DISCRETE SPECTRUM AND IRREDUCIBILITY ON DIRICHLET METRIC MEASURE SPACES (PRELIMINARY REPORT)

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joint research with
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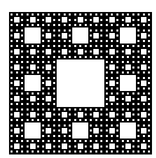
Kyoto University, January 10, 2025



Plan:

(1) Spectral and sub-Riemannian heat kernel convergence: $\mathbf{SU\,(2)} \longrightarrow \mathbb{H}$

(2) Discrete spectrum and eigenfunction estimates for DMMS

(3) Irreducibility

(4) Fractals (... if time permits ...)

ASYMPTOTIC DILATIONS

Contraction $\Phi : \mathrm{SU}(2) \longrightarrow \mathbb{H}$

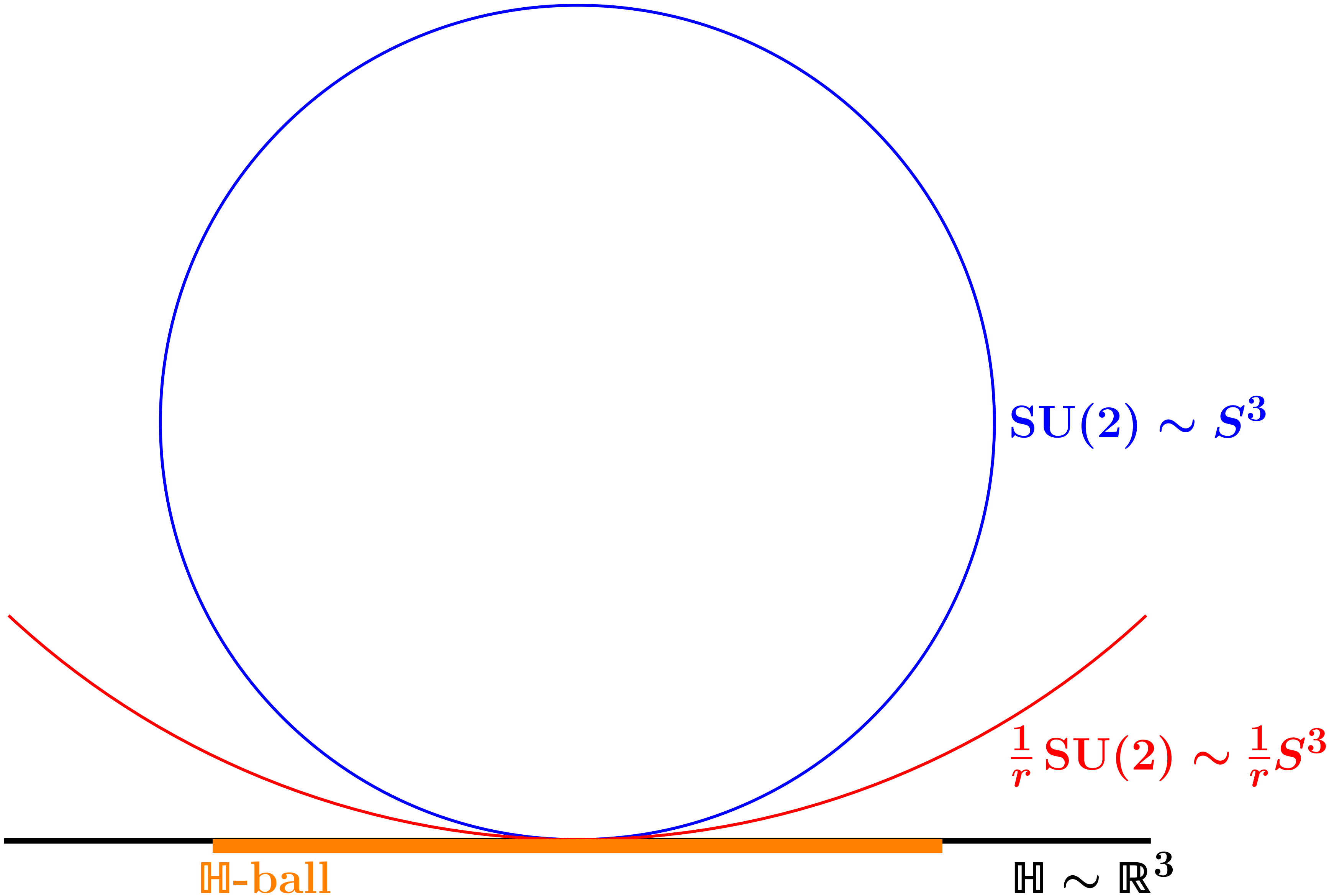
- Both groups are equipped with a sub-Riemannian structure
- Heisenberg group $\mathbb{H}$ viewed as a re-scaled limit of $\mathrm{SU}(2)$ near the identity

Convergence of the re-normalized spectrum in $\mathrm{SU}(2)$ to the spectrum in the unit ball of $\mathbb{H}$

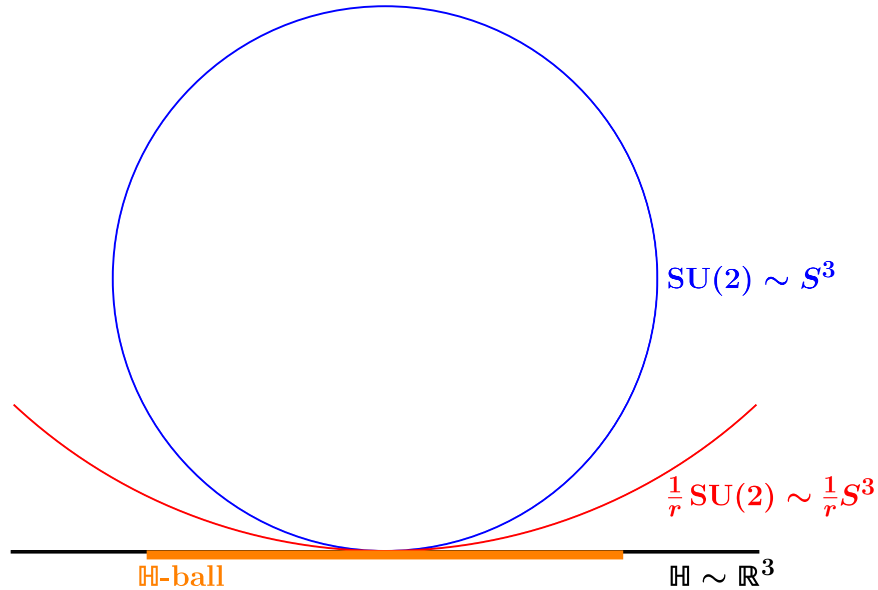
Theorem (Carfagnini, Gordina, Teplyaev)

