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VIBRATION SPECTRA OF FINITELY RAMIFIED, SYMMETRIC FRACTALS

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Abstract

We show how to calculate the spectrum of the Laplacian operator on fully symmetric, finitely ramified fractals. We consider well-known examples, such as the unit interval and the Sierpiński gasket, and much more complicated ones, such as the hexagasket and a non-post critically finite self-similar fractal. We emphasize the low computational demands of our method. As a conclusion, we give exact formulas for the limiting distribution of eigenvalues (the integrated density of states), which is a purely atomic measure (except in the classical case of the interval), with atoms accumulating to the Julia set of a rational function. This paper is the continuation of the work published by the same authors in Ref. 1.

Keywords: Eigenvalues and Eigenfunctions on Fractals; Self-Similar Laplacians; Spectral Decimation.

1. INTRODUCTION

In a previous paper[1] we gave a theoretical justification for a method by which the spectrum of the Laplacian operator on fully symmetric, finitely ramified fractals can be computed. The advantage of this method is that the computational demands are low. The purpose of this paper is to provide detailed explanations and examples of how this method can be applied to a range of fractals.

For the background on the analysis on fractals, one can consult the recent textbook,[2] with more advanced topics covered in monographs.[3–5] Among many applications of fractal structures we particularly note Refs. 6 and 7. The recent survey[8]

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describes relation between analysis on fractals and self-similar groups.

The first self-similar set that we examine is the unit interval[a] (Sec. 4). The purpose of this example is to show that when the finitely ramified fractal is also a set that is amenable to classical analysis, the spectrum that we obtain is the same as the one obtained from the classical analysis. Next we examine the Sierpiński gasket (Sec. 5) as one of the simplest and best known fractals that does not admit a classical analysis of the Laplacian. From here we go on to analyze the Hexagasket (Sec. 6) which is a more complicated and to a novice less familiar fractal than the Sierpiński gasket, yet it is still very similar to the Sierpiński gasket. Lastly we analyze a fractal that is not post-critically finite (Sec. 7) to show how the process works in a more general setting.

Before continuing, we shall take a moment to describe what exactly it is that is vibrating. The physical intuition for these calculations is based on a thin piece of material placed horizontally that is struck and begins to vibrate vertically. One may think of a horizontal fractal membrane or plate which, when vibrating, moves up and down in the vertical direction. It is well known that the pure vibration modes correspond to the eigenmodes of the Laplacian. In particular, the frequency of the vibration corresponds to the eigenvalue, and the shape or amplitude corresponds to the eigenfunction. In the classical one-dimensional case of a string, meaning a unit interval in our terminology, the pure vibration modes are given by familiar sine curves. Figure 1 shows an example of an eigenfunction on the Sierpiński gasket with vastly exaggerated amplitude. Figures 2 and 3 show eigenfunctions on the level 3 Sierpiński gasket. These figures makes use of Neumann boundary conditions as do the basic calculations. For us the Neumann boundary conditions simply imply a reflecting boundary so that no energy is lost from the vibrating fractal, but do not impose any restrictions on how our fractal may vibrate. The case when the boundary points are fixed in space corresponds to Dirichlet, or zero, boundary conditions. In this case the calculations are very similar and so we omit them. Among many related computations of eigenmodes on fractals and in the domains with fractal boundary, we particularly note, Refs. 10–12 which also contain many references.

2. FRACTAL BASICS

The type of fractal that we consider in this paper consists of a compact subset of $\mathbb{R}^2$ which is the fixed set of a family of injective mappings. Note that there is nothing special about two-dimensional Euclidean space, just that it makes the pictures easier to draw. Denote the fractal as F and the set of injective mappings as $\{\phi_i\}_{i=1}^n$. Then the fractal is the unique compact subset such that $F = \bigcup_{i=1}^n \phi_i F$, that is, the fractal is the fixed point of the set of

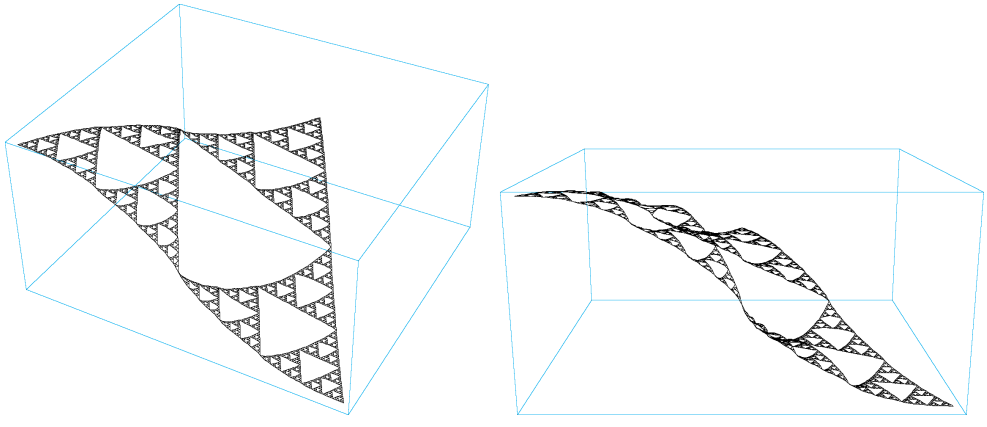

Fig. 1 A basic Neumann eigenfunction on the Sierpiński gasket, three-dimensional views.

[a]The unit interval admits a decomposition into smaller intervals, and so is the attractor of an iterated function system, but perhaps the interval cannot be called a fractal, according to the terminology introduced by Mandelbrot in Ref. 9. The term "fractal" is usually reserved for geometric objects not having simpler Euclidean descriptions. Rather than calling the unit interval a fractal, we call it a Euclidean set admitting a self-similar decomposition.

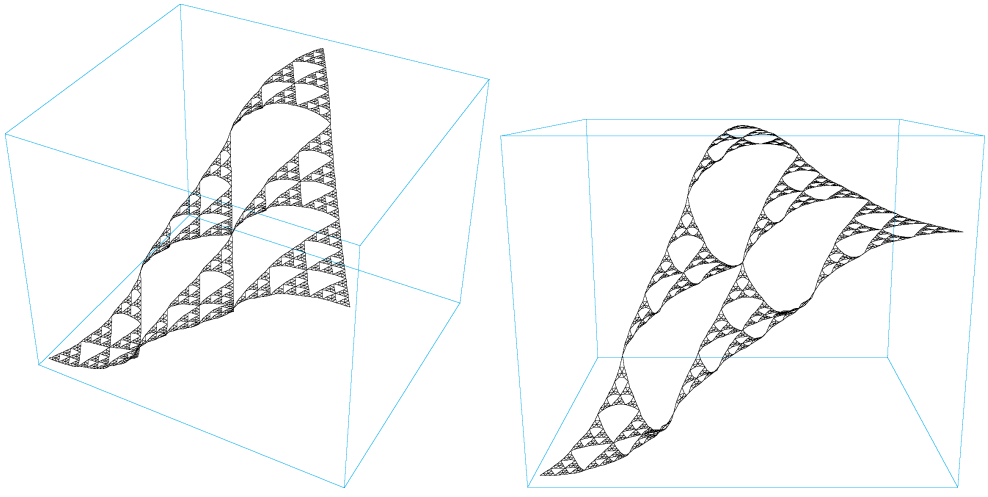

Fig. 2 A basic Neumann eigenfunction on the level 3 Sierpiński gasket, three-dimensional views.

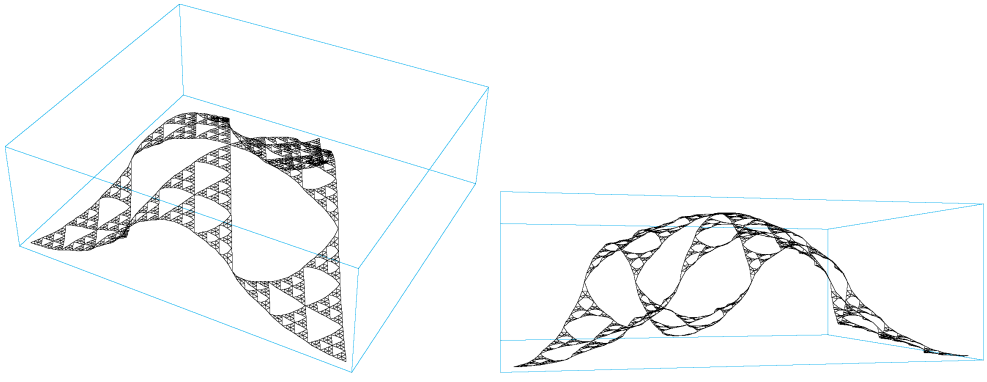

Fig. 3 A Neumann eigenfunction on the level 3 Sierpiński gasket, three-dimensional views.

mappings. We require that $\phi_i F \cap \phi_j F$ is, when $i \neq j$, at most a single point and quite possibly empty. The set of points which are fixed under one of the injections is called the boundary set, that is $x = \phi_i x$ for some i.

