ε for some $\varepsilon > 0$.

Now we consider the set of eigenvalues obtained by starting with λ_{m_0} satisfying $|\lambda_{m_0}| < 2$ and choosing $\varepsilon_m = -1$ for all $m > m_0$. It is clear that we produce the set $5^{m_0}\Phi(B_2)$, which contains $B_{5^{m_0}\varepsilon}$. There are two obstacles to repeating the previous arguments. First, in order to get started, we need a λ_{m_0}-eigenfunction of Δ_{m_0} with prescribed boundary values. By linear algebra this is possible as long as λ_{m_0} is not a Dirichlet eigenvalue of Δ_{m_0}. In the next section we will see that the smallest Dirichlet eigenvalue of Δ_{m_0} is 2 or 5, so this is no problem after all. The other obstacle is that λ might be a Dirichlet eigenvalue of Δ. This is a legitimate problem, so we postpone it to the next section. We can now summarize what we have found.

Theorem 3.2.2 *Suppose λ is not a Dirichlet eigenvalue of Δ. Then there exists m_0 (depending explicitly on $\log|\lambda|$) such that for any λ-eigenfunction u, $u\big|_{V_{m_0}}$ is an eigenfunction of Δ_{m_0}, and u is constructed by the spectral decimation recipe. The space of λ-eigenfunctions is exactly three-dimensional, and an eigenfunction is uniquely determined by its boundary values.*

It is worth pondering the similarities and differences between the functions $x(4-x)$ and $x(5-x)$ that govern the dynamics of the mapping $\lambda_m \to \lambda_{m-1}$ in the two cases, I and SG. If you are looking for strange numerology you could make the observation that this is, once again, just a matter of replacing 4 by 5 (we saw this in the pointwise definition of the Laplacian). Perhaps this is just a coincidence, perhaps not. But there is a big difference between these two quadratic polynomials. The Julia set of $x(4-x)$ is the interval $[0,4]$, while the Julia set of $x(5-x)$ is a Cantor set. This will have interesting implications for the spectrum of Δ on SG.

EXERCISES

3.2.1. Show that the matrix

$$\begin{pmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{pmatrix}$$

has eigenvalues 2 and 5 (multiplicity 2). Relate this fact to the solution of (3.2.3).

3.2.2. If $\lambda_m = 2$ use (3.2.4) to conclude that $u(x_0)+u(x_1)+u(x_2)=0$. If $\lambda_m = 5$ use (3.2.6) to conclude that $u(x_0)=u(x_1)=u(x_2)$.

3.2.3. If λ is a negative real, show that all λ_m are negative reals.

3.2.4. Prove (3.2.16), and use it to show that the limit in (3.2.15) exists under the condition that all but a finite number of $\varepsilon_m = -1$. Also show that the limit in (3.2.15) does not exist if there are an infinite number of $\varepsilon_m = 1$.

3.2.5. Show that the spectral decimation recipe always produces a uniformly continuous function on V_*, and $-\Delta u = \lambda u$.

3.2.6. Show that (3.2.18) defines an analytic function in the complement of $[\frac{25}{4}, \infty)$ with $\Phi(0)=0$, $\Phi'(0)\neq 0$.

3.2.7. Give an explicit estimate for m_0 in Theorem 3.2.2.

3.2.8. If $-\Delta u = \lambda u$, show that

$$-\Delta(u \circ F_w) = \frac{\lambda}{5^{|w|}} u \circ F_w.$$

3.2.9.* Show that if λ is negative and $-\Delta u = \lambda u$, then u is the minimizer of $\mathcal{E}(v,v) - \lambda \int_K v^2 d\mu$ among all functions v with the same boundary values. Also show that a minimizer exists.

3.3 EIGENVALUES AND MULTIPLICITIES

In this section we find all Dirichlet eigenvalues and eigenfunctions and their multiplicities. Not surprisingly, this will involve using the forbidden eigenvalues. The first problem we consider is to describe the spectrum of Δ_m. We will consider two kinds of eigenvalues, *initial* and *continued*. The continued eigenvalues will be those that arise from eigenvalues of Δ_{m-1} by the spectral decimation formula. Those that remain, the initial eigenvalues, must be some of the forbidden eigenvalues by Lemma 3.2.1.

So we begin with Δ_1. We find by inspection a one-dimensional Dirichlet eigenspace for $\lambda_1 = 2$ and a two-dimensional Dirichlet eigenspace for $\lambda_1 = 5$, shown in Figure 3.3.1. Since $\#(V_1 \setminus V_0) = 3$, this is the complete spectrum. Note that these eigenspaces transform under the symmetry group D_3 according to the trivial and two-dimensional representations. Incidentally, if we drop the Dirichlet condition we find two more eigenfunctions with eigenvalue 2, and just one more eigenfunction with eigenvalue 5, because (3.2.6) implies that all the boundary values are equal. See Figure 3.3.2. We also note that the forbidden eigenvalue 6 occurs with multiplicity 3 as shown in Figure 3.3.3, but it is not a Dirichlet eigenvalue.