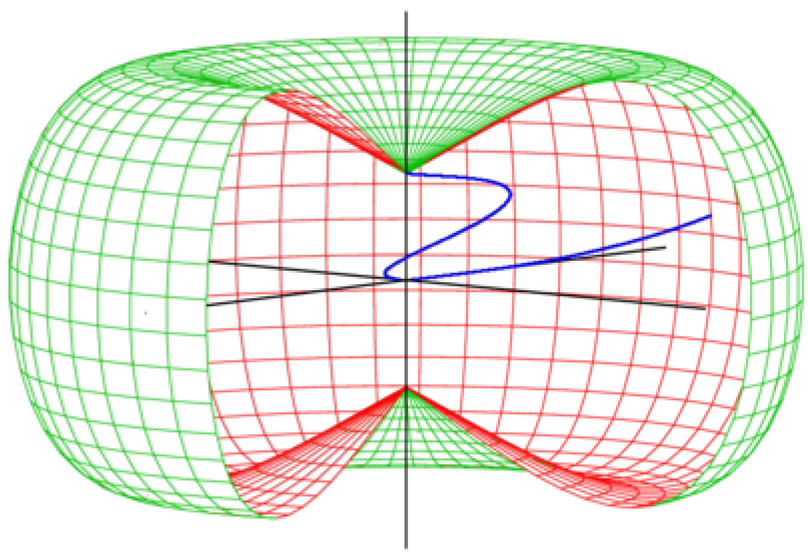
- $0 < \lambda_1^{\mathbb{H}} < \lambda_2^{\mathbb{H}} \leqslant \lambda_3^{\mathbb{H}} \leqslant \ldots$ Dirichlet eigenvalues in the unit ball $B_1^{\mathbb{H}}$ in $\mathbb{H}$, counted with multiplicity
- $0 < \lambda_1^{r} < \lambda_2^{r} \leqslant \lambda_3^{r} \leqslant \ldots$ Dirichlet eigenvalues in the r-ball $B_r^{\mathrm{SU}(2)}$ in $\mathrm{SU}(2)$, counted with multiplicity

$$\Longrightarrow \quad \lim_{r \to 0} r^2 \lambda_n^r = \lambda_n^{\mathbb{H}} \qquad n \geqslant 1$$



$\mathbf{SU(2)} \sim S^3$
$\frac{1}{r}\mathbf{SU(2)} \sim \frac{1}{r}S^3$
$\mathbb{H}$-ball
$\mathbb{H} \sim \mathbb{R}^3$





the Heisenberg ball [picture made by Nate Eldredge]

MOSCO CONVERGENCE, STRONG AND NORM RESOLVENT CONVERGENCE

- Mosco convergence is equivalent to the strong resolvent convergence, which does not imply the convergence of eigenvalues.

- The norm resolvent convergence is stronger than the strong resolvent convergence and it does imply the convergence of eigenvalues.

- - We aim at even stronger uniform convergence of resolvent and heat kernels and eigenfunctions using Dynkin-Hunt formula:

$$p_t^{\mathcal{U}}(x,y) := p_t(x,y) - \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{\mathcal{U}}<t\}}\, p_{t-\tau_{\mathcal{U}}}\left(X_{\tau_{\mathcal{U}}}, y\right)\right]$$

SELECTED RELATED ARTICLES

- David Croydon and Ben Hambly. **Local limit theorems for sequences of simple random walks on graphs**. Potential Analysis (2008).

- Alexander Grigor'yan and Naotaka Kajino. **Localized upper bounds of heat kernels for diffusions via a multiple Dynkin-Hunt formula**. Transactions of the American Mathematical Society (2017).

- Zhen-QingChen, Takashi Kumagai, Laurent Saloff-Coste, Jian Wang, and Tianyi Zheng. **Long range random walks and associated geometries on groups of polynomial growth**. Annales de l'Institut Fourier (2022).

- Zhen-Qing Chen, Takashi Kumagai, Laurent Saloff-Coste, Jian Wang, and Tianyi Zheng. **Limit theorems for some long range random walks on torsion free nilpotent groups**. Springer (2023).

CONVERGENCE OF THE DIRICHLET HEAT KERNELS

- $p_t^{\mathcal{U},\mathbb{H}}(\cdot,\cdot)$ Dirichlet heat kernel in $\mathcal{U} \subset \mathbb{H}$
- $p_t^{\mathcal{V},\mathrm{SU}(2)}(\cdot,\cdot)$ Dirichlet heat kernel in $\mathcal{V} \subset \mathrm{SU}\left(2\right)$

Lemma (Carfagnini, Gordina, Teplyaev) For each $t > 0$

$$\lim_{r\to 0} r^4 p_{r^2 t}^{\mathcal{V}_r,\mathrm{SU}(2)}\left(\Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right), \Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right)\right) = p_t^{\mathcal{U},\mathbb{H}}(x,y)$$

uniformly for $x, y \in \mathcal{U}$, which is an $\underline{\textit{arbitrary bounded open subset}}$ of $\mathbb{H}$. Here $\mathcal{V}_r := \Phi^{-1}\left(\delta_r^{\mathbb{H}}(\mathcal{U})\right) \subset \mathrm{SU}$.

Corollary (Carfagnini, Gordina, Teplyaev) For each $t > 0$

$$\lim_{r\to 0} r^4 p_{r^2 t}^{B_r^{\mathrm{SU}(2)}}\left(\Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right), \Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right)\right) = p_t^{B_1^{\mathbb{H}}}(x,y)$$

uniformly for $x, y \in B_1^{\mathbb{H}}$

LOCAL CONVERGENCE OF STOCHASTIC FLOWS

- g_s hypoelliptic Brownian motion on $\mathbf{SU}(2)$
- X_s hypoelliptic Brownian motion on $\mathbb{H}$

Lemma (Carfagnini, Gordina, Teplyaev) For small enough r there is a continuous stochastic process Y_s^r in $\mathbb{H}$ such that

$$Y_s^r := \delta_{1/r}^{\mathbb{H}} \Phi\left(g_{r^2 s}\right) \qquad s < \inf\{t : d_{\mathbb{H}}(I, Y_s^r) \geqslant 1\}$$

in the sense of distributions and

$$\lim_{r \to 0} \sup_{0 \leqslant s \leqslant T} |Y_s^r - X_s| = 0$$

in probability.

Proof. ... Kunita 1986 *Lectures on stochastic flows and applications*, ... plus geometric localization arguments. □

"Elliptic results" + pointed Gromov-Hausdorff convergence:
Hui-Chun Zhang and Xi-Ping Zhu. Weyl's law on RCD(K,N) metric measure spaces. Comm. Anal. Geom. 2019.

NASH INEQUALITY

- Carlen-Kusuoka-Stroock '87: Nash inequality $p_t(x,y) \leqslant ct^{-\nu/2}$

$$\|f\|_{L^2(\mathcal{X},\mu)}^{2+\frac{4}{\nu}} \leqslant C\mathcal{E}(f,f)\|f\|_{L^1(\mathcal{X},\mu)}^{\frac{4}{\nu}} \qquad f \in \mathcal{D}_{\mathcal{E}}$$

$$\|P_t f\|_{L^\infty(\mathcal{X},\mu)} \leqslant C\, t^{-\nu/2}\|f\|_{L^1(\mathcal{X},\mu)} \qquad f \in L^1(\mathcal{X},\mu), t>0$$

- **ultracontractivity**, Davies, Varopoulos et al

- M. Carfagnini, M. Gordina and A. Teplyaev: Riemannian manifolds with non-negative Ricci curvature, self-similar processes, not necessarily continuous; Brownian motion on fractals; Dirichlet forms on mms under Sturm's assumptions (complete closed balls, doubling, weak Poincaré, PHI); group action on metric measure spaces, convergence of spectra

DISCRETE SPECTRUM FOR DIRICHLET FORMS

Proposition [Carfagnini, Gordina, Teplyaev] Assuming ultracontractivity,

- $\mu(\mathcal{U}) < \infty \implies$ the spectrum of $A^{\mathcal{U}}$ is discrete and the heat kernel $p_t^U(x, y)$ has the usual eigenfunction expansion

- $\lambda_1 > 0$ if $\lim\limits_{t\to\infty} \operatorname*{ess\,sup}\limits_{(x,y)\in\mathcal{U}\times\mathcal{U}} p_t^{\mathcal{U}}(x, y) \to 0$

Example [Bounded domains, continuous spectrum] Note that there are well-known examples of bounded domains in $\mathbb{R}^2$ such that Neumann Laplacian has no discrete spectrum, e. g. B. Simon 1991, 1992, "jelly roll".