We choose to approximate these fractals by a sequence of finite graphs whose limit is the full fractal. The vertices of the depth n approximation are the images of the boundary points under n of the injections. So if $\{z_j\}_{j=1}^m$ is our boundary set then the vertices for the depth 3 approximation of F are $\phi_{i_3} \circ \phi_{i_2} \circ \phi_{i_1}(z_j)$ for all choices of $i_k \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Two vertices are connected by an edge if they are images of different boundary points under the same sequence of n injections. For example, if we look at the boundary points and the depth 0 graph, we get the complete graph with those vertices. If we look at the depth 1 graph we get the vertices $\phi_i(z_j)$ where we place edges between the vertices $\{\phi_i(z_j)\}_{j=1}^m$ and do this for each i. The most intuitive example of this is Fig. 6. We denote the set of vertices for the depth n graph V_n which we also use for the graph itself. Once we have this graph approximation we can consider the probabilistic Laplacian on V_n, whose matrix we give the name M_n.

We write $\sigma(M)$ for the spectrum of the matrix M which, since the matrices are finite-dimensional, is just the set of eigenvalues. If $z \in \sigma(M_n)$ then we write $mult_n z$ for the multiplicity of z as an eigenvalue of M_n. We will spend a great deal of time considering $mult_n z$ as n increases to infinity in later sections.

3. EIGENFUNCTION EXTENSION

The theoretical underpinnings of our procedure are a collection of results of an entirely algebraic flavor that relate the eigenvalues of a given matrix

to the eigenvalues of a submatrix. The relation is described through two rational functions and in all of our examples the submatrix that we look at is the identity matrix which is the reason we have such low computational demands.

The Laplacian matrix that we start with is the depth 1 Laplacian matrix, $M_1 = M$. It is written as a square matrix with ones on the diagonal. Off-diagonal entries are determined by $a_{ij} = \frac{-1}{deg(x_i)}$ where x_i is a vertex in V_1 if x_i and x_j are joined by an edge and $a_{ij} = 0$ when x_i and x_j are not joined by an edge. These matrices have the nice property that when summing across a row the sum is always zero. We define A to be the square block in the upper left hand corner that has as many rows and columns as there are boundary points in the fractal. This matrix is always just a copy of the identity matrix. To begin the iterative process of extending eigenvalues from one Laplacian to a deeper one we have to do the base case first, of extending the eigenvalues of M_0 to those of M. We begin by writing M is block form:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

We consider the Schur complement of $M - zI$ which is

$$S(z) = (A - zI) - B(D - z)^{-1}C.$$

In Ref. 1 we showed, using ideas from Refs. 13 and 14, that

$$S(z) = \phi(z)(M_0 - R(z)).$$

Where $\phi(z)$ and $R(z)$ are scalar valued rational functions, providing that $z \notin \sigma(D)$. If we let N_0 be the number of boundary points then M_0 has entries $a_{ii} = 1$ and $a_{ij} = \frac{-1}{N_0 - 1}$. We can look at specific entries in this matrix valued equation to get the following two scalar valued equations:

$$\phi(z) = -(N_0 - 1)S_{1,2}(z)$$

and

$$R(z) = 1 - \frac{S_{1,1}(z)}{\phi(z)}.$$

Already we can see two types of z which would cause problems for us, either $z \in \sigma(D)$ or $\phi(z) = 0$, we collectively call these points exceptional and the set of them $E(M, M_0)$, the exceptional set.

[b]http://www.math.uconn.edu/~teplyaev/fractals/.

Let us dispose of non-exceptional values of z first:

Theorem 3.1. *Suppose that z is not an eigenvalue of D, and not a zero of ϕ. Then z is an eigenvalue of M with eigenvector v if and only if $R(z)$ is an eigenvalue of M_0 with eigenvector v_0, and $v = \begin{bmatrix} v_0 \\ v' \end{bmatrix}$ where*

$$v' = -(D - zI)^{-1}Cv_0.$$

This implies that there is a one-to-one map from the eigenspace of M_0 corresponding to $R(z)$ onto the eigenspace of M corresponding to z.

Theorem 3.1, in particular, defines the eigenfunction extension map $v' \mapsto -(D - zI)^{-1}Cv_0$. Using this map iteratively one can compute eigenfunctions with very high accuracy and efficiency. The programs implementing this hierarchical iterative procedure do not involve large matrix calculations,[b] and the results are illustrated in Figs. 1 to 3. The first numerical calculations on the Sierpiński gasket were done in Ref. 10, with more topics considered in Refs. 2, 12 and references therein. The level 3 Sierpiński gasket was studied in Refs. 1, 2, 4, 15 and other works.

Theorem 3.1 is proved in Ref. 1. It leaves the exceptional values to be dealt with, which are addressed in the next proposition. In this proposition $mult_D(z)$ is the multiplicity of z as an eigenvalue of D, a similar usage to that of $mult_n$. Here and throughout dim_n is the dimension of the function space on V_n since V_n is a finite collection of points the space of functions on V_n is just the number of points in V_n.

Proposition 3.1. 1. *If $z \notin E(M_0, M)$, then*

$$mult_n(z) = mult_{n-1}(R(z)), \tag{1}$$

and every corresponding eigenfunction at depth n is an extension of an eigenfunction at depth $n - 1$.

2. *If $z \notin \sigma(D)$, $\phi(z) = 0$ and $R(z)$ has a removable singularity at z, then*

$$mult_n(z) = dim_{n-1}, \tag{2}$$

and every corresponding eigenfunction at depth n is localized.

3. *If $z \in \sigma(D)$, both $\phi(z)$ and $\phi(z)R(z)$ have poles at z, $R(z)$ has a removable singularity at z, and*

$\frac{d}{dz}R(z) \neq 0$, *then*

$$mult_n(z) = m^{n-1}mult_D(z) - dim_{n-1} + mult_{n-1}(R(z)), \quad (3)$$

and every corresponding eigenfunction at depth n *vanishes on* V_{n-1}.

4. *If* $z \in \sigma(D)$, *but* $\phi(z)$ *and* $\phi(z)R(z)$ *do not have poles at* z, *and* $\phi(z) \neq 0$, *then*

$$mult_n(z) = m^{n-1}mult_D(z) + mult_{n-1}(R(z)). \quad (4)$$

In this case $m^{n-1}mult_D(z)$ *linearly independent eigenfunctions are localized, and* $mult_{n-1}(R(z))$ *more linearly independent eigenfunctions are extensions of corresponding eigenfunction at depth* $n-1$.

5. *If* $z \in \sigma(D)$, *but* $\phi(z)$ *and* $\phi(z)R(z)$ *do not have poles at* z, *and* $\phi(z) = 0$, *then*

$$mult_n(z) = m^{n-1}mult_D(z) + mult_{n-1}(R(z)) + dim_{n-1} \quad (5)$$

provided $R(z)$ *has a removable singularity at* z. *In this case there are* $m^{n-1}mult_D(z) + dim_{n-1}$ *localized and* $mult_{n-1}(R(z))$ *non-localized corresponding eigenfunctions at depth* n.

6. *If* $z \in \sigma(D)$, *both* $\phi(z)$ *and* $\phi(z)R(z)$ *have poles at* z, $R(z)$ *has a removable singularity at* z, *and* $\frac{d}{dz}R(z) = 0$, *then*

$$mult_n(z) = mult_{n-1}(R(z)), \quad (6)$$

provided there are no corresponding eigenfunctions at depth n *that vanish on* V_{n-1}. *In general we have*

$$mult_n(z) = m^{n-1}mult_D(z) - dim_{n-1} + 2mult_{n-1}(R(z)). \quad (7)$$

7. *If* $z \notin \sigma(D)$, $\phi(z) = 0$ *and* $R(z)$ *has a pole* z, *then* $mult_n(z) = 0$ *and* z *is not an eigenvalue.*

8. *If* $z \in \sigma(D)$, *but* $\phi(z)$ *and* $\phi(z)R(z)$ *do not have poles at* z, $\phi(z) = 0$, *and* $R(z)$ *has a pole* z, *then*

$$mult_n(z) = m^{n-1}mult_D(z) \quad (8)$$

and every corresponding eigenfunction at depth n *vanishes on* V_{n-1}.