Next we consider Δ_2. First we observe that there are no Dirichlet eigenfunctions corresponding to $\lambda_2 = 2$. This follows from (3.2.4), which says that the sum of the

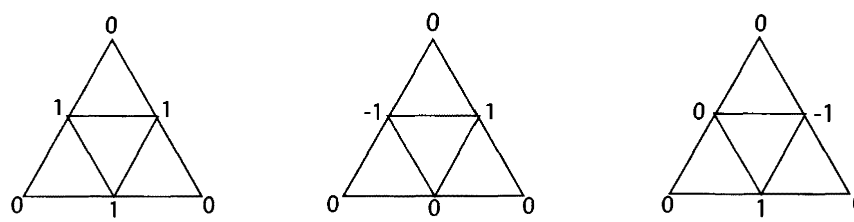

Figure 3.3.1

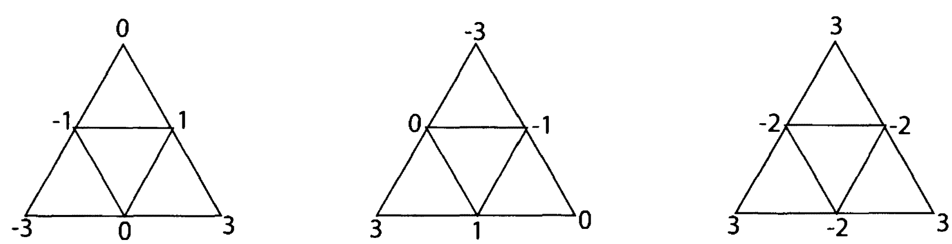

Figure 3.3.2

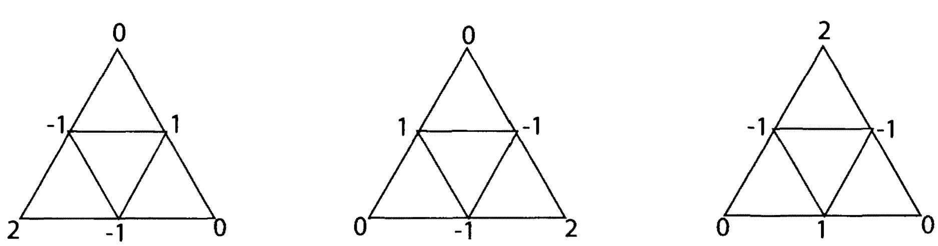

Figure 3.3.3

boundary values on each of the three 1-cells $F_i V_1$ must vanish. So if u vanishes on V_0, we have

$$u(x_0)+u(x_1)=u(x_1)+u(x_2)=u(x_0)+u(x_2)=0,$$

and so u vanishes on V_1. That means u restricted to each 1-cell $F_i V_1$ must be a multiple of the Dirichlet 2-eigenvalue in Figure 3.3.1. It is easy to see that it is impossible to make the 2-eigenvalue equation hold at the points in $V_1 \setminus V_0$ unless all the multiples are 0, so $u \equiv 0$. This argument may be used inductively to prove that there are no $\lambda_m = 2$ Dirichlet eigenvalues of Δ_m for any $m > 1$. We leave the details to the exercises.

Moving on to the $\lambda_2 = 5$ case, we look for ways to duplicate the construction in Figure 3.3.1 on a smaller level. We note that (3.2.6) implies that u must vanish on V_1 for a Dirichlet eigenfunction, so this is the only possible approach. Note that we have a two-dimensional space of possibilities for the restriction of u to each 1-cell $F_i V_1$, and within this six-dimensional space there are three linear constraints expressing the 5-eigenvalue equation at each point in $V_1 \setminus V_0$. So we expect to find a three-dimensional space of Dirichlet eigenfunctions, and indeed we find them as shown in Figure 3.3.4 (to make these figures more legible we omit all labels of values 0). One nice metaphor for this construction is a "chain of batteries." Each battery has a positive and negative terminal, and it must be aligned with neighboring batteries positive to negative, or a charged terminal may abut a boundary point. If we take the sum of the three eigenfunctions in Figure 3.3.4, we obtain a circuit of three batteries around the central empty upside-down triangle, as shown