Example [Unbounded domains, discrete spectrum] Note that there are well-known examples of Schr̈odinger operators on unbounded metric measure spaces of infinite measure with purely discrete spectrum, e. g. B. Simon 2009.

GENERALIZED HEAT CONTENT

$$Q_{\mathcal{U}}(t) := \int_{\mathcal{U}} \mathbb{P}^x\left(\tau_{\mathcal{U}} > t\right) dm(x) = \int_{\mathcal{U}} u(t,x)dx$$

Theorem (C-G-T) Under ultracontractivity and other usual assumptions for any open set $\mathcal{U}$ of finite measure

$$\lim_{t\to\infty} e^{\lambda_1 t} Q_{\mathcal{U}}(t) = \sum_{k=1}^{M_1} c_k^2,$$

where $c_k := \int_{\mathcal{U}} \phi_k(x)dm(x)$, and M_1 is the multiplicity of λ_1.

Again, no regularity of the boundary is assumed.

ESTIMATES OF EIGENFUNCTIONS

Theorem (C-G-T) Under the usual assumptions and the Nash inequality, for any open set $\mathcal{U}$ of finite measure, the spectrum is discrete and eigenfunctions satisfy

$$\|\varphi_n\|_{L^\infty} \leqslant c\lambda_n^\delta \|\varphi_n\|_{L^2}$$

where c is a constant depending on $\mathcal{U}$ and δ.
Again, no regularity of the boundary is assumed.

- *This inequality was obtained by Jun Kigami* **in the case of self-similar p.c.f. fractals**, formula (4.5.1).

Our article contains more detailed estimates in more general ultracontractive cases and under more specific heat kernel bounds. Usually

$$\delta = \frac{\alpha}{\beta} = \frac{\nu}{2}$$

where the space is Alhfors α-regular and β is the time scaling exponent if the process is *(distance-)self-similar*:

$$d(X^x_{t\varepsilon}, x) \overset{(d)}{=} \varepsilon^{\frac{1}{\beta}} d(X^x_t, x).$$

SMALL DEVIATIONS

Theorem [Carfagnini, Gordina, Teplyaev] Assume that

- $P_t^{B_1(x)}$ is irreducible for some $x \in \mathcal{X}$
- the heat kernel $p_t^{B_1(x)}(x, y)$ exists for all t and for all $x, y \in \mathcal{X}$ and that

$$p_t(x, y) \leqslant c\, t^{-\frac{\alpha}{\beta}} \text{ for any } t, x, y$$

- there exists a t_0 such that $p_{t_0}^{B_1(x)}(x, y)$ is continuous for $x, y \in \mathcal{X}$
- X_t^x is self-similar

$$\Longrightarrow$$

- $$\lim_{\varepsilon \to 0} e^{\lambda_1 \frac{t}{\varepsilon^\beta}} \mathbb{P}^x \left(\sup_{0 \leqslant s \leqslant t} d(X_s, x) < \varepsilon \right) = c_1 \varphi_1(x),$$

where $\lambda_1 > 0$ is the spectral gap of $A^{B_1(x)}$ with zero boundary conditions outside of the unit ball $B_1(x)$, and φ_1 is the corresponding positive eigenfunction, $c_n := \int_U \varphi_n(y) \mu\,(dy)$

GROUP ACTIONS ON DMMS

Let G be a topological group acting measurably on a metric measure space $(\mathcal{X}, d, \mu)$, that is, there is a measurable map

$$\Phi : G \times \mathcal{X} \longrightarrow \mathcal{X}, (g, x) \longrightarrow \Phi_g(x) =: x_g$$

such that

$$\Phi_e(x) = x \text{ for } \mu - \text{a.e. } x \in \mathcal{X}$$
$$\Phi_g(\Phi_h(x)) = \Phi_{gh}(x) \text{ for all } g, h \in G, \text{ and } \mu - \text{a.e. } x \in \mathcal{X}.$$

Here e is the identity element in G.

Such a group action induces an action on L^2-functions on G as follows

$$\widetilde{\Phi} : G \times L^2(\mathcal{X}, \mu) \longrightarrow L^2(\mathcal{X}, \mu), \ (\widetilde{\Phi}_g f)(x) = f(\Phi_g(x)).$$

Let us denote by $(\Phi_g)_* m$ the pushforward of m under $\Phi_g : \mathcal{X} \to \mathcal{X}$.

Definition The action of a group G on a Dirichlet metric measure space $(\mathcal{X}, d, \mu, \mathcal{E})$ is said to be a *group action preserving the Dirichlet space class* if it is a measurable action on $(\mathcal{X}, d, \mu)$ such that

$$\widetilde{\Phi}_g(\mathcal{D}_{\mathcal{E}}) = \mathcal{D}_{\mathcal{E}}, \quad \widetilde{\Phi}_g(\mathcal{D}_A) = \mathcal{D}_A,$$

$(\Phi_g)_*\mu$ and μ are mutually absolutely continuous for all $g \in G$, and the Radon-Nikodym derivative

$$J_g := \frac{d(\Phi_g)_*\mu}{d\mu}(x)$$

is independent of $x \in \mathcal{X}$.

Definition We say that a Dirichlet metric measure space $(\mathcal{X}, d, \mu, \mathcal{E})$ admits a G*-dilation structure* if the action of G on $(\mathcal{X}, d, \mu, \mathcal{E})$ preserves the Dirichlet space class and if there exists $\kappa = \kappa(g)$ such that

$$\mathcal{E}(f \circ \Phi_g, h \circ \Phi_g) = J_g^{\kappa}\mathcal{E}(f, h), \tag{1}$$

for any $f, h \in \mathcal{D}_{\mathcal{E}}$.

IRREDUCIBILITY

A Borel set $A \in \mathcal{B}(\mathcal{X})$ is *P_t-invariant* if $P_t(\mathbb{1}_A f) = 0$ μ-a.e. on A for every $t > 0$ and $f \in L^2(\mathcal{X}, \mu)$.

The semigroup $\{P_t\}_{t \geqslant 0}$ is called *irreducible* if for any P_t-invariant set A either $\mu(A) = 0$ or $\mu(A^c) = 0$.

► Suppose a diffusion has an ***a.e. positive heat kernel***.
Is this diffusion **irreducible in each path-connected open set** (killed at exiting this open set)?

(As usual, no regularity of the boundary is assumed.)

... the intuitive answer is "yes" ...

▶ If a diffusion has a **positive heat kernel**, then is this diffusion **irreducible in each path-connected open set** (killed at exiting this open set)? ... there are examples with the negative answer.

Example [Reducible local Dirichlet form on a connected set] Let $\mathcal{X} = \mathbb{R}^2$ with the Euclidean metric, and μ be the Lebesgue measure on $\mathcal{X}$, S be a non-trivial straight line segment in $\mathbb{R}^2$. Consider the Dirichlet form corresponding to the standard two-dimensional Brownian motion X_t killed at the first time it reaches S. Then P_t is irreducible in $\mathbb{R}^2$, but it is reducible in any disk $\mathcal{U}$ which is separated by S into two non-empty parts.

Example [Reducible non-local Dirichlet form on a connected set] Let

$$A := \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : |x - y| < 1 \text{ and } [x, y] \cap S = \emptyset\},$$

where $[x, y]$ denotes a straight line interval connecting x and y. Let

$$\mathcal{E}(f, g) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} (f(x) - f(y)) (g(x) - g(y)) \mathbb{1}_A(x, y) \, dx dy.$$

Then $\mathcal{E}$ is irreducible, but $\mathcal{E}^{\mathcal{U}}$ is reducible if $\mathcal{U}$ is again any disk which is separated by S into two non-empty parts.