The proof of this proposition can be found in Ref. 1 and in a large part depends on the Schur complement formula and taking inverses of matrices in block form. This proposition does the heavy lifting for us as we recursively extend eigenvalues from the V_1 approximation to the full fractal F. We are finally at a point where we can fully write down the description of our method:

1. Identify the self-similar structure of a finitely ramified fractal, create the V_1 and V_2 approximations to the fractal. Note how many cells the fractal is thought of as having (when we consider the unit interval we actually do so three times with different numbers of cells). Write the matrix of the Laplacian.
2. Identify M_0, D, and find their eigenvectors. Calculate $R(z)$ and $\phi(z)$. List $\sigma(M_0)$ as the level zero eigenvalues. List $\sigma(D)$ and the poles of $\phi(z)$ as $E(M_0, M)$, the exceptional values.
3. Do the inductive calculations using Proposition 3.1 to find the multiplicities of the eigenvalues for any V_n approximation of the fractal, and finally take the limiting distribution of those multiplicities.

The choice of examples in this paper show off to full advantage the utility of this process. The example of the one-dimensional interval shows that the number of cells we see the fractal as having is non-canonical and that the method adapts to this, and also it reinforces that classically known results about Laplacians on Euclidean spaces are compatible with our results. The Sierpiński gasket example shows how our method works in a non-trivial fractal that many readers are already familiar with. Then on to the Hexagasket which has a construction very similar to the Sierpiński gasket to reinforce the methods before going onto the non-p.c.f. fractal in the final example. The most salient detail about this fractal is that the vertices have unbounded degree as we consider higher and higher level approximations to the fractal.

4. ONE-DIMENSIONAL INTERVAL AS A SELF-SIMILAR SET

In this section we show how our results allow us to recover classically known information about the spectrum of the discrete Laplacians that approximate the usual one-dimensional continuous Laplacian. The unit interval $[0, 1]$ can be represented as a self-similar set in various ways. Here we consider three cases: when it subdivided into two, three or four subintervals of equal length. In our notation this means that m is 2, 3, or 4. The depth 1 networks for these cases are shown in Fig. 4. The first two cases were also discussed in Ref. 16. Note that in each case the function $R(z)$ is the same as the Chebyshev polynomial of degree m for the interval

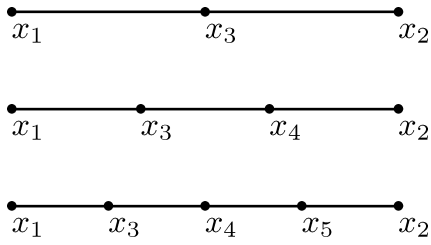

Fig. 4 V_1 networks for the interval in cases $m = 2, 3, 4$, respectively.

$[0, 2]$, which is the smallest interval that contains the spectrum of the matrices M_n. It is shown in Ref. 16, in particular, that the iterations of these polynomials are related in a natural way with the Riemann zeta function. The proof that $R(z)$ is given by the Chebyshev polynomials can be found in Ref. 17.

Case $m = 2$. The matrix of the depth 1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D - z)^{-1}C = \begin{pmatrix} \frac{1}{2(z-1)} & \frac{1}{2(z-1)} \end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{1}{2(1-z)}$$

and

$$R(z) = 2z(2 - z).$$

The only eigenvalue of D is $\sigma(D) = \{1\}$. One can also compute $\sigma(M) = \{2, 1, 0\}$ with the corresponding eigenvectors $\{\{-1, -1, 1\}, \{-1, 1, 0\}, \{1, 1, 1\}\}$. It is easy to see that $\phi(z) \neq 0$. Thus, the exceptional set is

$$E(M_0, M) = \{1\}.$$

To begin the analysis of the exceptional value, note that $R(z)$ does not have any poles (Fig. 5). We are interested in the value of $R(z)$ at the exceptional point, which is $R(1) = 2$. It is easy to see that 1 is a pole of $\phi(z)$, $R(z)$ has a removable singularity at 1, and $\frac{d}{dz}R(1) = 0$. So for all n we can use Proposition 3.1(6) to compute its multiplicity

$$mult_n(1) = 1.$$

Case $m = 3$. The matrix of the depth 1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D - z)^{-1}C = \begin{pmatrix} \frac{2(z-1)}{3-8z+4z^2} & \frac{1}{8z-4z^2-3} \\ \frac{1}{8z-4z^2-3} & \frac{2(z-1)}{3-8z+4z^2} \end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{1}{4\left(z - \frac{3}{2}\right)\left(z - \frac{1}{2}\right)}$$

and

$$R(z) = z(3 - 2z)^2.$$

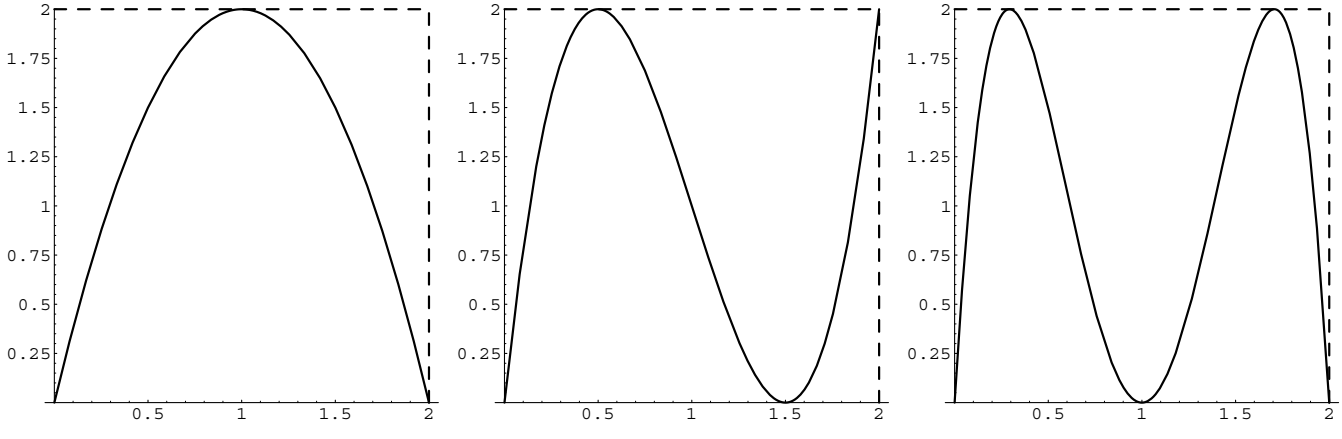

Fig. 5 The graph of $R(z)$ for $F = [0, 1]$ with $m = 2$, $m = 3$ and $m = 4$, respectively.

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{3}{2}, \frac{1}{2}\right\}$$

with corresponding eigenvectors $\{\{-1, 1\}, \{1, 1\}\}$. One can also compute

$$\sigma(M) = \left\{2, \frac{3}{2}, \frac{1}{2}, 0\right\}$$

with the corresponding eigenvectors $\{\{1, -1, -1, 1\}, \{-2, -2, 1, 1\}, \{-2, 2, -1, 1\}, \{1, 1, 1, 1\}\}$. It is easy to see that $\phi(z) \neq 0$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\frac{3}{2}, \frac{1}{2}\right\}.$$

Again, note that $R(z)$ does not have any poles. We are interested in the values of $R(z)$ at the exceptional points, which are

$$R\left(\frac{3}{2}\right) = 0, \quad R\left(\frac{1}{2}\right) = 2.$$

Since $\frac{d}{dz}R(z) = 0$ at these points, we can use Proposition 3.1(6) to obtain

$$mult_n\left(\frac{3}{2}\right) = mult_n\left(\frac{1}{2}\right) = 1$$

for all n.

Case $m = 4$. The matrix of the depth 1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ -\frac{1}{2} & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} \frac{3-8z+4z^2}{4(-1+5z-6z^2+2z^3)} & \frac{1}{4(-1+5z-6z^2+2z^3)} \\ -\frac{1}{2-8z+4z^2} & -\frac{1}{2-8z+4z^2} \\ \frac{1}{4(-1+5z-6z^2+2z^3)} & \frac{3-8z+4z^2}{4(-1+5z-6z^2+2z^3)} \end{pmatrix}.$$

We compute that

$$\phi(z) = \frac{1}{4 - 20z + 24z^2 - 8z^3}$$

and

$$R(z) = 8z(z-2)(1-z)^2.$$

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{1}{2}\left(2+\sqrt{2}\right), 1, \frac{1}{2}\left(2-\sqrt{2}\right)\right\}$$

with corresponding eigenvectors

$$\{\{1, -\sqrt{2}, 1\}, \{-1, 0, 1\}, \{1, \sqrt{2}, 1\}\}.$$

One can also compute

$$\sigma(M) = \left\{2, \frac{1}{2}(2+\sqrt{2}), 1, \frac{1}{2}(2-\sqrt{2}), 0\right\}.$$

It is easy to see that $\phi(z) \neq 0$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\frac{1}{2}(2+\sqrt{2}), 1, \frac{1}{2}(2-\sqrt{2})\right\}.$$

To begin the analysis of the exceptional values, note that $R(z)$ does not have any poles. We are interested in the values of $R(z)$ at the exceptional points, which are

$$R\left(\frac{1}{2}(2+\sqrt{2})\right) = 2, \quad R(1) = 0,$$

$$R\left(\frac{1}{2}(2-\sqrt{2})\right) = 2.$$

Once again, $\frac{d}{dz}R(z) = 0$ at these points, and by Proposition 3.1(6) we have

$$mult_n\left(\frac{1}{2}(2+\sqrt{2})\right) = mult_n(1) = mult_n\left(\frac{1}{2}(2-\sqrt{2})\right) = 1$$

for all n.