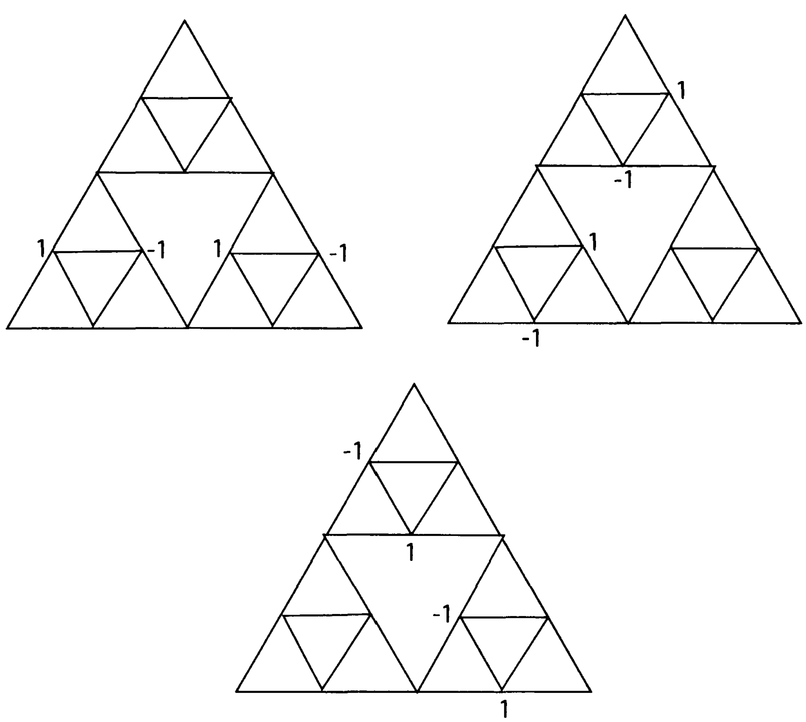

Figure 3.3.4

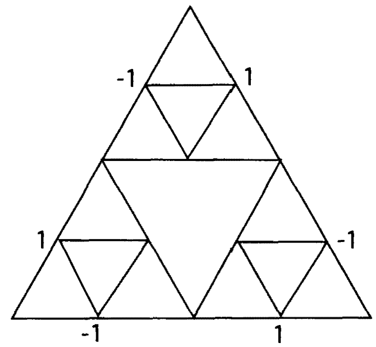

Figure 3.3.5

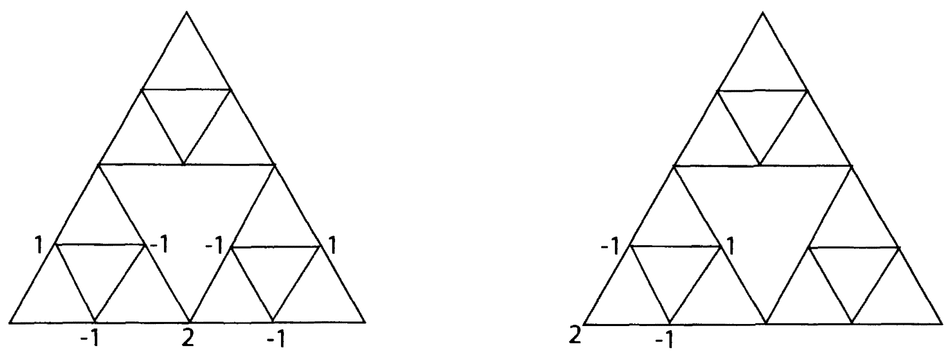

Figure 3.3.6

in Figure 3.3.5. In order to generalize this construction to higher values of m, we will take as our basis the eigenfunction in Figure 3.3.5 together with the first two in Figure 3.3.4, even though this breaks the symmetry.

Next we look for the forbidden eigenvalue $\lambda_2 = 6$. It is convenient to drop the Dirichlet condition and first look for all solutions of the 6-eigenvalue equation. Clearly the restriction to each 1-cell $F_i V_1$ must be a 6-eigenfunction on V_1, and these are described in Figure 3.3.3. Since we must glue these together at the points in $V_1 \setminus V_0$, there is at most one eigenfunction with values prescribed at the points in V_1. Miraculously they all turn out to satisfy the 6-eigenvalue equation at the points of $V_1 \setminus V_0$. (An explanation of the miracle will come later.) So we obtain a six-dimensional space of eigenfunctions, with a basis indexed by points in V_1, with $u_j(x_k) = \delta_{jk}$ if $V_1 = \{x_1, \ldots, x_6\}$. We show two of these eigenfunctions in Figure 3.3.6, and the others are obtained by rotation. By inspection we see that there is a three-dimensional space of Dirichlet eigenfunctions indexed by the points in $V_1 \setminus V_0$. From $\lambda_1 = 2$ we obtain $\lambda_2 = \frac{5 \pm \sqrt{17}}{2}$ with multiplicities 1, and from $\lambda_1 = 5$ we obtain $\lambda_2 = \frac{5 \pm \sqrt{5}}{2}$ with multiplicities 2 (the multiplicity of $\lambda_1 = 5$).

Now for some arithmetic. We have found initial eigenvalues $\lambda_2 = 5$ and $\lambda_2 = 6$ with multiplicities 3 and 3. We also have continued eigenvalues obtained from $\lambda_1 = 2, 5$ by spectral decimation. From $\lambda_1 = 2$ we obtain $\lambda_2 = \frac{5 \pm \sqrt{17}}{2}$ with multiplicities 1, and from $\lambda_1 = 5$ we obtain $\lambda_2 = \frac{5 \pm \sqrt{5}}{2}$ with multiplicities 2 (the multiplicity of $\lambda_1 = 5$). The grand total is $3 + 3 + 1 + 1 + 2 + 2 = 12$, and $12 = \#(V_2 \setminus V_0)$, the dimension of the space for the Dirichlet Laplacian Δ_2. This verifies that we have found the complete Dirichlet spectrum of Δ_2. In increasing order it reads

$$\left(\frac{5-\sqrt{17}}{2}, \frac{5-\sqrt{5}}{2}, \frac{5+\sqrt{5}}{2}, \frac{5+\sqrt{17}}{2}, 5, 6\right)$$

with multiplicities $(1, 2, 2, 1, 3, 3)$. We also observe that Lemma 3.2.1 does imply that the entire spectrum of Δ_2 must arise in this way, so the dimension count is just a check. In the general case it will be more difficult to prove that we have correctly computed the multiplicities of the initial eigenvalue 5, so we will rely on a dimension count to complete the argument.