Proposition [Irreducibility of the semigroup] Assume that the heat kernel $p_t(x,y)$ exists for all t and all $x,y \in \mathcal{X}$ and $\mathcal{U}$ is an open path-connected set in $\mathcal{X}$. The semigroup $P_t^{\mathcal{U}}$ is irreducible if for any $y \in \mathcal{U}$ and r small enough and $x \in B_r(y)$ there exists $t_0 = t_0(x,y,r)$ such that for any $z \in \mathcal{U}^c$ and any $s < t < t_0$

$$p_t(x,y) - p_s(z,y) > 0.$$

Corollary Let $(\mathcal{X}, d, \mu)$ satisfy the chain condition, and $\mathcal{U}$ is path-connected,

$$c_1\, t^{-\frac{\alpha}{\beta}}\, \Phi\left(c_2 \frac{d(x,y)}{t^{\frac{1}{\beta}}}\right) \leqslant p_t(x,y) \leqslant c_3 \left(t^{-\frac{\alpha}{\beta}} + 1\right)\, \Phi\left(c_4 \frac{d(x,y)}{t^{\frac{1}{\beta}}}\right)$$

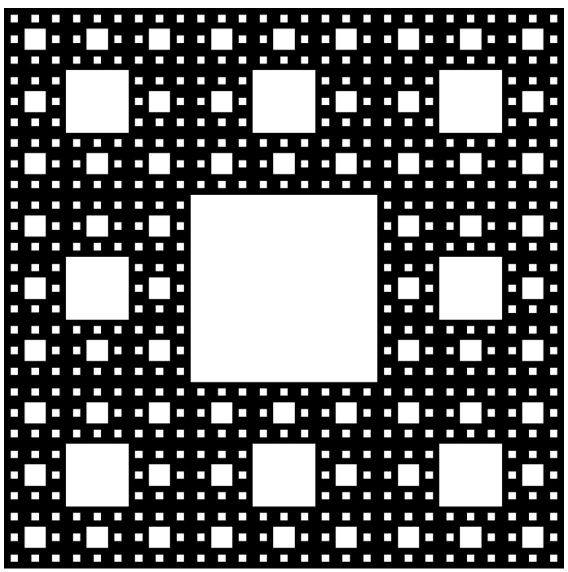
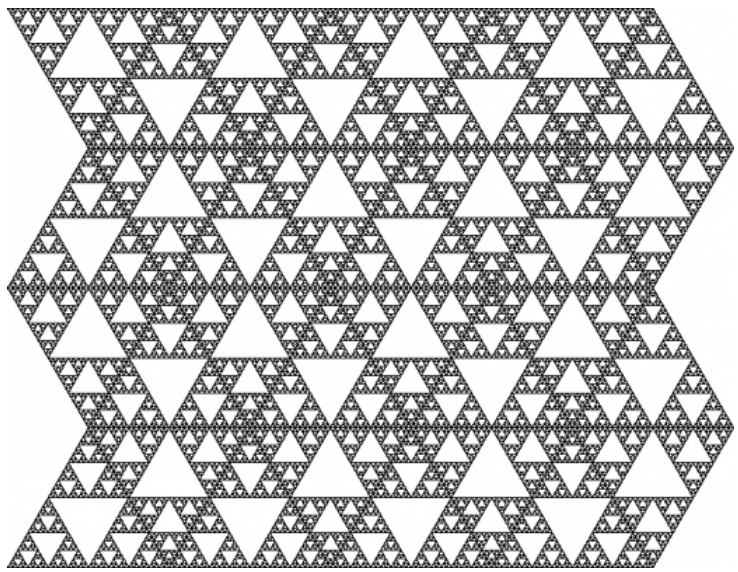
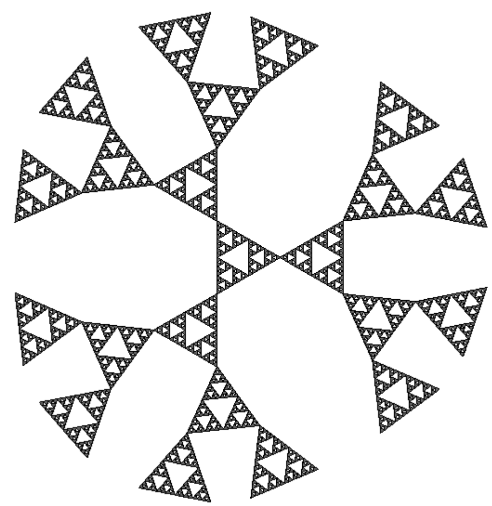
where Φ is a positive decreasing function on $[0,\infty)$ with

$$\lim_{r\to\infty} r^{\alpha}\Phi(r) = 0$$

then $P_t^{\mathcal{U}}$ is irreducible.

FRACTALS (OR FRACTAFOLDS*)

- *Strichartz: *A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold.*
  - *A "fractafold" is to a fractal what a manifold is to a Euclidean half-space.*
    - There is no generally agreed upon definition of "fractal", other than "I know one when I see one":

Motivation: Strichartz'89, *Harmonic analysis as spectral theory of Laplacians*

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

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Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at