5. SIERPIŃSKI GASKET

Spectral analysis on the Sierpiński gasket originates from the physics literature including Refs. 18 and 19 and is well known.[2,10,13,20–22] In this section we show how one can study it using our methods. Note that Sierpiński lattices recently appeared as the Schreier graphs of so-called Hanoi tower groups[5,23–25] (see also Ref. 26).

Figure 6 shows the depth 1 approximation to the Sierpiński gasket. The depth 1 Laplacian matrix $M = M_1$, which is obtained from the above figure, is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{4} & 1 \end{pmatrix}.$$

The eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} \frac{1}{-5+2(7-4z)z} & \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{2(-1+z)}{5+2z(-7+4z)} \\ \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{1}{-5+2(7-4z)z} & \frac{2(-1+z)}{5+2z(-7+4z)} \\ \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{2(-1+z)}{5+2z(-7+4z)} & \frac{1}{-5+2(7-4z)z} \end{pmatrix}.$$

From these we have that (Fig. 7)

$$\phi(z) = \frac{3-2z}{5-14z+4z^2}$$

and

$$R(z) = (5-4z)z.$$

The eigenvalues of M written with multiplicities are

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{4}, \frac{3}{4}, 0\right\}$$

and the corresponding eigenvectors are $\{-1, -1, 0, 0, 0, 1\}$, $\{-1, 0, -1, 0, 1, 0\}$, $\{0, -1, -1, 1, 0, 0\}$, $\{2, 0, -2, -1, 0, 1\}$, $\{2, -2, 0, -1, 1, 0\}$, $\{1, 1, 1, 1, 1, 1\}$. The eigenvalues of D written with multiplicities are

$$\sigma(D) = \left\{\frac{5}{4}, \frac{5}{4}, \frac{1}{2}\right\}$$

and the corresponding eigenvectors are $\{-1, 0, 1\}, \{-1, 1, 0\}, \{1, 1, 1\}$. The equation $\varphi = 0$ has as its solution $\{\frac{3}{2}\}$ so the exceptional set is

$$E(M_0, M) = \left\{\frac{5}{4}, \frac{1}{2}, \frac{3}{2}\right\}.$$

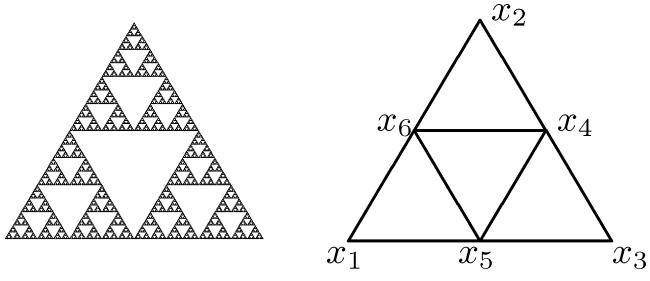

Fig. 6 The Sierpiński gasket and its V_1 network.

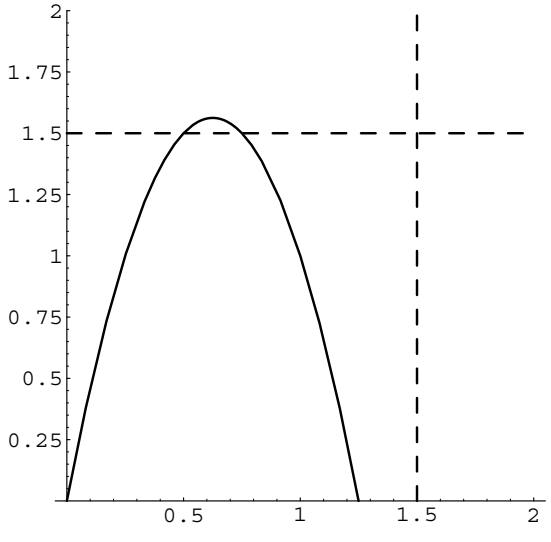

Fig. 7 The graph of $R(z)$ for the Sierpiński gasket.

We can find the multiplicities of these exceptional values by using Proposition 3.1.

For the value $\frac{5}{4}$, which is a pole of $\phi(z)$ and in $\sigma(D)$, we use Proposition 3.1(3) to find the multiplicities:

$$mult_1\left(\frac{5}{4}\right) = 2-3+1 = 0,$$

$$mult_2\left(\frac{5}{4}\right) = 6-6+1 = 1,$$

$$mult_3\left(\frac{5}{4}\right) = 18-15+1 = 4.$$

For the value $\frac{1}{2}$, which is also a pole of $\phi(z)$ and in $\sigma(D)$, we again use Proposition 3.1(3) to find the multiplicities:

$$mult_1\left(\frac{1}{2}\right) = 1-3+2 = 0,$$

$$mult_2\left(\frac{1}{2}\right) = 3-6+3 = 0,$$

$$mult_3\left(\frac{1}{2}\right) = 9-15+6 = 0.$$

For the value $\frac{3}{2}$, since $\frac{3}{2} \notin \sigma(D)$ and $\phi(\frac{3}{2}) = 0$, we use Proposition 3.1(2) to find the multiplicities.

Table 1 Ancestor-Offspring Structure of the Eigenvalues on the Sierpiński Gasket.

$z\in\sigma(M_0)$	0					$\frac{3}{2}$																										
$mult_0(z)$	***1***					***2***																										
$z\in\sigma(M_1)$	0		$\frac{5}{4}$			$\frac{3}{4}$				$\frac{1}{2}$		$\frac{3}{2}$																				
$mult_1(z)$	***1***					***2***						***3***																				
$z\in\sigma(M_2)$	0		$\frac{5}{4}$			$\xi_1(\frac{3}{4})$		$\xi_2(\frac{3}{4})$				$\frac{3}{4}$		$\frac{1}{2}$		$\frac{3}{2}$		$\frac{5}{4}$														
$mult_2(z)$	***1***					***2***		***2***				***3***				***6***		***1***														
$z\in\sigma(M_3)$	0	$\frac{5}{4}$				$*$	$*$	$*$	$*$			$*$	$*$			$\frac{3}{4}$	$\frac{1}{2}$	$*$	$*$	$\frac{3}{2}$	$\frac{5}{4}$											
$mult_3(z)$	***1***					***2***	***2***	***2***	***2***			***3***	***3***			***6***		***1***	***1***	***15***	***4***											

Here the multiplicity of $\frac{3}{2}$ in the nth depth is equal to the dimension at depth $n-1$.

$$mult_1\left(\frac{3}{2}\right) = 3, \quad mult_2\left(\frac{3}{2}\right) = 6,$$

$$mult_3\left(\frac{3}{2}\right) = 15.$$

Table 1 shows the ancestor-offspring structure of the eigenvalues of the Sierpiński gasket. The symbol * indicates the branches

$$\xi_1(z) = \frac{5 - \sqrt{25 - 16z}}{8}$$

and

$$\xi_2(z) = \frac{5 + \sqrt{25 - 16z}}{8}$$

of the inverse function $R^{-1}(z)$ computed at the ancestor value z. By Proposition 3.1(1) the ancestor and the offspring have the same multiplicity. The empty columns represent exceptional values. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right hand part of the same row.

By induction one can obtain the following proposition, which is known in the case of the Sierpiński gasket (see Refs. 2, 13 and 21).

Notation R_{-n} is used for the pre-image of a set A under the nth composition power of the function R.

Proposition 5.1. (i) $\sigma(M_0) = \{0, \frac{3}{2}\}$.

(ii) *For any* $n \geqslant 0$

$$\sigma(M_n) \subset \bigcup_{m=0}^{n} R_{-m}\left\{0, \frac{3}{2}\right\}$$

and for any $n \geqslant 1$ *we have*

$$\sigma(M_n) = \left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\left\{0, \frac{3}{4}\right\}\right).$$

In particular, for $n \geqslant 2$

$$\sigma(M_n) = \left\{0, \frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\left\{\frac{3}{4}\right\}\right) \bigcup \left(\bigcup_{m=0}^{n-2} R_{-m}\left\{\frac{5}{4}\right\}\right).$$

(iii) *For any* $n \geqslant 0$, $dim_n = \frac{3^{n+1}+3}{2}$.