Now we are ready for the general case. We know that $\lambda_m = 2$ is not a Dirichlet eigenvalue, so $\lambda_m = 5$ and $\lambda_m = 6$ are the initial eigenvalues, and we need to compute their multiplicities $M_m(5)$ and $M_m(6)$. By repeating the analysis of the 6-eigenfunctions of Δ_2, we see that $M_m(6) = \#(V_{m-1} \setminus V_0) = \frac{3^m-3}{2}$ (see Exercise 1.1.1) with an eigenfunction associated to each point x in $V_{m-1} \setminus V_0$, namely the first eigenfunction in Figure 3.3.6 reduced and rotated to fit the point. If $x = F_w q_i = F_{w'} q_{i'}$ for $|w| = |w'| = m-1$, then the eigenfunction u is supported in $F_w V_1 \cup F_{w'} V_1$. This is the first example of a completely localized eigenfunction. We will discuss this phenomenon in detail in the next section.

Using the battery metaphor we may describe a large set of linearly independent 5-eigenfunctions. This will give us a lower bound for $M_5(m)$, and later the dimension count will show that this lower bound is the correct value. The idea is based on the 1-homology of the graph Γ_{m-1}. If you look at the graph you see a bunch of 1-cycles going around each of the empty upside-down triangles of different sizes. Clearly these cycles are not boundaries, and it is not hard to see that they are linearly independent. (If you delete the boundary points, these will generate the entire 1-homology group.) For each cycle we may put a string of batteries around it, the number of batteries depending on the size of the cycle. In this way we obtain 5-eigenfunctions associated to each cycle. We also want to consider two more, a string of batteries joining q_0 to q_1, and a string of batteries joining q_1 to q_2. It is possible to show that this yields a linearly independent set of 5-eigenfunctions, but we leave the details to the exercises. Note that we would not want to adjoin another string of batteries joining q_2 to q_0, for then we would not have a linearly independent set (the total sum, with appropriate signs, would be zero). The total number of cycles, counting down in size, is $1 + 3 + 3^2 + \cdots + 3^{m-2} = \frac{3^{m-1}-1}{2}$, so our lower bound is $M_5(m) \geq \frac{3^{m-1}+3}{2}$.

So that is the story for initial eigenvalues. What about continued eigenvalues? It is essentially the same as for $m = 2$, except there is one new twist. If $\lambda_{m-1} = 6$, then the two solutions (3.2.14) for λ_m are 2 and 3, and 2 is a forbidden eigenvalue. (On the other hand, we never find $\lambda_m = 5$ or 6 in (3.2.14), because that would make $\lambda_{m-1} = 0$ or -6, and all Dirichlet eigenvalues are positive.) So every eigenvalue λ_{m-1} bifurcates into two choices of λ_m, by the choice $\varepsilon_m = \pm 1$, except $\lambda_{m-1} = 6$, which just yields the single choice $\lambda_m = 3$ corresponding to $\varepsilon_m = 1$.

The total space of Dirichlet eigenfunctions for Δ_{m-1} has dimension $\frac{3^m-3}{2}$, and for Δ_m it is $\frac{3^{m+1}-3}{2}$ (see Exercise 1.1.1). Of the $\frac{3^m-3}{2}$ eigenfunctions of Δ_{m-1}, we know that $M_6(m-1) = \frac{3^{m-1}-3}{2}$ of them correspond to eigenvalue $\lambda_{m-1} = 6$, while the remaining $\frac{3^m-3^{m-1}}{2} = 3^{m-1}$ of them correspond to other eigenvalues, leading to a space of continued eigenfunctions of dimension $\frac{3^{m-1}-3}{2} + 2 \cdot 3^{m-1}$. If we add to this $M_6(m) = \frac{3^m-3}{2}$ and the lower bound $\frac{3^{m-1}+3}{2}$ for $M_5(m)$, we obtain

$$\frac{3^{m-1}-3}{2} + \frac{4 \cdot 3^{m-1}}{2} + \frac{3^m-3}{2} + \frac{3^{m-1}+3}{2} = \frac{9 \cdot 3^{m-1}-3}{2} = \frac{3^{m+1}-3}{2}$$

as desired. So by induction we have the complete spectrum.

Having found the Dirichlet eigenvalues and eigenfunctions for Δ_m, it follows by routine limiting arguments that the Dirichlet spectrum of Δ is obtained in the limit as $m \to \infty$. Note that each Dirichlet eigenvalue λ of Δ enters as λ_m in the Dirichlet spectrum of Δ_m with the correct multiplicity. We leave the details to the exercises.