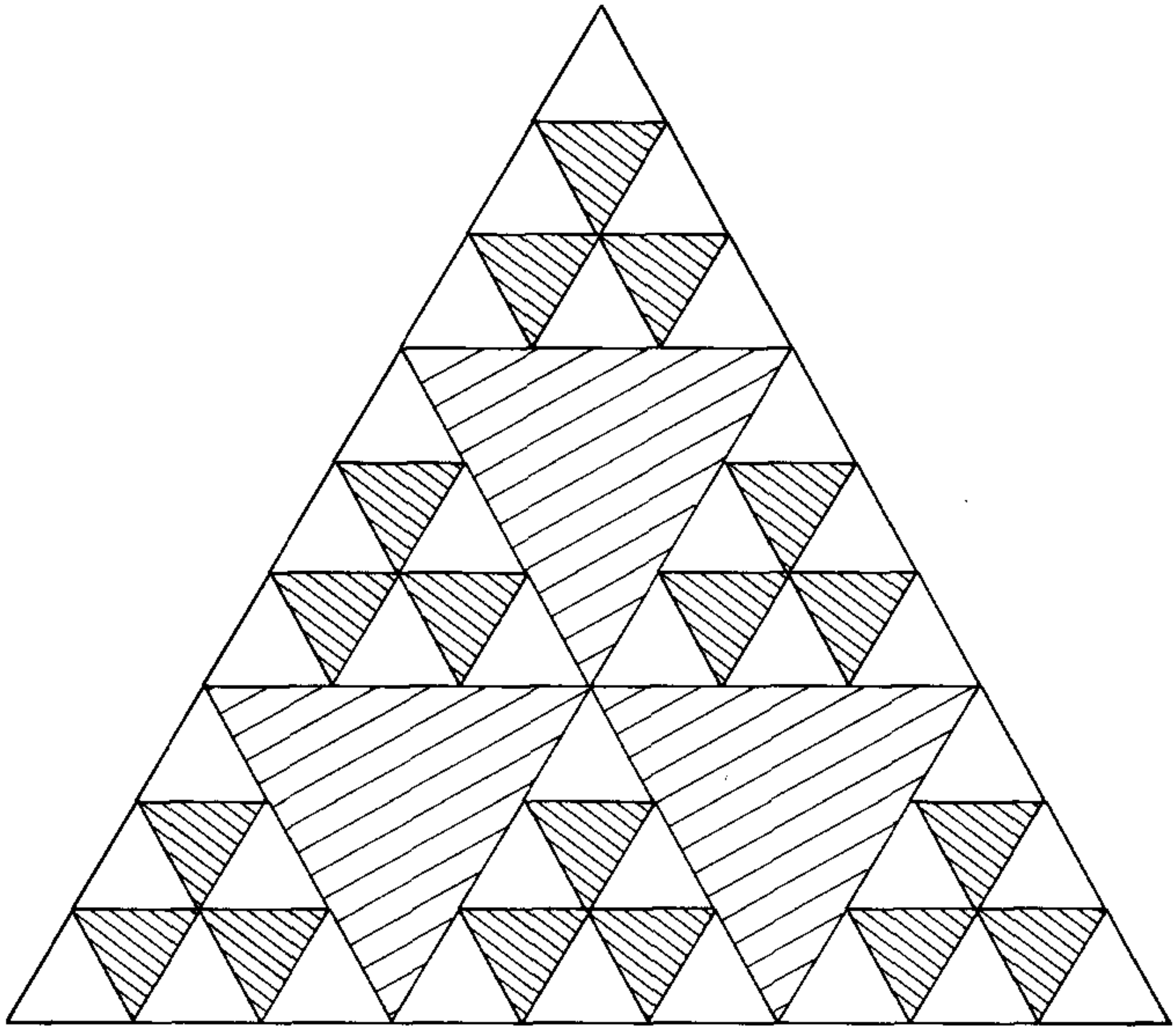

Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

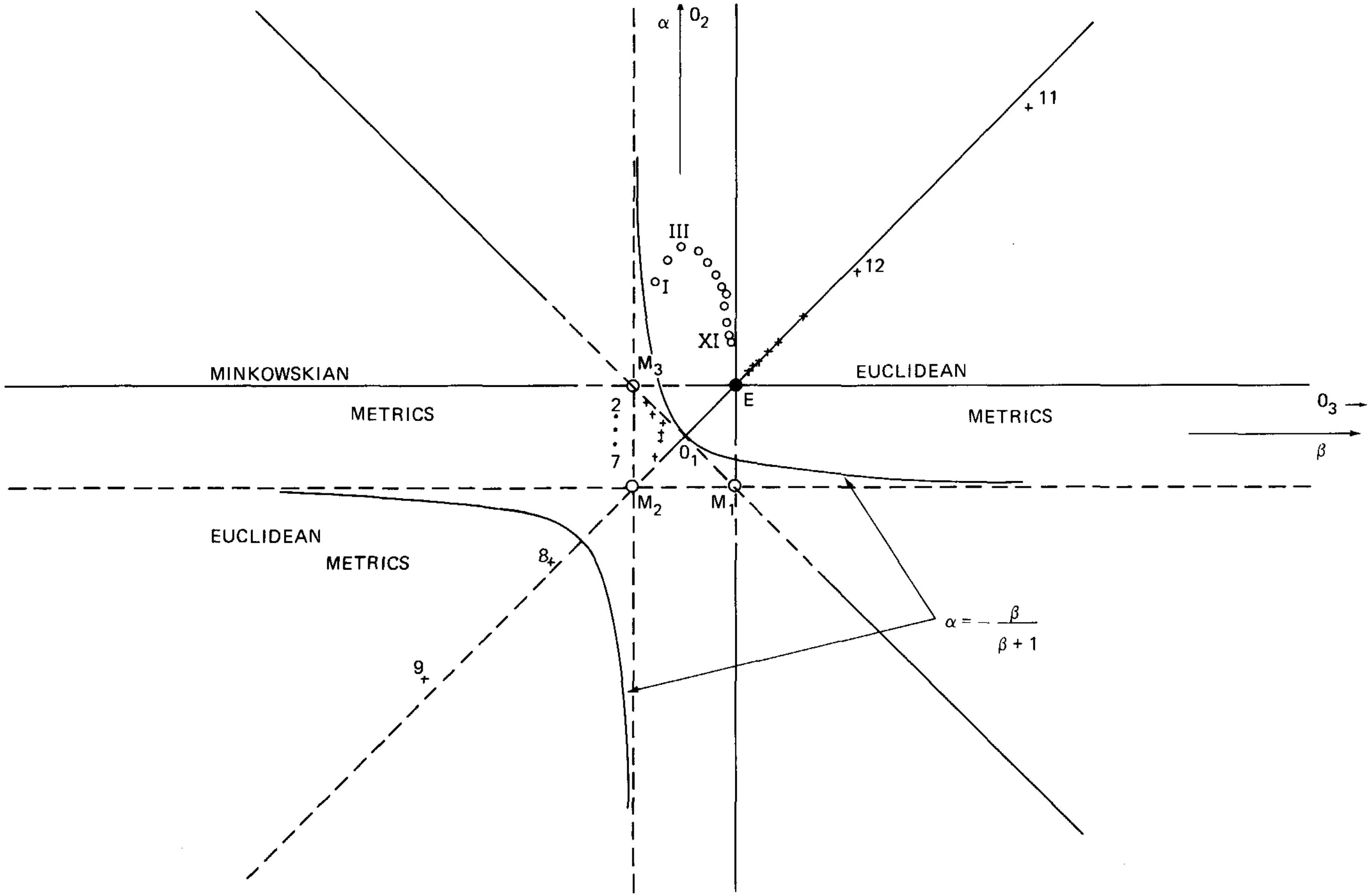

Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

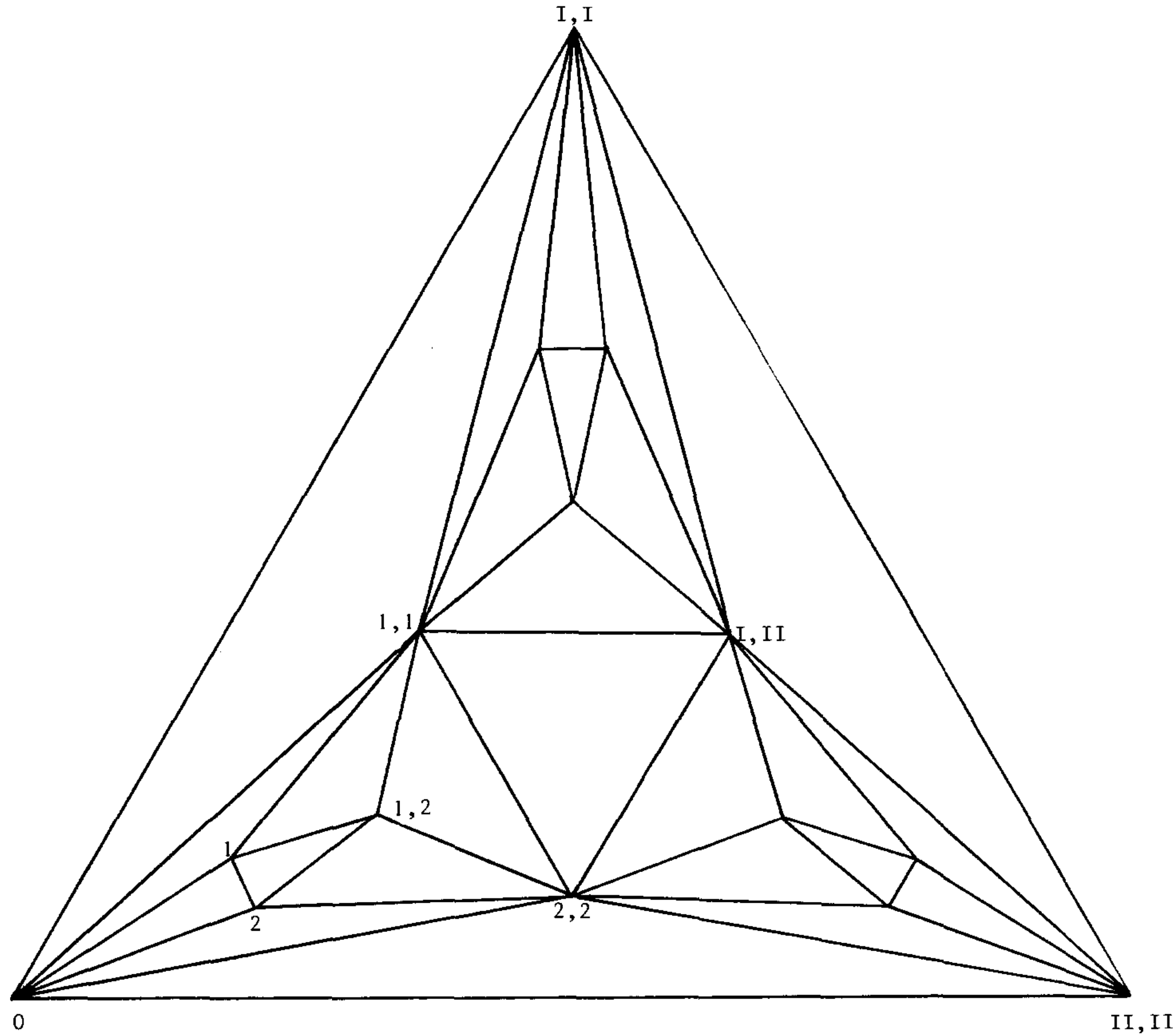

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

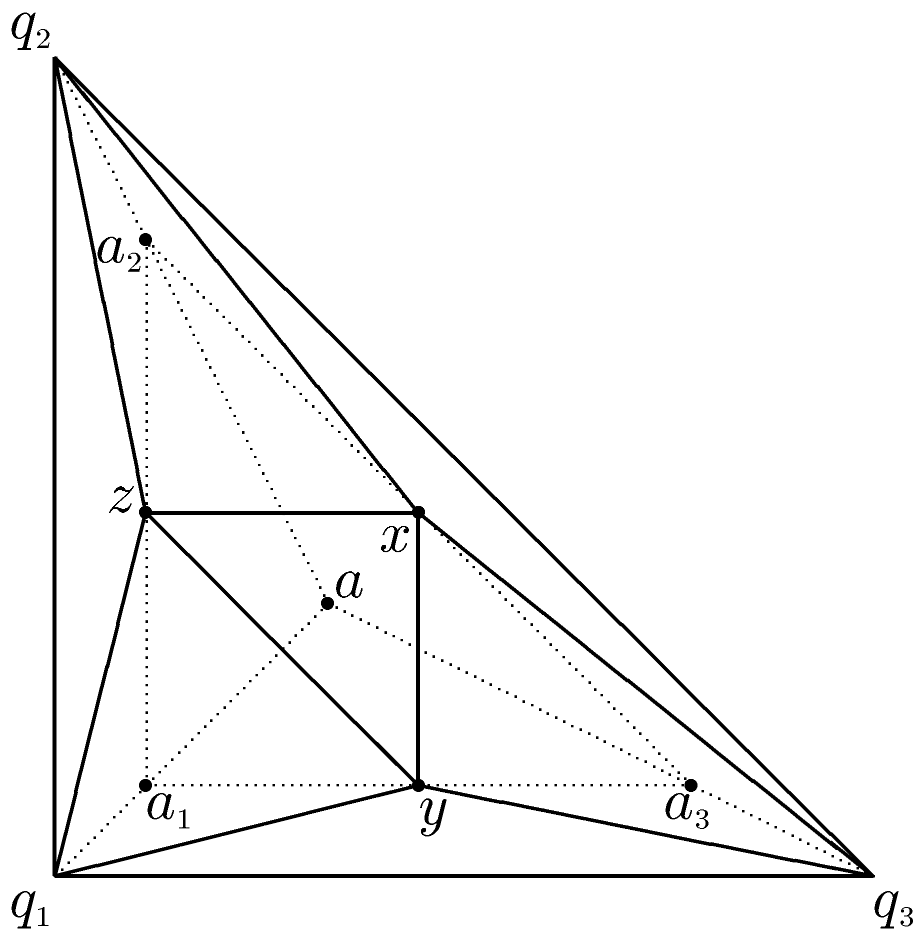

Figure 6.4. Geometric interpretation of Proposition 6.1.