(iv) *For any* $n \geqslant 0$, $mult_n(0) = 1$.

(v) *For any* $n \geqslant 0$, $mult_n\left(\frac{3}{2}\right) = \frac{3^n+3}{2}$.

(vi) *If* $z \in R_{-k}\{\frac{3}{4}\}$ *then* $mult_n(z) = \frac{3^{n-k-1}+3}{2}$ *for* $n \geqslant 1$, $0 \leqslant k \leqslant n-1$.

(vii) *If* $z \in R_{-k}\{\frac{5}{4}\}$ *then* $mult_n(z) = \frac{3^{n-k-1}-1}{2}$ *for* $n \geqslant 2$, $0 \leqslant k \leqslant n-2$.

Corollary 5.2. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure* κ *with the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\left\{\frac{3}{4}\right\}\right) \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\left\{\frac{5}{4}\right\}\right).$$

Moreover,

$$\kappa\left(\left\{\frac{3}{2}\right\}\right) = \frac{1}{3},$$

and

$$\kappa(\{z\}) = 3^{-m-1}$$

if $z \in R_{-m}\{\frac{3}{4}, \frac{5}{4}\}$.

The second and third claim of the corollary follow from the proposition by taking the multiplicity of an eigenvalue at depth n and dividing it by the number of eigenvalues of the depth n Laplacian, counting multiplicities, then limiting n to ∞.

6. HEXAGASKET

The hexagasket, or the hexakun, is a fractal which in different situations (Refs. 2, 4, 27–31 and references therein) is called a polygasket, a six-gasket, or a (2, 2, 2)-gasket. The depth 1 approximation to it is shown in Fig. 8.

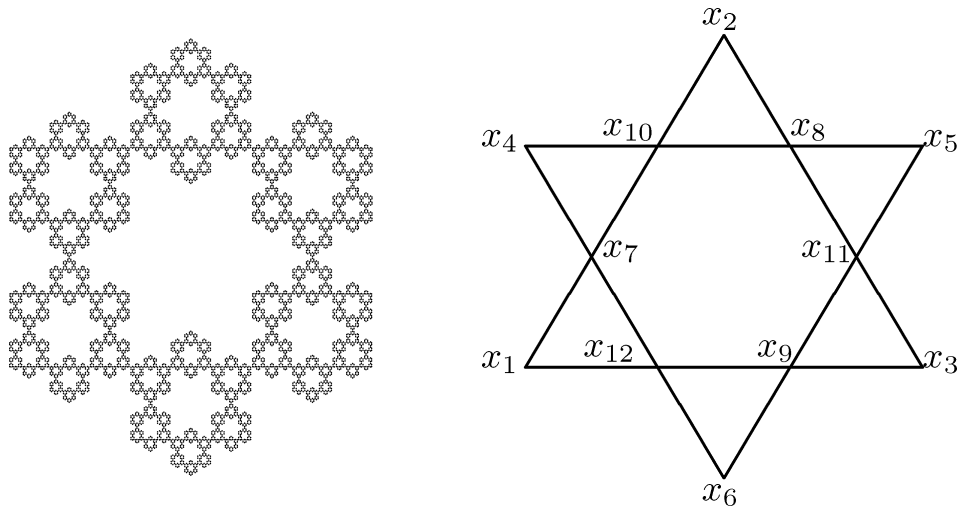

Fig. 8 The hexagasket and its V_1 network.

The matrix of the depth 1 Laplacian $M_1 = M$ is

$$\begin{pmatrix}
1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 \\
-\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 \\
0 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 0 \\
0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} \\
0 & 0 & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 & -\frac{1}{4} \\
0 & 0 & -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & -\frac{1}{2} & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 1
\end{pmatrix}$$

and the eigenfunction extension map $(D - z)^{-1}C$ is the matrix

$$\begin{pmatrix}
-4 + z(23 + 4z(-9 + 4z)) & -1 + z & -2 + (7 - 4z)z \\
-4 + z(23 + 4z(-9 + 4z)) & -2 + (7 - 4z)z & -1 + z \\
-2 + (7 - 4z)z & -4 + z(23 + 4z(-9 + 4z)) & -1 + z \\
-1 + z & -4 + z(23 + 4z(-9 + 4z)) & -2 + (7 - 4z)z \\
-1 + z & -2 + (7 - 4z)z & -4 + z(23 + 4z(-9 + 4z)) \\
-2 + (7 - 4z)z & -1 + z & -4 + z(23 + 4z(-9 + 4z)) \\
-3 + 4(3 - 2z)z & -1 & -3 + 4(3 - 2z)z \\
-3 + 4(3 - 2z)z & -3 + 4(3 - 2z)z & -1 \\
-1 & -3 + 4(3 - 2z)z & -3 + 4(3 - 2z)z
\end{pmatrix}$$

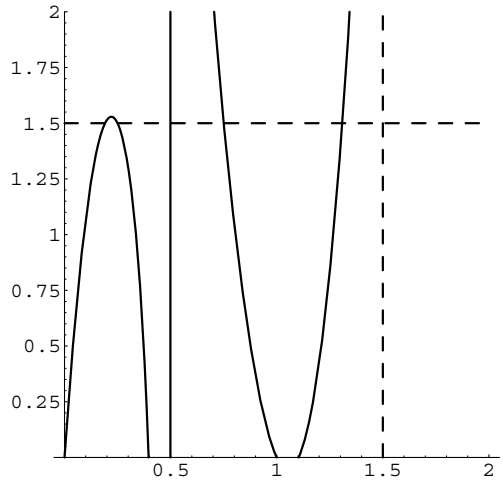

Fig. 9 The graph of $R(z)$ for the hexagasket.

divided by $(1-6z+4z^2)(7-24z+16z^2)$. Moreover, we compute that (Fig. 9)

$$\phi(z) = \frac{3+4(z-2)z}{(1-6z+4z^2)\,(7-24z+16z^2)}$$

and

$$R(z) = \frac{2z(z-1)(7-24z+16z^2)}{2z-1}.$$

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{1}{4}(3\pm\sqrt{5}), \frac{1}{4}(3\pm\sqrt{2}), \frac{1}{4}\left(3\pm\sqrt{2}\right)\right\}.$$

One can also compute

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{4}, \frac{1}{4}, \frac{1}{4}, 0\right\}$$

with the corresponding eigenvectors $\{0, 1, 0, 0, 0, 0, -1, 0, 0, 0, 0, 1\}$, $\{1, 0, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0\}$, $\{1, 0, 0, -1, 0, 0, 0, 0, 0, 1, 0, 0\}$, $\{1, 0, -1, -1, 0, 0, 0, 0, 1, 0, 0, 0\}$, $\{0, 1, -1, 0, 0, 0, -1, 1, 0, 0, 0, 0\}$, $\{1, -1, 0, 0, -1, 1, 0, 0, 0, 0, 0, 0\}$, $\{-1, -1, -1, 0, 0, 0, 0, 0, 0, 1, 1, 1\}$, $\{1, -1, 0, 0, \frac{1}{2}, -\frac{1}{2}, 0, \frac{1}{2}, -\frac{1}{2}, -1, 0, 1\}$, $\{0, -1, 1, -\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, \frac{1}{2}, 0, -1, 1, 0\}$, $\{-1, 1, 0, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, 0, 1\}$, $\{0, 1, -1, -\frac{1}{2}, \frac{1}{2}, 1, \frac{1}{2}, -\frac{1}{2}, -1, -1, 1, 0\}$, $\{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1\}$.

It is easy to see that $\phi(z) = 0$ has two solutions $\frac{1}{2}$ and $\frac{3}{2}$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\frac{3}{2}, \frac{1}{4}(3\pm\sqrt{5}), \frac{1}{4}(3\pm\sqrt{2}), \frac{1}{2}\right\}.$$

To begin the analysis of the exceptional values, note that $\frac{1}{2}$ is the only pole of $R(z)$ and therefore is not an eigenvalue by Proposition 3.1(7).