One more small technical point remains. We have shown how to construct all Dirichlet eigenfunctions corresponding to a Dirichlet eigenvalue λ, and the spectral decimation recipe is still valid, starting with a generation of birth m_0 where $\lambda_{m_0} = 2, 5$, or 6. Can we say the same for all eigenfunctions corresponding to that eigenvalue λ? The proof of Theorem 3.2.2 does not answer this question. We can show that the answer is yes by a case-by-case analysis. If $\lambda_{m_0} = 6$, then there are three more eigenfunctions corresponding to the vertices in V_0 (see Figure 3.3.3 for $m_0 = 1$ and the right side of Figure 3.3.6 for $m_0 = 2$). This means that spectral decimation constructs λ-eigenfunctions with arbitrary boundary values. Therefore, an arbitrary λ-eigenfunction is a sum of one of these and a Dirichlet eigenfunction, and hence is constructed by spectral decimation. The case $\lambda_1 = 2$ or 5 is more delicate. Note that the eigenfunctions shown in Figures 3.3.1 and 3.3.2 together span a three-dimensional space in each case, but we cannot prescribe boundary values freely. In the case $\lambda_1 = 2$ we have the constraint $u(q_0) + u(q_1) + u(q_2) = 0$, and in the case $\lambda_1 = 5$ we have the constraint $u(q_0) = u(q_1) = u(q_2)$. These constraints were derived from the spectral decimation recipe. How do we know they hold for all λ-eigenfunctions? A direct proof would be difficult, so we argue indirectly. Each eigenfunction u is made up by gluing together the functions $u \circ F_i$, which are $\frac{\lambda}{5}$-eigenfunctions. Since $\frac{\lambda}{5}$ is not a Dirichlet eigenvalue, the functions $u \circ F_i$ are determined by their boundary values $u(F_i q_j)$ by Theorem 3.2.2. So there are six values to be prescribed at the points in V_1. From our gluing theorem we know that we get an eigenfunction if and only if the matching conditions for normal derivatives hold at the points in $V_1 \setminus V_0$. Note that this condition is independent of the spectral decimation recipe. We have not explained how to compute these normal derivatives yet. A priori they might all vanish, but we can show that this doesn't happen. Indeed, if all the normal derivatives vanished, then the matching conditions would be automatic, so there would be a three-dimensional space of Dirichlet eigenfunctions. But we know this is false. Next, by using symmetry considerations, we can show that the matching conditions for normal derivatives at a vertex in $V_1 \setminus V_0$ is just a multiple of the 5-eigenvalue equation for Δ_1 at that point. We leave the details to the exercises. So every λ-eigenfunction on K restricts to a λ_1-eigenfunction on V_1, and the spectral decimation recipe works. Similar reasoning works in the case $\lambda_{m_0} = 5$ for any m_0.

In summary, the first conclusion of Theorem 3.2.2 remains true if λ is a Dirichlet eigenvalue; the only differences are that the space of all eigenfunctions need not have dimension 3, and boundary values cannot be freely prescribed.

We will examine the Dirichlet spectrum more closely in later sections. Here we will just make a few observations. We may classify eigenvalues into three series, which we call the 2-series, 5-series, and 6-series, depending on the value of λ_{m_0}. The eigenvalues in the 2-series all have multiplicity 1, while the eigenvalues in the other series all exhibit higher multiplicity. Also, if λ is an eigenvalue in the 5-series or 6-series, then $5^m\lambda$ is also an eigenvalue, corresponding to a generation of birth m_0+m, with the same choice of ε's (suitably reindexed). The lowest eigenvalue, or *bottom of the spectrum*, belongs to the 2-series with all $\varepsilon_m = -1$. The corresponding eigenfunction, called the *ground state*, is positive (see the exercises).

Next we give a brief discussion of the Neumann spectrum of Δ. As indicated in Section 3.1, we want to impose a Neumann condition on the graph Γ_m by imagining that it is embedded in a larger graph by reflecting in each boundary vertex and imposing the λ_m-eigenvalue equation on the even extension of u. This just means that we impose the equation

$$(4-\lambda_m)\,u\,(q_0) = 2u\left(F_0^m q_1\right) + 2u\left(F_0^m q_2\right) \tag{3.3.1}$$

at q_0, and so on. Then the Neumann λ_m-eigenvalue equations consist of exactly $\#V_m$ equations in $\#V_m$ unknowns. It is even convenient to allow $m = 0$, in which case there are three equations (3.3.1) and no others. In particular, on V_0 we find eigenvalues $\lambda_0 = 0$ corresponding to the constant function, and $\lambda_0 = 6$ corresponding to the two-dimensional space of functions satisfying $u(q_0)+u(q_1)+u(q_2) = 0$.

When $m = 1$, we find a three-dimensional space with $\lambda_1 = 6$, namely the ones described in Figure 3.3.3, since we can check that these satisfy condition (3.3.1). This is the secret behind the ease with which we glued small copies of these eigenvalues together: They have normal derivatives equal to zero, so the matching condition for normal derivatives is automatic. This is the only initial eigenvalue. There are unique ways to continue $\lambda_0 = 0$ and $\lambda_0 = 6$, namely $\lambda_1 = 0$ (5 is forbidden) and $\lambda_1 = 3$, so this completes the spectrum of Δ_1.

The general case is very similar to the Dirichlet spectrum, with only a few changes:

(1) The constant function is a Neumann eigenfunction with all $\lambda_m = 0$ and $\lambda = 0$.
(2) There is no 2-series.
(3) The 6-series has multiplicity increased by 3, namely the eigenfunctions associated to the boundary points.
(4) The 5-series has multiplicity reduced by 2, since we retain all eigenfunctions associated with loops, but discard the two extra ones that chain from one boundary point to another (these do not satisfy (3.3.1)). In particular, the 5-series begins with $m_0 = 2$. We leave it to the exercises to verify that the dimensions add up correctly.

We still have to justify (3.3.1). If it holds for all m, then since $\lambda_m = O(\frac{1}{5^m})$ for large m we can use it to obtain $\partial_n u(q_i) = 0$, the usual expression of the Neumann boundary condition. Conversely, $\partial_n u(q_i) = 0$ allows us to glue the function to its even reflection, and that leads to (3.3.1).