The Spectral Dimension of the Universe is Scale Dependent

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We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

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Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

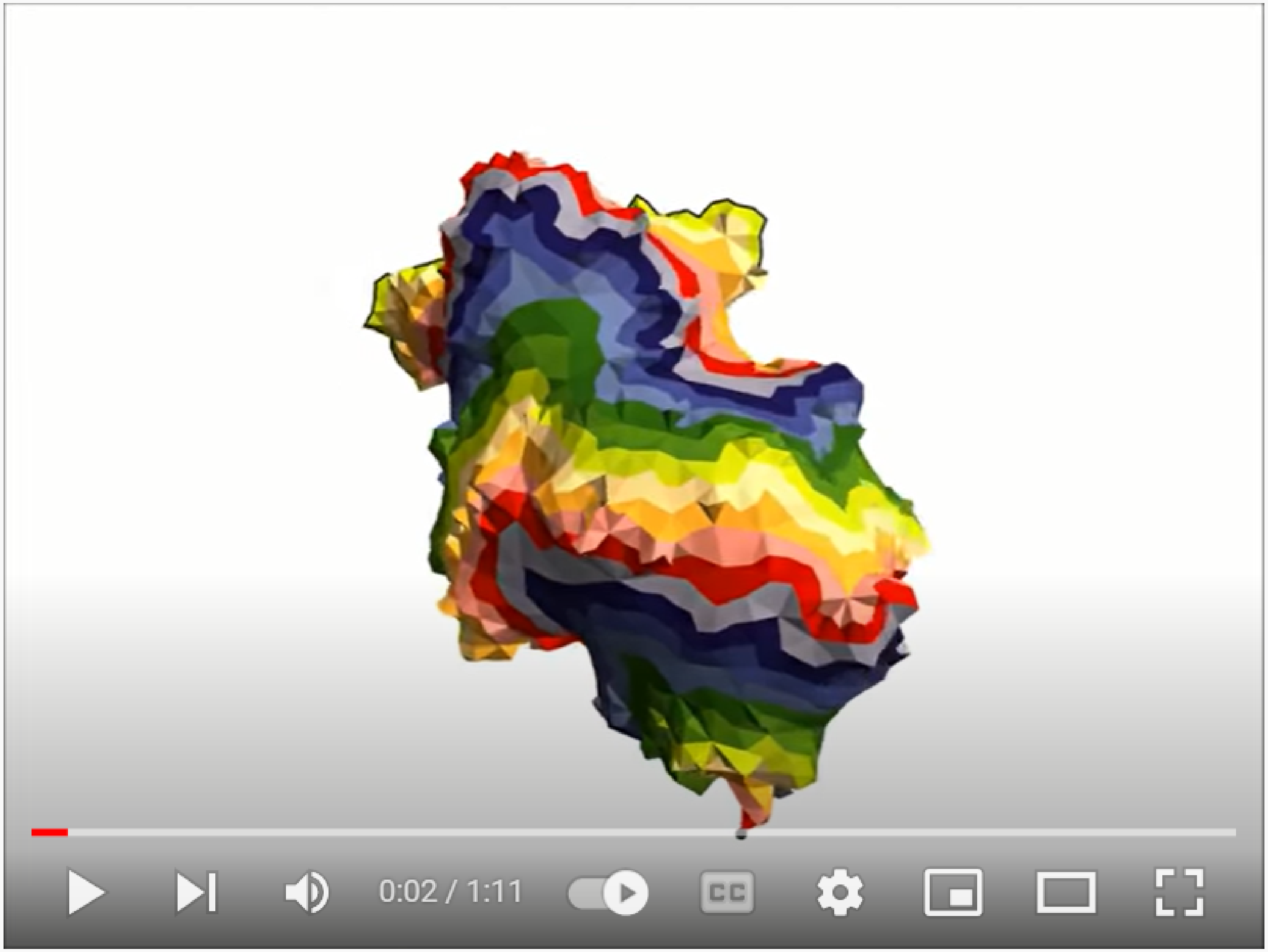
tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data [Martin Reuter, Frank Saueressig]:

Three scaling regimes of the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG):
(1) a classical regime $d_s = d, d_w = 2$,
(2) a semi-classical regime $d_s = 2d/(2+d),\ d_w = 2+d$,
(3) the UV-fixed point regime $d_s = d/2,\ d_w = 4$.
On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is in very good agreement with the data and provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

- *Quasisymmetric uniformization and heat kernel estimates* by Mathav Murugan: $d_w = d_f$ which is consistent with $d_s = 2d_f/d_w = 2$

Timothy Budd, Radboud University Nijmegen



Causal dynamical triangulations

25,971 views Jan 26, 2013 Causal dynamical triangulation (CDT) is a lattice model of quantum gravity. In two space-time dimensions (instead of the four we live in) it

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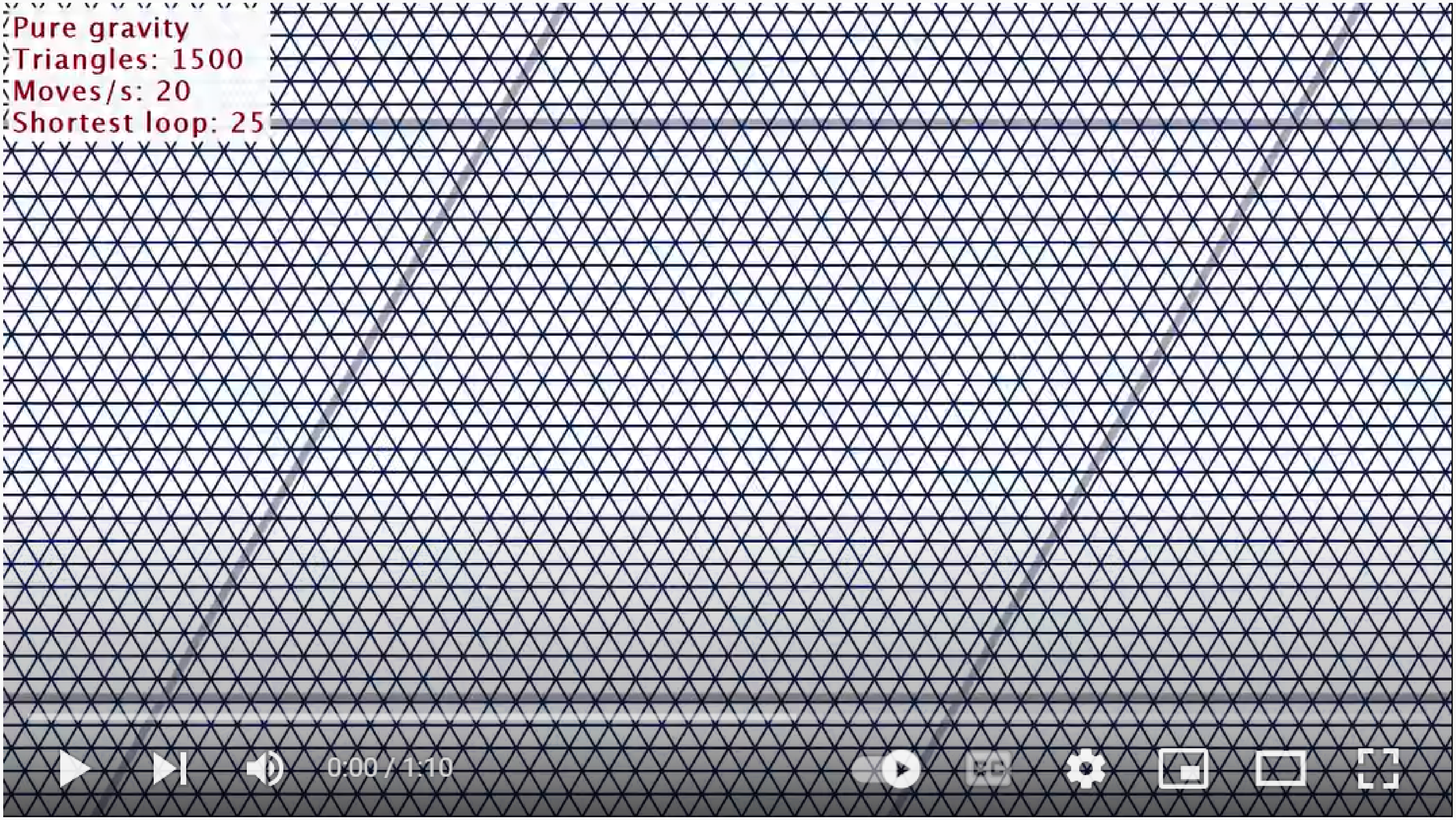

Dynamical triangulation of the 2-torus

1,435 views Sep 7, 2013 This video illustrates a Monte Carlo simulation for two-dimensional quantum gravity on a torus. Starting with a regular triangulation of the torus repeatedly a so-called flip move is performed on a randomly chosen edge. The triangulations obtained after a large