It is easy to see that $\frac{1}{4}(3\pm\sqrt{2})$ and $\frac{1}{4}(3\pm\sqrt{5})$ are the four poles of $\phi(z)$ and so we can use Proposition 3.1(3) to compute the multiplicities. We obtain

$$mult_1\left(\frac{1}{4}(3\pm\sqrt{2})\right) = 6^0\cdot 2 - 3 + 1 = 0,$$

$$mult_2\left(\frac{1}{4}(3\pm\sqrt{2})\right) = 6^1\cdot 2 - 12 + 1 = 1,$$

$$mult_1\left(\frac{1}{4}(3\pm\sqrt{5})\right) = 6^0\cdot 1 - 3 + 2 = 0,$$

$$mult_2\left(\frac{1}{4}(3\pm\sqrt{5})\right) = 6^1\cdot 1 - 12 + 6 = 0.$$

The exceptional value $\frac{3}{2}$ is in the spectrum $\sigma(D)$, not a pole of $\phi(z)$ and $\phi(\frac{3}{2}) = 0$. For this reason we can use Proposition 3.1(5) to compute the multiplicities.

$$mult_1\left(\frac{3}{2}\right) = 6^0\cdot 3 + 0 + 3 = 6,$$

$$mult_2\left(\frac{3}{2}\right) = 6^1\cdot 3 + 0 + 12 = 30.$$

As in the other sections, the multiplicities of all eigenvalues at depths 0, 1 and 2 are shown in Table 2. The following theorem and corollary summarize the absolute and relative multiplicities of eigenvalues on the hexagasket.

Theorem 6.1. (i) $\sigma(M_0) = \{0, \frac{3}{2}\}$.

(ii) *We have that* $\sigma(M_1) = \left\{0, \frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}\right\}$ *and for* $n \geqslant 2$ *we have*

$$\sigma(M_n)\left\{0, \frac{3}{2}\right\} \cup \left(\bigcup_{m=0}^{n-1} R_{-m}\left\{1, \frac{1}{4}, \frac{3}{4}\right\}\right) \cup \left(\bigcup_{m=0}^{n-2} R_{-m}\left\{\frac{3\pm\sqrt{2}}{4}\right\}\right).$$

(iii) *For any* $n \geqslant 0$ *we have* $dim_n = \frac{6+9\cdot 6^n}{5}$.

(iv) *For any* $n \geqslant 0$, $mult_n(0) = 1$ *and* $mult_n(\frac{3}{2}) = \frac{6+4\cdot 6^n}{5}$.

(v) *For any* $n \geqslant 1$ *and* $0 \leqslant k < n-1$ *we have that if* $z \in R_{-k}(1)$ *then* $mult_n(z) = 1$.

(vi) *For any* $n \geqslant 1$ *and* $0 \leqslant k < n-1$ *we have that if* $z \in R_{-k}\{\frac{1}{4}, \frac{3}{4}\}$ *then*

$$mult_n(z) = \frac{6 + 4\cdot 6^{n-k-1}}{5}.$$

Table 2 Ancestor-Offspring Structure of the Eigenvalues on the Hexagasket.

<table>
<tr><td>$z{\in}\sigma(M_0)$</td><td colspan="8">0</td><td colspan="9">$\frac{3}{2}$</td><td colspan="5"></td></tr>
<tr><td>$mult_0(z)$</td><td colspan="8">***1***</td><td colspan="9">***2***</td><td colspan="5"></td></tr>
<tr><td>$z{\in}\sigma(M_1)$</td><td colspan="3">0</td><td colspan="4">1</td><td>$\frac{3\pm\sqrt{2}}{4}$</td><td colspan="4">$\frac{1}{4}$</td><td colspan="4">$\frac{3}{4}$</td><td>$\frac{3\pm\sqrt{5}}{4}$</td><td colspan="3">$\frac{3}{2}$</td><td colspan="2"></td></tr>
<tr><td>$mult_1(z)$</td><td colspan="3">***1***</td><td colspan="4">***1***</td><td></td><td colspan="4">***2***</td><td colspan="4">***2***</td><td></td><td colspan="3">***6***</td><td colspan="2"></td></tr>
<tr><td>$z{\in}\sigma(M_2)$</td><td>0</td><td>1</td><td>$\frac{3\pm\sqrt{2}}{4}$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td></td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td>$*$</td><td></td><td>$\frac{1}{4}$</td><td>$\frac{3}{4}$</td><td>$\frac{3\pm\sqrt{5}}{4}$</td><td>$\frac{3}{2}$</td><td>$\frac{3\pm\sqrt{2}}{4}$</td></tr>
<tr><td>$mult_2(z)$</td><td>***1***</td><td>***1***</td><td></td><td>***1***</td><td>***1***</td><td>***1***</td><td>***1***</td><td></td><td>***2***</td><td>***2***</td><td>***2***</td><td>***2***</td><td>***2***</td><td>***2***</td><td>***2***</td><td>***2***</td><td></td><td>***6***</td><td>***6***</td><td></td><td>***30***</td><td>***1*** | ***1***</td></tr>
</table>

(vii) *For any* $n \geqslant 2$ *and* $0 \leqslant k < n-2$ *we have that if* $z \in R_{-k}(\frac{3\pm\sqrt{2}}{4})$ *then*

$$mult_n(z) = \frac{6^{n-k-1}-1}{5}.$$

(viii) *For* $n \geqslant 0$ *we have* $mult_n(\frac{3\pm\sqrt{5}}{4}) = 0$.

Proof. For this fractal we have $\sigma(D) = \{0, \frac{3}{2}\}$ with $mult_0\left(\frac{3}{2}\right) = 2$ and, for the purposes of Proposition 3.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from Proposition 3.1(1) because 0 is a fixed point of $R(z)$, and from Proposition 3.1(5).

Items (v) and (vi) follow from Proposition 3.1(1).

Items (vii) and (viii) follow from Proposition 3.1(3). □

Corollary 6.1. *The normalized limiting distribution of eigenvalues* (*the integrated density of states*) *is a pure point measure* κ *with the set of atoms*

$$\left\{\frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{\infty} R_{-m}\left\{\frac{1}{4}, \frac{3}{4}, \frac{3\pm\sqrt{2}}{4}\right\}\right).$$

Moreover, $\kappa\left(\{\frac{3}{2}\}\right) = \frac{4}{9}$, *and*

$$\kappa(\{z\}) = \frac{4}{9}6^{-m-1} \quad \text{if } z \in R_{-m}\left\{\frac{1}{4}, \frac{3}{4}\right\};$$

$$\kappa(\{z\}) = \frac{1}{9}6^{-m-1} \quad \text{if } z \in R_{-m}\left\{\frac{3\pm\sqrt{2}}{4}\right\}.$$

7. A NON-P.C.F. ANALOG OF THE SIERPIŃSKI GASKET

Several non-p.c.f. analogs of the Sierpiński gasket were introduced in Ref. 29. Here we analyze the simplest one of them. It is finitely ramified but not p.c.f. in the sense of Kigami. This fractal can be constructed as a self-affine fractal in $\mathbb{R}^2$ using six affine contractions, as shown in Ref. 29. One affine contraction has the fixed point $(0,0)$ and the matrix

$$\begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

and the other five affine contractions can be obtained by combining this one with the symmetries of the equilateral triangle with vertices $(0,0)$, $(1,0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. Figure 10 shows the fractal and the V_1 network for it. It is proved in Ref. 29 that this particular embedding into $\mathbb{R}^2$ has the advantage that the set restrictions of $C^1(\mathbb{R}^2)$ functions to the fractal is dense in the domain of the energy form $\mathcal{E}$. It is significantly more difficult to describe the domain of the Laplacian, as explained in Ref. 32. Note that Ref. 29 considers the same energy (Dirichlet) form, but a different Laplacian.

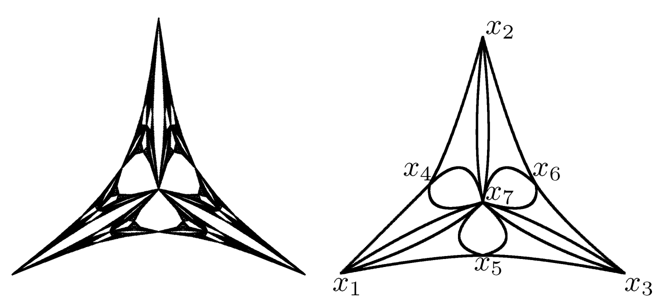

Fig. 10 The non-p.c.f. analog of the Sierpiński gasket and its V_1 network.

The matrix of the depth 1 Laplacian $M_1 = M$ is

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{1}{4} & 0 & -\frac{1}{4} & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{4} & -\frac{1}{4} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{4} & -\frac{1}{4} & 1 & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & 0 & -\frac{1}{4} & 0 & 1 & 0 & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{4} & 0 & 0 & 0 & 1 & -\frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & 1 \end{pmatrix}$$

and the eigenfunction extension map is

$$(D-z)^{-1}C = \begin{pmatrix} -\frac{1}{6-18z+12z^2} & \frac{-5+6z}{12(1-3z+2z^2)} & \frac{-5+6z}{12(1-3z+2z^2)} \\ \frac{-5+6z}{12(1-3z+2z^2)} & -\frac{1}{6-18z+12z^2} & \frac{-5+6z}{12(1-3z+2z^2)} \\ \frac{-5+6z}{12(1-3z+2z^2)} & \frac{-5+6z}{12(1-3z+2z^2)} & -\frac{1}{6-18z+12z^2} \\ \frac{1}{-3+6z} & \frac{1}{-3+6z} & \frac{1}{-3+6z} \end{pmatrix}.$$

Moreover, we compute that

$$\phi(z) = \frac{15-14z}{24-72z+48z^2}$$

and

$$R(z) = -\frac{24z(z-1)(2z-3)}{14z-15}.$$

The eigenvalues of D, written with multiplicities, are

$$\sigma(D) = \left\{\frac{3}{2}, 1, 1, \frac{1}{2}\right\}$$

with corresponding eigenvectors $\{-1,-1,-1,1\}$, $\{-1,0,1,0\}$, $\{-1,1,0,0\}$, $\{1,1,1,1\}$. One can also compute

$$\sigma(M) = \left\{\frac{3}{2}, \frac{3}{2}, \frac{5}{4}, \frac{5}{4}, \frac{3}{4}, \frac{3}{4}, 0\right\}$$

with the corresponding eigenvectors $\{-1,-1,-1,0,0,0,1\}$, $\{-1,-1,-1,1,1,1,0\}$, $\{-1,0,1,-1,0,1,0\}$, $\{-1,1,0,-1,1,0,0\}$, $\{1,0,-1,-1,0,1,0\}$, $\{1,-1,0,-1,1,0,0\}$, $\{1,1,1,1,1,1,1\}$ (Fig. 11).

It is easy to see that $\phi(z) = 0$ has one solution $\{\frac{15}{14}\}$. Thus, the exceptional set is

$$E(M_0, M) = \left\{\frac{3}{2}, 1, \frac{1}{2}, \frac{15}{14}\right\}.$$

To begin the analysis of the exceptional values, note that $\frac{15}{14}$ is a pole of $R(z)$ and therefore is not an eigenvalue by Proposition 3.1(7). We are interested in the values of $R(z)$ in the other exceptional points, which are

$$R(1) = R\left(\frac{3}{2}\right) = 0 \quad \text{and} \quad R\left(\frac{1}{2}\right) = \frac{3}{2}.$$

It is easy to see that 1 and $\frac{1}{2}$ are poles of $\phi(z)$ and so we can use Proposition 3.1(3) to compute the multiplicities. We obtain

$$mult_1(1) = 2 - 3 + 1 = 0,$$

$$mult_1\left(\frac{1}{2}\right) = 1 - 3 + 2 = 0,$$

$$mult_2(1) = 12 - 7 + 1 = 6,$$

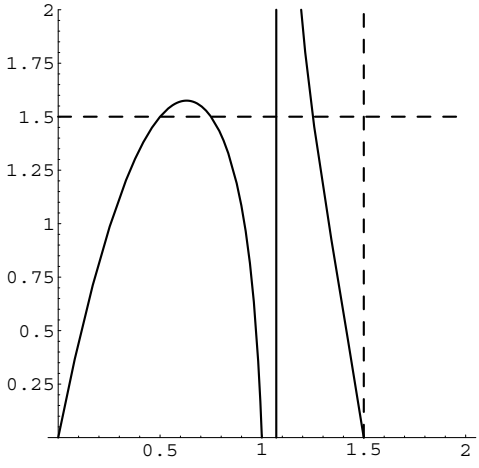

Fig. 11 The graph of $R(z)$ for the non-p.c.f. analog of the Sierpiński gasket.

Table 3 Ancestor-Offspring Structure of the Eigenvalues on the Non-p.c.f. Analog of the Sierpiński Gasket.

$z\in\sigma(M_0)$	0						$\frac{3}{2}$											
$mult_0(z)$	***1***						***2***											
$z\in\sigma(M_1)$	0			1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{4}$			$\frac{5}{4}$			$\frac{3}{2}$					
$mult_1(z)$	***1***						***2***			***2***			***2***					
$z\in\sigma(M_2)$	0	1	$\frac{3}{2}$				*	*	*	*	*	*	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{4}$	$\frac{1}{2}$	1	$\frac{3}{2}$
$mult_2(z)$	***1***						***2***	***2***	***2***	***2***	***2***	***2***		***2***	***2***	***1***	***6***	***7***

and

$$mult_2\left(\frac{1}{2}\right) = 6 - 7 + 2 = 1.$$

Since $\frac{3}{2}$ is not a pole of $\phi(z)$, we can use Proposition 3.1(4) to compute the multiplicities

$$mult_1\left(\frac{3}{2}\right) = 1 + 1 = 2$$

and

$$mult_2\left(\frac{3}{2}\right) = 6 + 1 = 7.$$

The ancestor-offspring structure of the eigenvalues on the non-p.c.f. analog of the Sierpiński gasket is shown in Table 3. The symbol * indicates branches of the inverse function $R^{-1}(z)$ computed at the ancestor value. The multiplicity of the ancestor is the same as that of the offspring by Proposition 3.1(1). The empty columns correspond to the exceptional values. If they are eigenvalues of the appropriate M_n, then the multiplicity is shown in the right hand part of the same row. We have the following theorem and corollary to summarize the results at all depths.

Theorem 7.1. (i) *For any $n \geqslant 0$ we have that $\sigma(\Delta_n) \subset \bigcup_{m=0}^{n} R_{-m}(\{0, \frac{3}{2}\})$ and $\sigma(\Delta_1) = \{0, \frac{3}{4}, \frac{5}{4}, \frac{3}{2}\}$.*

(ii) *For $n \geqslant 2$ we have that*

$$\sigma(\Delta_n) = \left\{0, \frac{3}{2}\right\} \bigcup \left(\bigcup_{m=0}^{n-1} R_{-m}\left\{\frac{3}{4}, \frac{5}{4}\right\}\right) \bigcup \left(\bigcup_{m=0}^{n-2} R_{-m}\left\{\frac{1}{2}, 1\right\}\right).$$

(iii) *For any $n \geqslant 0$ we have $dim_n = \frac{11+4\cdot 6^n}{5}$.*

(iv) *For any $n \geqslant 0$ we have $mult_n(0) = 1$.*

(v) *For any $n \geqslant 1$ we have $mult_n\left(\frac{3}{2}\right) = 6^{n-1} + 1$.*

(vi) *For any $n \geqslant 1$ and $z \in R_{1-n}\{\frac{3}{4}, \frac{5}{4}\}$ we have that $mult_n(z) = 2$.*

(vii) *For any $0 \leqslant m \leqslant n-2$ and $z \in R_{-m}\{\frac{3}{4}, \frac{5}{4}\}$ we have that*

$$mult_n(z) = mult_{n-m-1}\left(\frac{3}{2}\right) = 6^{n-m-2} + 1.$$

(viii) *For any $0 \leqslant m \leqslant n-2$ and $z \in R_{-m}\{\frac{1}{2}\}$ we have $mult_n\left(\frac{1}{2}\right) = \frac{11\cdot 6^{n-m-2}-6}{5}$.*

(ix) *For any $0 \leqslant m \leqslant n-2$ and $z \in R_{-m}\{1\}$ we have $mult_n(1) = \frac{6^{n-m}-6}{5}$.*

Proof. For this fractal we have $\sigma(\Delta_0) = \{0, \frac{3}{2}\}$ with $mult_0\left(\frac{3}{2}\right) = 2$ and, for the purposes of Proposition 3.1, $m = 6$.

Item (i) is obtained above in this section.

Item (ii) follows from the subsequent items.

Item (iii) is straightforward by induction.

Item (iv) follows from Proposition 3.1(1) because 0 is a fixed point of $R(z)$.

Item (v) easily follows from Proposition 3.1(4).

Items (vi) and (vii) follows from the items above.

Items (viii) and (ix) follows from Proposition 3.1(3) because

$$mult_n\left(\frac{1}{2}\right) = 6^{n-1}\cdot 1 - \frac{11 + 4\cdot 6^{n-1}}{5} + 6^{n-2} + 1 = \frac{11\cdot 6^{n-2} - 6}{5},$$

$$mult_n(1) = 6^{n-1}\cdot 2 - \frac{11 + 4\cdot 6^{n-1}}{5} + 1 = \frac{6^n - 6}{5}.$$

□

Corollary 7.1. *The normalized limiting distribution of eigenvalues (the integrated density of states) is a pure point measure κ with the set of atoms*

$$\bigcup_{m=0}^{\infty} R_{-m}\left\{\frac{3}{2}, 1\right\},$$

where $\kappa\left(\{\frac{3}{2}\}\right) = \frac{5}{24}$ and

$$\kappa(\{z\}) = \frac{5}{4}6^{-m-2} \quad \text{if } z \in R_{-m}\left\{\frac{3}{4}, \frac{5}{4}\right\};$$

$$\kappa(\{z\}) = \frac{11}{4}6^{-m-2} \quad \text{if } z \in R_{-m}\left\{\frac{1}{2}\right\};$$

$$\kappa(\{z\}) = \frac{1}{4}6^{-m} \quad \text{if } z \in R_{-m}\{1\}.$$

8. CONCLUSION

While the method we have described and used several times in this paper does require some steps that are not readily automated, the very low computational demands make it practical to apply this to even very complicated fractals provided there is enough symmetry and a nice enough cell structure. With the availability of computer algebra systems the computations to calculate the functions $R(z)$ and $\phi(z)$ for a given fractal this step is only time consuming if the fractal has dozens of boundary points. The one computational step that we have not considered is giving an approximation of the Julia set of $R(z)$ for any of these fractals. This is because, despite the aesthetic value of the pictures, there is no novelty in such calculations.

One of the applications of our results is to produce examples of Laplacians on fractals with large spectral gaps and nicer analogs of Fourier series (i.e. eigenfunction expansions), according to Ref. 33 (see also Refs. 34 and 35). Another set of applications is related to the localization of eigenfunctions (see Refs. 13, 36 and 37) and quantum and metric graphs (see Refs. 29, 38–42).

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REFERENCES

1. N. Bajorin, T. Chen, A. Dagan, C. Emmons, M. Hussein, M. Khalil, P. Mody, B. Steinhurst and A. Teplyaev, Vibration modes of 3n-gaskets and other fracals, *J. Phys. A Math. Theor.* **41** (2008) 015101 (21 pp.).
2. R. S. Strichartz, *Differential Equations on Fractals: A Tutorial* (Princeton University Press, 2006).
3. M. T. Barlow, Diffusions on fractals, in *Lectures on Probability Theory and Statistics*, Saint-Flour, 1995, *Lecture Notes in Mathematics*, Vol. 1690 (Springer, Berlin, 1998).
4. J. Kigami, Analysis on fractals, in *Cambridge Tracts in Mathematics*, Vol. 143 (Cambridge University Press, 2001).
5. V. Nekrashevych, Self-similar groups, in *Mathematical Surveys and Monographs*, Vol. 117 (AMS, 2005).
6. C. Puente-Baliarda, J. Romeu, Ra. Pous and A. Cardama, On the behavior of the Sierpinski multiband fractal antenna, *IEEE Trans. Antennas Propagation* **46** (1998) 517–524.
7. Y. Gefen, A. Aharony and B. B. Mandelbrot, Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices, *J. Phys. A* **16** (1983) 1267–1278; **17** (1984) 435–444 and 1277–1289.
8. V. Nekrashevych and A. Teplyaev, Groups and analysis on fractals, to appear in *Analysis on Graphs and Its Applications, Proceedings of Symposia in Pure Mathematics*, American Mathematical Society.
9. B. B. Mandelbrot, *The Fractal Geometry of Nature* (W. H. Freeman and Co., San Francisco, California, 1982).
10. K. Dalrymple, R. S. Strichartz and J. P. Vinson, Fractal differential equations on the Sierpinski gasket, *J. Fourier Anal. Appl.* **5** (1999) 203–284.
11. M. L. Lapidus, J. W. Neuberger, R. J. Renka and C. A. Griffith, Snowflake harmonics and computer graphics: numerical computation of spectra on fractal drums, *Int. J. Bifurc. Chaos Appl. Sci. Eng.* **6** (1996) 1185–1210.
12. R. S. Strichartz and M. Usher, Splines on fractals, *Math. Proc. Cambridge Philos. Soc.* **129** (2000) 331–360.
13. A. Teplyaev, Spectral analysis on infinite Sierpiński gaskets, *J. Funct. Anal.* **159** (1998) 537–567.
14. L. Malozemov and A. Teplyaev, Self-similarity, operators and dynamics, *Math. Phys. Anal. Geom.* **6** (2003) 201–218.
15. S. Drenning and R. Strichartz, Spectral Decimation on Hamblys Homogeneous Hierarchical Gaskets, preprint.
16. A. Teplyaev, Spectral zeta functions of fractals and the complex dynamics of polynomials, *Trans. Am. Math. Soc.* **359** (2007) 4339–4358.
17. B. Krön and E. Teufl, Asymptotics of the transition probabilities of the simple random walk on self-similar graph, *Trans. Am. Math. Soc.* **356** (2003) 393–414.
18. R. Rammal, Spectrum of harmonic excitations on fractals, *J. Phys.* **45** (1984) 191–206.
19. R. Rammal and G. Toulouse, Random walks on fractal structures and percolation clusters, *J. Phys. Lett.* **44** (1983) L13–L22.
20. J. Béllissard, Renormalization group analysis and quasicrystals, in *Ideas and Methods in Quantum and Statistical Physics*, Oslo, 1988, (Cambridge University Press, Cambridge, 1992), pp. 118–148.
21. M. Fukushima and T. Shima, On a spectral analysis for the Sierpiński gasket, *Potential Anal.* **1** (1992) 1–35.
22. T. Shima, On eigenvalue problems for Laplacians on p.c.f. self-similar sets, *Japan J. Indust. Appl. Math.* **13** (1996) 1–23.
23. R. Grigorchuk and Z. Sunic, Asymptotic aspects of Schreier graphs and Hanoi Towers groups,

C. R. Math. Acad. Sci. Paris **342** (2006) 545–550.
24. R. Grigorchuk and V. Nekrashevych, Self-similar groups, operator algebras and Schur complement. *J. Mod. Dyn.* **1**(3) (2007) 323–370.
25. A. Teplyaev, Asymtotic and spectral analysis on a Hanoi Towers group and the Sierpinski gasket, preprint.
26. L. Bartholdi, R. Grigorchuk and V. Nekrashevych, From fractal groups to fractal sets, in *Fractals in Graz 2001, Trends in Mathematics* (Birkhäuser, 2003), pp. 25–118.
27. B. Adams, S. A. Smith, R. Strichartz and A. Teplyaev, The spectrum of the Laplacian on the pentagasket, in *Fractals in Graz 2001, Trends in Mathematics* (Birkhäuser, 2003).
28. B. Boyle, D. Ferrone, N. Rifkin, K. Savage and A. Teplyaev, Electrical resistance of N-gasket fractal networks, *Pac. J. Math.* **233** (2007) 15–40.
29. A. Teplyaev, Harmonic coordinates on fractals with finitely ramified cell structure, *Can. J. Math.* **60** (2008) 457–480.
30. J. Tyson and J.-M. Wu, Characterizations of snowflake metric spaces, *Ann. Acad. Sci. Fenn. Math.* **30** (2005) 313–336.
31. J. Tyson and J.-M. Wu, Quasiconformal dimensions of self-similar sets, *Rev. Mat. Iberoam.* **22** (2006) 205–258.
32. O. Ben-Bassat, R. S. Strichartz and A. Teplyaev, What is not in the domain of the Laplacian on Sierpinski gasket type fractals, *J. Funct. Anal.* **166** (1999) 197–217.
33. R. S. Strichartz, Laplacians on fractals with spectral gaps have nicer Fourier series, *Math. Res. Lett.* **12** (2005) 269–274.
34. J. Schenker and M. Aizenman, The creation of spectral gaps by graph decoration, *Lett. Math. Phys.* **53** (2000) 253–262.
35. D. Zhou, Spectral analysis of Laplacians on certain fractals, Ph.D. thesis, University of Waterloo (2007).
36. L. Malozemov and A. Teplyaev, Pure point spectrum of the Laplacians on fractal graphs, *J. Funct. Anal.* **129** (1995) 390–405.
37. M. T. Barlow and J. Kigami, Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets, *J. Lond. Math. Soc.* **56** (1997) 320–332.
38. E. Akkermans, A. Comtet, J. Desbois, G. Montambaux and C. Texier, Spectral determinant on quantum graphs, *Ann. Phys.* **284** (2000) 10–51.
39. A. Comtet, J. Desbois and C. Texier, Functionals of Brownian motion, localization and metric graphs, *J. Phys. A* **38** (2005) R341–R383.
40. P. Exner, P. Hejcik and P. Seba, Approximation by graphs and emergence of global structures, *Rep. Math. Phys.* **57** (2006) 445–455.
41. P. Kuchment, Quantum graphs I. Some basic structures, *Waves Random Media* **14** (2004) S107–S128.
42. P. Kuchment, Quantum graphs II. Some spectral properties of quantum and combinatorial graphs, *J. Phys. A* **38** (2005) 4887–4900.