EXERCISES

3.3.1. Show that there are no Dirichlet eigenvalues with $\lambda_{m_0} = 2$ for $m_0 > 1$. Show that there are no Neumann eigenvalues with $\lambda_{m_0} = 2$ for any m_0.

3.3.2. Prove the linear independence of the $\frac{3^{m-1}+3}{2}$ Dirichlet 5-eigenfunctions for Δ_m constructed above.

3.3.3.* Prove that the Dirichlet (resp., Neumann) spectrum of Δ is the limit (in the sense of (3.2.15)) of the Dirichlet (resp., Neumann) spectra of Δ_m.

3.3.4.* Show directly that the matching conditions for normal derivatives for u at a point in $V_1 \setminus V_0$ is just a multiple of the λ_1-eigenvalue equation at that point if $\lambda_1 = 2$ or 5.

3.3.5. Show that the ground state Dirichlet eigenfunction is positive on $K \setminus V_0$.

3.3.6. Verify that the sum of the multiplicities of the Neumann eigenvalues of Δ_m listed above adds to $\#V_m = \frac{3^{m+1}+3}{2}$.

3.3.7. Show that every Dirichlet eigenfunction in the 2-series is actually constant along the upside-down triangle joining the points in $V_1 \setminus V_0$. (Hint: Use (3.2.7) and induction.)

3.3.8.* Show that every Dirichlet eigenfunction in the 2-series is invariant under the discontinuous map that reflects each cell $F_i K$ about the axis through q_i. Then show by induction that it is invariant under an infinite sequence of discontinuous maps that reflect the $3 \cdot 2^{m-1}$ m-cells that line the upside-down triangle in Exercise 3.3.7. Use this to give a different proof of Exercise 3.3.7.

3.3.10.* Let $\tilde{\lambda}_1 < \tilde{\lambda}_2 < \cdots$ denote the distinct Dirichlet eigenvalues of Δ. Show that they have the following "octave" structure: Of the eigenvalues $\tilde{\lambda}_{7k+1}, \tilde{\lambda}_{7k+2}, \ldots, \tilde{\lambda}_{7k+7}$, the first four have generation of birth $m_0 = 1$, with $\tilde{\lambda}_{7k+1}$ and $\tilde{\lambda}_{7k+4}$ belonging to the 2-series, while $\tilde{\lambda}_{7k+2}$ and $\tilde{\lambda}_{7k+3}$ belong to the 5-series; $\tilde{\lambda}_{7k+5}$ and $\tilde{\lambda}_{7k+7}$ also belong to the 5-series but with $m_0 > 1$, and $\tilde{\lambda}_{7k+6}$ belongs to the 6-series with generation of birth $m_0 - 1$.

3.3.11. Show that the sequence $\{\frac{3}{2}5^m \lambda_m\}$ is increasing to λ for every Dirichlet eigenvalue.

3.4 LOCALIZED EIGENFUNCTIONS

We observed in the last section that there exist localized eigenfunctions on SG, meaning eigenfunctions with small support. This is true for the entire basis of eigenfunctions in the 6-series with large value of m_0, the support being the union of two adjacent m_0-cells. It is also true for some of the basis functions in the 5-series for large m_0, namely those associated to small loops. Since the existence of localized eigenfunctions is unprecedented in all of smooth mathematics, it is worth while trying to understand what makes it possible. We observe from the start that localized eigenfunctions are associated with eigenvalues of high multiplicities, and also eigenfunctions that are simultaneously Dirichlet and Neumann. (By the way, on I the nonzero eigenvalues are all both Dirichlet and Neumann, but there are no joint Dirichlet and Neumann eigenfunctions, and of course no localized eigenfunctions.)

Both I and SG are self-similar and highly symmetric. One aspect of the sine and cosine eigenfunctions on I that we did not emphasize in Section 3.1 is that they are all locally equivalent. For example, $\sin \pi kx$ is obtained from $\sin \pi x$ by composing with the mappings $x \to kx - j$ on the interval $[\frac{j}{k}, \frac{j+1}{k}]$, multiplying by $(-1)^j$ and gluing. Note that this exploits the self-similarity of I with respect to more mappings than just the dyadic similarities. It is not true on SG that all eigenfunctions may be constructed in this fashion, but it is true for some of them. The key idea is that if we start with a λ-eigenfunction u that is both a Dirichlet and Neumann eigenfunction, then $u \circ F_w^{-1}$ on $F_w K$ satisfies the $5^{|w|}\lambda$-eigenvalue equation on $F_w K$ and may be glued to the zero function outside $F_w K$ to yield a joint Dirichlet–Neumann (D–N for short) eigenfunction on K with eigenvalue $5^{|w|}\lambda$ and with support in $F_w K$. If we fix m and look at all words with $|w| = m$, we see that there are at least 3^m localized D–N eigenfunctions (supported in an m-cell) associated to the eigenvalue $5^m\lambda$. So we have simultaneously created localized eigenfunctions and high multiplicities. In addition, we have shown that 5 is a spectral multiplication factor, at least for a portion of the spectrum.

For example, if we start with $m_0 = 2$ and $\lambda_2 = 6$, the three Dirichlet eigenfunctions of Δ_2 (illustrated on the left in Figure 3.3.6) are also Neumann eigenfunctions, so all eigenfunctions of Δ generated by spectral decimation from these (there are infinitely many, since we have the choices of ε_m for $m \geq 4$) are D–N eigenfunctions. The above procedure produces eigenvalues with $m_0 = 2 + m$ and $\lambda_{2+m} = 6$. Note, however, that it does not generate all the localized eigenfunctions with the given eigenvalue. We had one Dirichlet eigenfunction for each point x in V_{m+1}, supported in the two $(m+1)$-cells touching at x. If $x \notin V_m$, then the two cells lie in a single m-cell, and these eigenfunctions are the ones constructed above. But if $x \in V_m$, then the two $(m+1)$-cells lie in distinct m-cells, and the eigenfunction is not of the above type.

Similarly, we see that there are localized eigenfunctions in the 5-series, since there is a D–N eigenfunction with $m_0 = 2$ and $\lambda_2 = 5$, namely the one shown in Figure 3.3.5. By localizing this eigenfunction we get all the eigenfunctions associated with loops of minimal size (for the given value of m_0). There are also localized eigenfunctions associated with larger loops that are not simply localizations of this single eigenfunction.

So far we have seen that the existence of a D–N eigenfunction gives us a seed for constructing some of the localized eigenfunctions we know about. But why should there exist D–N eigenfunctions? One answer is that high multiplicities require it. Specifically, if λ is a Dirichlet eigenvalue of multiplicity at least 4, then there must be a D–N eigenfunction in the eigenspace. This is just linear algebra: The Neumann conditions are just three homogeneous linear equations looking for a nontrivial solution. So the existence of one eigenvalue with multiplicity at least 4 implies not only localized eigenfunctions but also arbitrarily large multiplicities higher up in the spectrum.

Although this is an interesting observation, it does not appear to be very useful, because it is difficult to find an a priori argument for multiplicities above 2. But there is a reason for the existence of D–N eigenfunctions based solely on the D_3 symmetry of SG. The dihedral group has a nontrivial one-dimensional representation, called the alternating representation. This is just a funny way of saying that there are nontrivial functions that are invariant under rotations and skew-symmetric under reflections. We say these functions *transform according to the alternating representation* of D_3. The function in Figure 3.3.5 is one example. Now a general principle in group theory says that there must be infinitely many eigenfunctions of Δ within this class of functions. But the skew-symmetry under reflections implies that both the function and its normal derivative must vanish at boundary points. So all the eigenfunctions of this type are D–N. This argument applies to all fractal Laplacians that have dihedral symmetry of any order, even without spectral decimation. We will return to this in Chapter 4.

How many localized eigenfunctions are there? We don't mean this question literally, but rather: What is the proportion of linearly independent localized eigenfunctions among all Dirichlet (or Neumann) eigenfunctions with eigenvalues going up to a fixed value? Because we also have to quantify the degree of localization, the answer is complicated, so we look at a simpler question: How many eigenfunctions are D–N? The surprising answer is: almost all!

How do we figure out what a bottom part of the Dirichlet spectrum looks like? One simple way is to look at the spectrum of Δ_m. It consists of $\frac{3^{m+1}-3}{2}$ eigenvalues, counting multiplicity, and of these $\frac{3^m-3}{2}$ correspond to $\lambda_m = 6$. If we extend these eigenvalues by choosing $\varepsilon_{m'} = -1$ for all $m' > m$ (except $\varepsilon_{m+1} = 1$ if $\lambda_m = 6$), then we will obtain the smallest continued eigenvalues, and in fact eigenvalues smaller than the initial eigenvalues for any $m_0 > m$. (Remember that initial eigenvalues for $m_0 \geq 2$ are either 5 or 6, and $\varphi_\pm(x) = \frac{5\pm\sqrt{25-4x}}{2} < 5$ for $x > 0$.) Therefore, we obtain the lowest $\frac{3^{m+1}-3}{2}$ eigenvalues in the spectrum of Δ in the limit. We are not concerned here with the quantitative values of these eigenvalues, but rather with the qualitative features of the corresponding eigenfunctions, which are identical to the qualitative features of the eigenfunctions of Δ_m.

So the proportion of D–N eigenfunctions in the bottom part of the Dirichlet spectrum consisting of the first $\frac{3^{m+1}-3}{2}$ eigenvalues is identical to the proportion of D–N eigenfunctions in the Dirichlet spectrum of Δ_m. This is a question we can answer rather exactly.

Let a_m denote the total number of D–N eigenfunctions in the Dirichlet spectrum of Δ_m. For the initial eigenvalue $\lambda_m = 6$, all $\frac{3^m-3}{2}$ eigenfunctions are D–N. For the initial eigenvalue $\lambda_m = 5$, all those associated to loops are D–N, so the number is $\frac{3^{m-1}-1}{2}$. For the continued eigenvalues, we note that there are $a_{m-1} - \frac{3^{m-1}-3}{2}$ D–N eigenfunctions of Δ_{m-1} with $\lambda_{m-1} \neq 6$, and these bifurcate to produce $2a_{m-1} - 3^{m-1} + 3$ D–N eigenfunctions of Δ_m. The remaining $\frac{3^{m-1}-3}{2}$ D–N eigenfunctions of Δ_{m-1} with $\lambda_{m-1} = 6$ do not bifurcate, and so only produce $\frac{3^{m-1}-3}{2}$ D–N eigenfunctions of Δ_m. Adding everything up yields

$$a_m = \frac{3^m-3}{2} + \frac{3^{m-1}-1}{2} + 2a_{m-1} - 3^{m-1} + 3 + \frac{3^{m-1}-3}{2}$$

(3.4.1')

$$= \frac{3^m-1}{2} + 2a_{m-1}.$$

If we let $b_m = \frac{3^{m+1}-3}{2} - a_m$ denote the number of remaining eigenfunctions, then (3.4.1) implies

$$\begin{aligned} b_m &= 3^m - 1 - 2a_{m-1} \\ &= 3^m - 1 - 2\left(b_{m-1} - \frac{3^m-3}{2}\right) \\ &= 2b_{m-1} + 2, \end{aligned}$$

hence

(3.4.2) $$b_m = c2^m - 2$$

and $c = \frac{5}{2}$ because $b_1 = 3$. Clearly, the proportion $b_m / \frac{3^{m+1}-3}{2}$ of non-D–N eigenfunctions goes to zero at an exponential rate, so the proportion of D–N eigenfunctions goes to one.

This counting perspective also shows that the high multiplicities may be truly huge. If we look at the largest eigenvalue among the first $\frac{3^{m+1}-3}{2}$ Dirichlet eigenvalues ($m_0 = m$ and $\lambda_m = 6$), it has multiplicity $\frac{3^m-3}{2}$, so it accounts for almost $\frac{1}{3}$ of all these eigenvalues. We will return to this in the next section.

The existence of localized eigenfunctions has some strange consequences. Consider the heat equation

(3.4.3) $$\frac{\partial}{\partial t} u(x,t) = \Delta u(x,t),$$

where u is now a function on a subset of $SG \times \mathbb{R}$. This has solutions

(3.4.4) $$u(x,t) = e^{-\lambda t} u_\lambda(x),$$

where u_λ is a λ-eigenfunction. If u_λ is localized, then all the heat stays in supp u_λ and never leaks out to the rest of the world. This would indeed make a cozy cell, except that the temperature must be negative somewhere in the cell and it goes to zero in the long run! Similarly, the wave equation

(3.4.5) $$\frac{\partial^2}{\partial t^2} u(x,t) = \Delta u(x,t)$$

has solutions

(3.4.6) $$u(x,t) = \left(a \cos\sqrt{\lambda} t + b \sin\sqrt{\lambda} t\right) u_\lambda(x).$$

For u_λ localized this is a mode of vibration that only involves the cells in supp u_λ. Living in SG, you might not hear your noisy neighbors!

In smooth analysis, localized eigenfunctions cannot exist because eigenfunctions are analytic functions, and nonzero analytic functions cannot vanish on open sets (this assumes that the underlying manifold and Riemannian metric are analytic). So on SG, something must break down in the above reasoning. Actually there is a theory of analytic functions, but eigenfunctions are not necessarily globally analytic, although they are locally analytic.

EXERCISES

3.4.1.* Show that a priori there must be Dirichlet eigenvalues of multiplicity at least 2 because D_3 has an irreducible representation of dimension 2.

3.4.2. If γ_m denotes the largest Dirichlet eigenvalue among the first $\frac{3^{m+1}-3}{2}$ eigenvalues, show that $\gamma_m = c5^m$.

3.4.3. Find a formula for the number of distinct Dirichlet eigenvalues of Δ_m.

3.4.4.* Show that there are Dirichlet eigenfunctions in the 6-series that transform according to the alternating representation of D_3 with $m_0 \geq 3$, but not $m_0 = 2$.

3.4.5.* Show that there are no Dirichlet eigenfunctions in the 5-series that are invariant under D_3.

3.5 SPECTRAL ASYMPTOTICS

We have seen in the last section that the bottom $\frac{3^{m+1}-3}{2}$ eigenvalues of the Dirichlet spectrum of SG are generated from the spectrum of Δ_m. The largest of these eigenvalues (if $m \geq 2$) has $m_0 = m$, $\lambda_m = 6$, $\varepsilon_{m+1} = 1$, and $\varepsilon_{m'} = -1$ for $m' > m+1$. The eigenvalue $\lambda = \frac{3}{2} \lim_{m' \to \infty} 5^{m'} \lambda_{m'}$ is thus equal to a certain constant times 5^m. If we define the Dirichlet eigenvalue counting function

$$\rho(x) = \#\{\lambda \in \Lambda_D : \lambda \leq x\}, \tag{3.5.1}$$

where Λ_D denotes the Dirichlet spectrum (repeated according to multiplicity), then we have

$$\rho\left(c_1 5^m\right) = \frac{3^{m+1}-3}{2} \tag{3.5.2}$$

for the appropriate choice of c_1. This suggests an asymptotic growth rate

$$\rho(x) \sim x^{\log 3/\log 5}. \tag{3.5.3}$$

In fact the discussion shows that we expect more or less identical asymptotic behavior for the Neumann eigenvalue counting functions or the D–N eigenvalue counting function. We can ask, more precisely, for the behavior of the ratio

$$\rho(x)/x^{\log 3/\log 5} \tag{3.5.4}$$

as $x \to \infty$.

In analogy with the Weyl asymptotic law in smooth analysis, we might surmise that (3.5.4) converges to a limit, but we can immediately see that this is impossible because of the very high multiplicity of the eigenspace that gives