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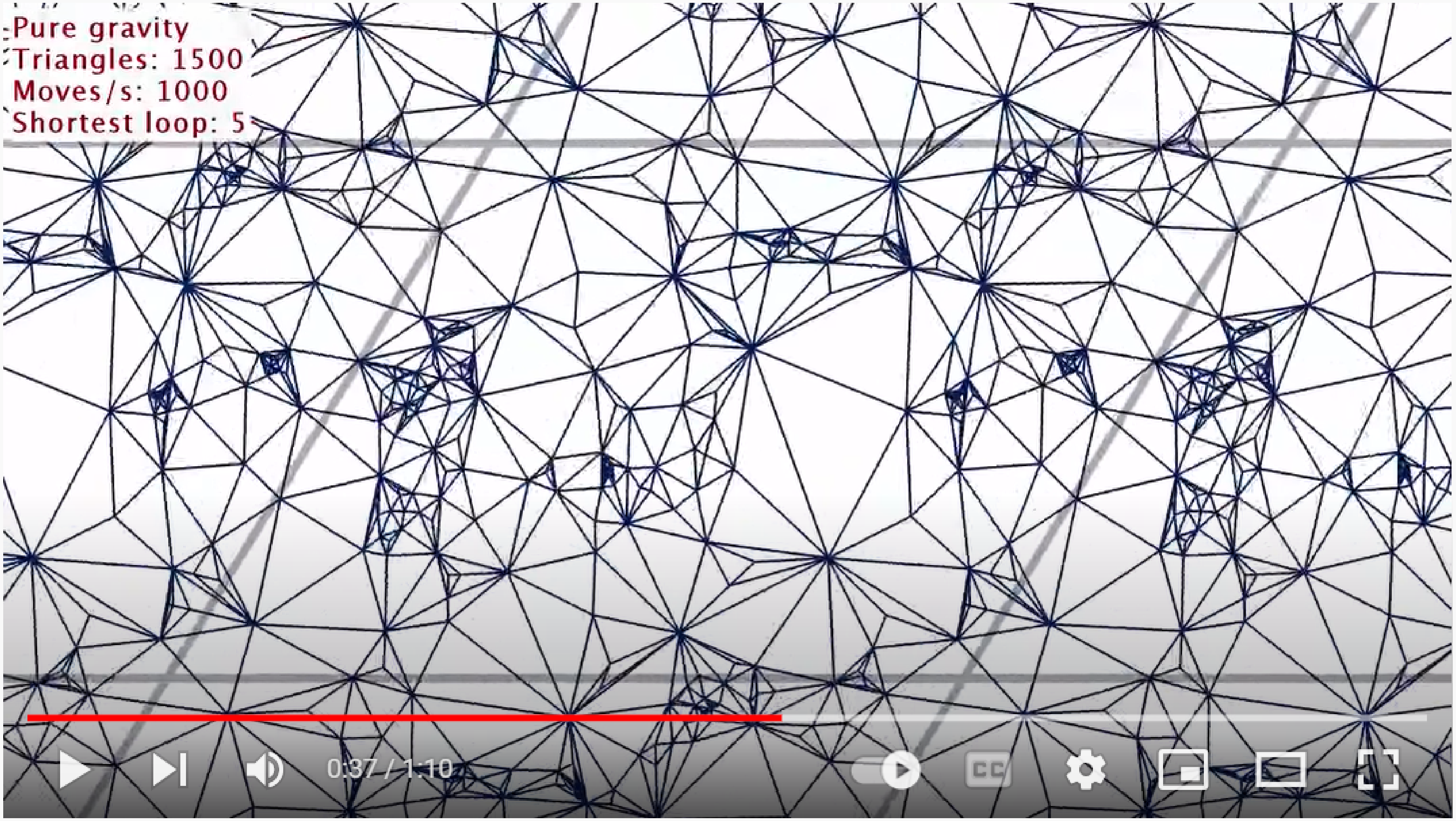

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sub-Gaussian heat kernel estimates (sGHKE)

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right) \tag{2}$$

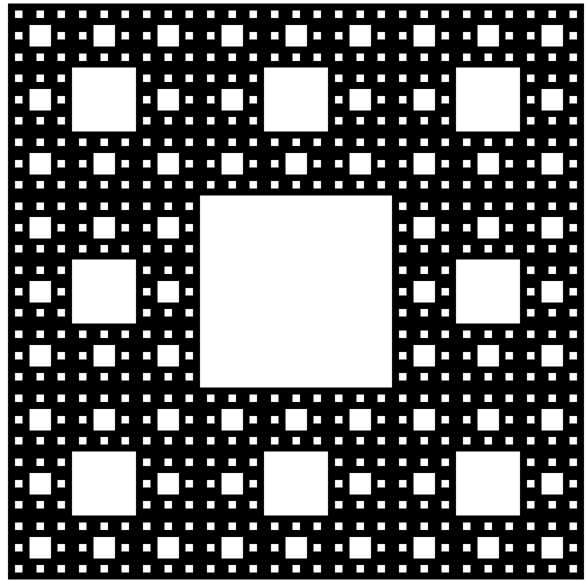
$$distance \sim (time)^{\frac{1}{d_w}}$$

$d_f =$ Hausdorff dimension

$\frac{1}{\gamma} = d_w =$ "walk dimension" (γ=diffusion index)

$\frac{2d_f}{d_w} = d_S =$ "spectral dimension" (diffusion dimension)

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)



$$1 = d_t = d_{mart} < d_{tH} = \frac{\ln 2}{\ln 3} + 1 < d_S < d_f = \frac{\ln 8}{\ln 3} < 2 < d_w$$

For Sierpinski carpets there exists a unique Dirichlet form and diffusion process due to [Barlow and Bass 1998, 1999] (see also [Barlow-Bass-Kumagai-T 2010]).

$d_{mart} = 1$ is a deep result of Kusuoka-Hino, see also Kajino-Murugan.

Here $d_{tH} = \dfrac{\ln 2}{\ln 3} + 1$ is the ***topological-Hausdorff dimension*** of the Sierpinski carpet defined in Theorem 5.4 in:

[**R.Balka, Z.Buczolich, M.Elekes.** ***A new fractal dimension: the topological Hausdorff dimension***. Adv. Math. 2015.]

Roughly speaking:

$$d_{tH} := 1 + \inf\{\text{Hausdorff dim. of boundaries of a base of open sets}\}$$

Open questions:

On the Sierpinski carpet,

$$\boldsymbol{\kappa = d_W - d_f + d_{tH} - 1 = d_W - d_f + \frac{\log 2}{\log 3}}$$

would give the best Hölder exponent for harmonic functions?
[*Numerical results: L.Rogers et al*]

Note that $(\boldsymbol{d_W - d_f})$ –Hölder continuity is known: Martin Barlow. Diffusions on fractals. In Lectures on probability theory and statistics (Saint-Flour, 1995), volume 1690 of Lecture Notes in Math. Springer, 1998. Heat kernels and sets with fractal structure. In Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), volume 338 of Contemp. Math., pages 11–40. Amer. Math. Soc., Providence, RI, 2003.

BV and weak Bakry-Émery non-negative curvature [P.Alonso-Ruiz, F.Baudoin, L.Chen, L.Rogers, N.Shanmugalingam, A.T.]

Definition. $BV(X) := KS^{\lambda_1^{\#},1}(X) = \mathrm{B}^{1,\alpha_1^{\#}}(X)$ with $\alpha_1^{\#} = \frac{\lambda_1^{\#}}{d_W}$ the L^1–Besov critical exponent, and for $f \in BV(X)$

$$\mathrm{Var}(f) := \liminf_{r \to 0^+} \iint_{\Delta_r} \frac{|f(y) - f(x)|}{r^{\lambda_1^{\#}} \mu(B(x,r))} \, d\mu(y) \, d\mu(x).$$

Definition. We say that $(X, \mu, \mathcal{E}, \mathcal{F})$ satisfies the weak-Bakry-Émery non-negative curvature condition $wBE(\kappa)$ if there exist a constant $C > 0$ and a parameter $0 < \kappa < d_W$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,

$$|P_t g(x) - P_t g(y)| \le C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}. \tag{3}$$

- If (X, d, μ) satisfies $wBE(\kappa)$ with $\kappa = \frac{d_W}{2}$, then the form $\mathcal{E}$ admits a carré du champ operator, which means that $d_w = 2$ by [**Kajino-Murugan 2019** Ann. Probab. 48, 2020]

- $\kappa \leqslant 1$ because $d(x, y)$ has to be essentially equivalent to a geodesic metric [Corollary 1.8, Theorem 2.11 **Mathav Murugan** JFA 2020]

- For nested fractals, p.c.f. with sGHKE (2)

$$\lambda_1^{\#} = \lambda_1^* = d_W \alpha_1^* = d_f$$

- For the Sierpinski carpet we <u>conjectured</u>

$$\lambda_1^{\#} = \lambda_1^* = d_f - d_{tH} + 1$$

where $d_{tH} = \dfrac{\ln 2}{\ln 3} + 1$ is the topological-Hausdorff dimension of the Sierpinski carpet.

THANK YOU FOR YOUR ATTENTION!

June 16-20, 2025: 8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals.