Non-Smooth Boundary Value Problems

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joint research with

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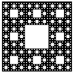

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Abstract:

- The lectures will begin with a review of the general functional analysis framework, covering the **Hille–Yosida theorem**, the spectral theory of self-adjoint operators as developed by von Neumann, the theory of positive quadratic forms, **Dirichlet forms** and Markov operators by Beurling and Deny, and the related theory of symmetric Markov stochastic processes (Kolmogorov, Levy, Doob, Hunt, Dynkin).
- Next, I will discuss applications of the theory of **ultra-contractive semigroups**, based on recent joint work with Carfagnini and Gordina, following the work of E.B. Davis.
- After that, I will present recent progress in non-smooth **Dirichlet, Neumann, and Robin Boundary Value Problems**, which are the result of joint work with Hinz, Magoulès, and Rozanova-Pierrat.
- Another application of the general theory will deal with non-smooth **Wentzell Boundary Value Problems**, a joint work with Hinz, Lancia, and Vernole.
- If time permits, I will also discuss recent advancements in **non-smooth layer potentials and Riemann-Hilbert problems** in a joint work with Claret and Rozanova-Pierrat.

Lectures 1 and 2 – Functional Analysis

1. Yosida approximations
2. Spectral theory of self-adjoint semigroups by von Neumann
3. Hille–Yosida theorem (Feller–Miyadera–Phillips theorem)
4. Lumer–Phillips theorem
5. Positive quadratic forms, Dirichlet forms and Markov operators
6. Symmetric Markov stochastic processes
7. Ultra-contractive semigroups, following the work of E.B. Davis
8. Discrete spectrum for Dirichlet forms
9. Nash inequality
10. Small deviations
11. Mosco convergence, strong and norm resolvent convergence
12. Convergence of eigenvalues in fractal domains

1. Yosida approximations

Assume that the operator (A, D_A) generates a C_0-semigroup on a Hilbert space H denoted by e^{tA}, $t \geqslant 0$. We assume that there is $\beta \geqslant 0$ such that for all $x \in D_A$

$$\langle Ax, x \rangle \leqslant -\beta |x|_H^2.$$

- Klaus-Jochen Engel and Rainer Nagel, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics, vol. 194, Springer, 2000.
- Haim Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Math. Studies, No. 5. Notas de Matemática (50). Elsevier, 1973.
- Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, 2011.

Semigroups

If X is a Banach space, a one-parameter semigroup of operators on X is a family of operators indexed on the non-negative real numbers $\{T(t)\}_{t\in[0,\infty)}$ such that

$$T(0) = I,\ T(s+t) = T(s) \circ T(t), \quad \forall t, s \geq 0.$$

The semigroup is said to be strongly continuous, also called a (C_0) semigroup, if and only if the mapping $t \mapsto T(t)x$ is norm-continuous for all $x \in X$, where $t \in [0, \infty)$.

The infinitesimal generator of a one-parameter semigroup T is an operator A defined on a possibly proper subspace of X as follows:

$$\lim_{h\to 0^+} h^{-1}\Big(T(h)x - x\Big) := Ax$$

The domain of A is the set of $x \in X$ such that the limit exists.

In other words, Ax is the right-derivative at 0 of the function $t \mapsto T(t)x$. The infinitesimal generator of a strongly continuous one-parameter semigroup is a closed linear operator defined on a dense linear subspace of X.

Let $\rho(A)$ be the resolvent set, then the resolvent of A is defined as

$$R_\lambda(A) := (\lambda I - A)^{-1}, \quad \lambda \in \rho(A) \in B(H),$$
$$R_\lambda(A) : H \longrightarrow D_A.$$

For $\lambda > 0$ we have $\|R_\lambda(A)\| \leqslant 1/\lambda$. In addition,

$$\lambda R_\lambda(A)\, x \xrightarrow[\lambda \to +\infty]{} x, \quad x \in H. \tag{1}$$

Note that $AR_\lambda(A)\, x = R_\lambda(A)\, Ax, \quad x \in D_A$.
The **Yosida approximations** to A are defined by

$$A_\alpha x := \frac{1}{\alpha} AR_{\frac{1}{\alpha}}(A)\, x, x \in H. \tag{2}$$

The Yosida approximations A_α to A satisfy the following properties, see [Proposition 7.2, Brezis 2011], where

$$J_\alpha := (I - \alpha A)^{-1}, \tag{3}$$

$J_\alpha \in B(H)$, $\|J_\alpha\| \leqslant 1$.

$$\begin{aligned}
&A_\alpha x \xrightarrow[\alpha \to 0]{} Ax, \quad x \in D_A, \\
&|A_\alpha x|_H \leqslant |Ax|_H, \quad x \in D_A, \\
&A_\alpha x = J_\alpha Ax, \quad x \in D_A, \\
&A_\alpha \in B(H), \\
&\|A_\alpha\| \leqslant \tfrac{1}{\alpha}, \\
&A_\alpha = AJ_\alpha = \tfrac{1}{\alpha}(J_\alpha - I).
\end{aligned} \tag{4}$$

*** **Proposition:** $\|J_\alpha\| \leqslant 1/(1+\alpha\beta)$ and for all $x \in H$

$$\langle A_\alpha x, x\rangle \leqslant -\frac{\beta}{1+\alpha\beta}|x|_H^2$$

Proof Let $x \in H$ and $y := J_\alpha x$, that is $x = y - \alpha A y$. Then

$$|x|_H \cdot |y|_H \geqslant \langle x, y\rangle = \langle y - \alpha A y, y\rangle \geqslant (1+\alpha\beta)|y|_H^2,$$

which implies $|x| \geqslant (1+\alpha\beta)|y|$. To prove the second inequality, note that

$$\langle -A_\alpha x, x\rangle = \langle -Ay, y - \alpha A y\rangle \geqslant \beta|y|_H^2 + \alpha|Ay|_H^2$$

$$= \beta|y|_H^2 + \frac{1}{\alpha}|x-y|_H^2 \geqslant \frac{\beta|x|_H^2}{1+\alpha\beta},$$

where the last inequality is obtained by minimization over all $y \in H$.

*** *Note that the estimates in this Proposition are best possible.*

M. Gordina, M. Röckner, A. Teplyaev, Singular perturbations of Ornstein-Uhlenbeck processes: integral estimates and Girsanov densities,

Probability Theory and Related Fields 178(3), 861-891 (2020)

Yosida approximations to non-linear time-dependent m-dissipative maps

Denote by 2^H the power set of the Hilbert space H. Let $F(t,\cdot) : [0,\infty) \times D_F \to 2^H$ be a family of maps such that D_F is a non-empty Borel set in H. Furthermore, $F(t,\cdot)$ is an m-dissipative map, that is, for any $x_1, x_2 \in D_F$

$$\langle y_1 - y_2, x_1 - x_2 \rangle \leqslant 0, \text{ for any } y_1 \in F(t, x_1), y_2 \in F(t, x_2), t \in [0,\infty)$$

and for any $\alpha > 0$ and $t \in [0,\infty)$

$$\text{Range}\,(\alpha I - F(t,\cdot)) := \{\alpha x - y : y \in F(t,x), x \in D_F\} = H.$$

Fix $t \in [0, \infty)$ and set $F := F(t, \cdot)$. Then for any $\alpha > 0$ we define

$$F_\alpha := \frac{1}{\alpha}\left(J_\alpha(x) - x\right), x \in H, \tag{5}$$

where

$$J_\alpha(x) := (I - \alpha F)^{-1}(x), \; I(x) = x,$$

which is a nonlinear generalization of (3). Then each F_α is single-valued, dissipative, Lipschitz continuous with Lipschitz constant less than $\frac{2}{\alpha}$ and satisfies

$$\lim_{\alpha \to 0} F_\alpha(x) = F_0(x), x \in D_F, \tag{6}$$

$$|F_\alpha(x)|_H \leqslant |F_0(x)|_H, x \in D_F. \tag{7}$$

It is clear from the last inequality that for each $x_0 \in D_F$

$$|F_\alpha(t,x)|_H \leq |F_0(t,x_0)|_H + \frac{2}{\alpha}|x|_H \leq a(|x_0|_H) + \frac{2}{\alpha}|x|_H, \quad x \in H. \tag{8}$$

2. Spectral theory of self-adjoint semigroups by von Neumann

If A is a self-adjoint operator on a Hilbert space then

$$A = \int_{\mathbb{R}} \lambda dE_A(\lambda)$$

for the unique orthogonal projection-valued measure $E_A(\cdot)$.

Moreover, for any measurable function $f :\to$, there is a well defined self-adjoint operator

$$f(A) = \int_{\mathbb{R}} f(\lambda) dE_A(\lambda)$$

Proof of the Spectral Theorem

Cayley transform:

$$U = (A - iI)(A + iI)^{-1}$$

$$A = i(U + I)(U - I)^{-1}$$

Note that V is a unitary operator: $U^* = U^{-1}$

Use Gelfand transform on cyclic subspaces of A or U

Birman M. Sh., Solomjak M. Z.
Spectral theory of self-adjoint operators in Hilbert space

Reed M., Simon B. *I: Functional analysis*

Rudin W., *Functional analysis*

The Hille–Yosida theorem (general case)

Let $\boldsymbol{A}$ be a linear operator defined on a linear subspace $\boldsymbol{D}(\boldsymbol{A})$ of the Banach space $\boldsymbol{X}$, $\omega \in \mathbb{R}$, and M > 0. Then $\boldsymbol{A}$ generates a strongly continuous semigroup $\boldsymbol{T}$ that satisfies

$$\|\boldsymbol{T}(t)\| \leq \boldsymbol{M}\mathrm{e}^{\omega t}$$

if and only if:
$\boldsymbol{A}$ is closed and $\boldsymbol{D}(\boldsymbol{A})$ is dense in $\boldsymbol{X}$, every real $\lambda > \omega$ belongs to the resolvent set of $\boldsymbol{A}$ and for such λ and for all positive integers $\boldsymbol{n}$,

$$\|(\lambda \boldsymbol{I} - \boldsymbol{A})^{-n}\| \leq \frac{\boldsymbol{M}}{(\lambda - \omega)^{n}}.$$

In the special case of contraction semigroups ($\boldsymbol{M} = \mathbf{1}$ and $\omega = \mathbf{0}$) only the case $\boldsymbol{n} = \mathbf{1}$ has to be checked:

The Hille–Yosida theorem (contraction semigroups)

Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space X. Then A generates a strongly continuous semigroup T that satisfies

$$\|T(t)\| \leqslant 1$$

if and only if:
A is closed and $D(A)$ is dense in X, every real $\lambda > 0$ belongs to the resolvent set of A and for such λ,

$$\|(\lambda I - A)^{-1}\| \leqslant \frac{1}{\lambda}.$$

4. Lumer–Phillips theorem

Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space X. Then A generates a contraction semigroup if and only if $D(A)$ is dense in X, A is dissipative

$$\|(\lambda I - A)x\| \geq \lambda \|x\|$$

and $A - \lambda_0 I$ is surjective for some $\lambda_0 > 0$, where I denotes the identity operator. An operator satisfying the last two conditions is called maximally dissipative.

*** **Note:** the conditions that $D(A)$ is dense and that A is closed can be dropped if X is a **reflexive Banach space**. Moreover, in that case A generates a contraction semigroup if and only if A is closed and both A and its adjoint operator A^* are dissipative.

5. Positive quadratic forms, Dirichlet forms and Markov operators

A *Dirichlet form* $(\mathcal{E}, \mathcal{F})$ is a symmetric, bilinear and positive definite real valued form $\mathcal{E}$ on a subspace $\mathcal{F}$ that is dense in a real L^2-space $L^2(X, \mathcal{X}, \mu)$ over a σ-finite measure space $(X, \mathcal{X}, \mu)$ such that

- the space $\mathcal{F}$, together with the norm $f \mapsto (\mathcal{E}(f) + \|f\|^2_{L^2(X,\mathcal{X},\mu)})^{1/2}$, is a Hilbert space (the '*Dirichlet space*') and
- for any $f \in \mathcal{F}$ we have $(f \wedge 1) \vee 0 \in \mathcal{F}$ and $\mathcal{E}(f) \leq \mathcal{E}((f \wedge 1) \vee 0)$.

Here $\mathcal{E}(f) := \mathcal{E}(f, f)$.

There is a one-to-one correspondence of Dirichlet forms and non-positive definite self-adjoint operators on $L^2(X, \mathcal{X}, \mu)$ satisfying a certain Markov property.

The self-adjoint operator $(\mathcal{L}, \mathbf{dom}\,\mathcal{L})$ uniquely associated with $(\mathcal{E}, \mathcal{F})$ (and referred to as its *generator*) satisfies

$$\mathcal{E}(f, g) = -\langle \mathcal{L}f, g\rangle_{L^2(X,\mathcal{X},\mu)}, \quad f \in \mathbf{dom}\,\mathcal{L}, \; g \in \mathcal{F},$$

and is uniquely determined by this formula.

6. Symmetric Markov stochastic processes

By a theorem of Kolmororov (*Foundations of the Theory of Probability*, 1933) any self-adjoint positivity preserving semigroup $L^2(X, \mathcal{X}, \mu)$ corresponds to an essentially unique family of symmetric Markov processes X_t.

The semigroup property

$$P(t+s) = P(t)P(s)$$

is called the **Chapman–Kolmogorov equation** with

$$P(t)f(x) = \mathbb{E}^x(f(X_t))$$

Textbooks on symmetric Markov processes, semigroups and Dirichlet forms

Ethier SN, Kurtz TG. *Markov processes: characterization and convergence*. John Wiley & Sons 1986, 2nd Edition 2005

Bouleau N, Hirsch F. *Dirichlet forms and analysis on Wiener space*. Walter de Gruyter 1991

Ma ZM, Röckner M. *Introduction to the theory of (non-symmetric) Dirichlet forms*. Springer 1992

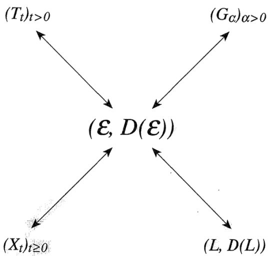
Fukushima M, Oshima Y, Takeda M. *Dirichlet forms and symmetric Markov processes*. Walter de Gruyter1994, 2nd Edtion 2010

Chen Z, Fukushima M. *Symmetric Markov processes, time change, and boundary theory* (LMS-35). Princeton University Press 2011

Zhi-Ming Ma   Michael Röckner

Introduction to the Theory of (Non-Symmetric)

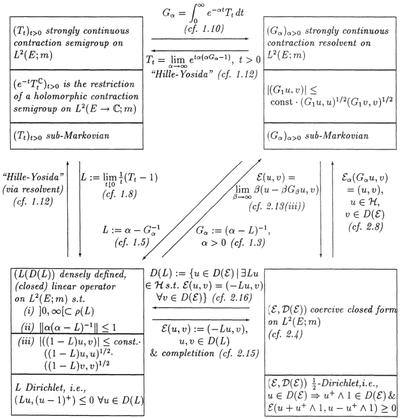
Dirichlet Forms



Springer-Verlag

Universitext

Diagram 3


$G_\alpha = \int_0^\infty e^{-\alpha t} T_t \, dt$ (cf. 1.10)
$T_t = \lim_{\alpha\to\infty} e^{t\alpha(\alpha G_\alpha - 1)}$, $t > 0$ "Hille-Yosida" (cf. 1.12)
$(T_t)_{t>0}$ strongly continuous contraction semigroup on $L^2(E;m)$
$(e^{-t}T_t^{\mathbb{C}})_{t>0}$ is the restriction of a holomorphic contraction semigroup on $L^2(E\to\mathbb{C};m)$
$(T_t)_{t>0}$ sub-Markovian
$(G_\alpha)_{\alpha>0}$ strongly continuous contraction resolvent on $L^2(E;m)$
$|(G_1 u, v)| \le$ const $\cdot (G_1 u, u)^{1/2} (G_1 v, v)^{1/2}$
$(G_\alpha)_{\alpha>0}$ sub-Markovian
"Hille-Yosida" (via resolvent) (cf. 1.12)
$L := \lim_{t\downarrow 0} \frac{1}{t}(T_t - 1)$ (cf. 1.8)
$L := \alpha - G_\alpha^{-1}$ (cf. 1.5)
$G_\alpha := (\alpha - L)^{-1}$, $\alpha > 0$ (cf. 1.3)
$\mathcal{E}(u,v) = \lim_{\beta\to\infty} \beta(u - \beta G_\beta u, v)$ (cf. 2.13(iii))
$\mathcal{E}_\alpha(G_\alpha u, v) = (u,v)$, $u \in \mathcal{H}$, $v \in D(\mathcal{E})$ (cf. 2.8)
$(L(D(L))$ densely defined, (closed) linear operator on $L^2(E;m)$ s.t.
(i) $]0,\infty[\subset \rho(L)$
(ii) $\|\alpha(\alpha - L)^{-1}\| \le 1$
(iii) $|((1-L)u, v)| \le$ const. $\cdot ((1-L)u,u)^{1/2} \cdot ((1-L)v,v)^{1/2}$
L Dirichlet, i.e., $(Lu, (u-1)^+) \le 0$ $\forall u \in D(L)$
$D(L) := \{u \in D(\mathcal{E}) \mid \exists Lu \in \mathcal{H}$ s.t. $\mathcal{E}(u,v) = (-Lu, v)$ $\forall v \in D(\mathcal{E})\}$ (cf. 2.16)
$\mathcal{E}(u,v) := (-Lu, v)$, $u, v \in D(L)$ & completion (cf. 2.15)
$(\mathcal{E}, D(\mathcal{E}))$ coercive closed form on $L^2(E;m)$ (cf. 2.4)
$(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ $\frac{1}{2}$-Dirichlet, i.e., $u \in D(\mathcal{E}) \Rightarrow u^+ \wedge 1 \in D(\mathcal{E})$ & $\mathcal{E}(u + u^+ \wedge 1, u - u^+ \wedge 1) \ge 0$

Theorem

Let $\{P_t\}_{t\geqslant 0}$ be a strongly continuous semigroup of symmetric operators on a Hilbert space H with the infinitesimal generator $\mathcal{L}$, then the following statements are equivalent.

1. P_t is compact for all $t > 0$;
2. P_{t_0} is compact for some $t_0 > 0$;
3. $\mathcal{L}$ has a discrete spectrum, that is, it has a pure point spectrum with isolated eigenvalues of finite multiplicity.

Carfagnini M, Gordina M, Teplyaev A. *Dirichlet metric measure spaces: spectrum, irreducibility, and small deviations*, arXiv:2409.07425

7. Ultra-contractive semigroups, following the work of E.B. Davis

Let P_t be a Markov semigroup on $L^2(\mathcal{X}, \mu)$, where μ is a σ-finite measure on a countably generated σ-algebra. We say that P_t is *ultracontractive* if

$$\|P_t f\|_{L^\infty} \leqslant c_t \|f\|_{L^2}, \tag{9}$$

where the corresponding norm is denoted by $\|P_t f\|_{2\to\infty} \leqslant c_t$.

Davies EB. *Heat kernels and spectral theory*. Cambridge University Press 1989:

Ultracontractivity is equivalent to the existence of an integral (heat) kernel for the semigroup P_t satisfying

$$0 \leqslant p_t(x, y) \leqslant a_t < \infty \tag{10}$$

almost everywhere on $\mathcal{X} \times \mathcal{X}$ for some $a_t \geqslant 0$.

Eigenfunction expansion of the Dirichlet heat kernel

Let $\{P_t\}_{t \geqslant 0}$ be a strongly continuous contraction semigroup on $L^2(\mathcal{X}, \mu)$, and let $\mathcal{U}$ be a measurable set in $\mathcal{X}$ with $0 < \mu(\mathcal{U}) < \infty$. If (10) is satisfied, then the series

$$p_t^{\mathcal{U}}(x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y)$$

converges uniformly on $\mathcal{U} \times \mathcal{U} \times [\varepsilon, \infty)$ for any $\varepsilon > 0$. Moreover,

$$\mathbb{P}^x \left(\tau_{\mathcal{U}} > t \right) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \int_{\mathcal{U}} \varphi_n(y) d\mu(y)$$

for any $x, y \in \mathcal{U}$, and $t > 0$.

8. Discrete spectrum for Dirichlet forms

Let $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ be a regular Dirichlet form with associated strongly continuous contraction semigroup $\{P_t\}_{t\geqslant 0}$ on $L^2(\mathcal{X}, \mu)$. . Assume that $p_t(x, y)$ exists for all t and for μ-a.e. $x, y \in \mathcal{X}$.

Theorem (Carfagnini, Gordina, Teplyaev)

Let $\mathcal{U}$ be an open bounded subset of $\mathcal{X}$, and $P_t^{\mathcal{U}}$ be the semigroup associated to $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ with the infinitesimal generator $A^{\mathcal{U}}$, where we impose zero boundary conditions outside of $\mathcal{U}$.

- *If $\mu(\mathcal{U}) < \infty$ then the spectrum of $A^{\mathcal{U}}$ is discrete and the associated heat kernel $p_t^{\mathcal{U}}(x, y)$ has the usual eigenfunction expansion.*
- *If there exists a $t_{\mathcal{U}} > 0$ such that*

$$M_{\mathcal{U}}(t_{\mathcal{U}}) = \operatorname*{ess\,sup}_{(x,y)\in\mathcal{U}\times\mathcal{U}} p_{t_{\mathcal{U}}}^{\mathcal{U}}(x, y) < \frac{1}{\mu(\mathcal{U})^2}, \tag{11}$$

then $\lambda_1 > 0$.

Note that $p_t^{\mathcal{U}}(x, y) \leqslant p_t(x, y)$.

9. Nash inequality

We say that the Dirichlet from $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ on $L^2(\mathcal{X}, \mu)$ satisfies a *Nash inequality* with the parameter ν if there is a $C > 0$ such that

$$\|f\|_{L^2(\mathcal{X},\mu)}^{2+\frac{4}{\nu}} \leqslant C\mathcal{E}(f, f)\|f\|_{L^1(\mathcal{X},\mu)}^{\frac{4}{\nu}} \text{ for all } f \in \mathcal{D}_{\mathcal{E}}. \tag{12}$$

Carlen EA, Kusuoka S, Stroock DW. *Upper bounds for symmetric Markov transition functions*. InAnnales de l'IHP Probabilités et statistiques 1987:

(12) is equivalent to the $L^1 \to L^\infty$ ultracontractivity of the heat semigroup with the specific power function depending on the parameter ν

$$\|P_t f\|_{L^\infty(\mathcal{X},\mu)} \leqslant C\, t^{-\frac{\nu}{2}} \|f\|_{L^1(\mathcal{X},\mu)},$$

for all $f \in L^1(\mathcal{X}, \mu)$ and $t > 0$, or equivalently

$$\operatorname*{ess\,sup}_{(x,y)\in\mathcal{X}\times\mathcal{X}} p_t(x, y) \leqslant Ct^{-\frac{\nu}{2}},$$

10. Small deviations

Theorem (Carfagnini, Gordina, Teplyaev)

Let $x \in \mathcal{X}$ and assume that $P_t^{B_1(x)}$ is irreducible. Assume that the heat kernel $p_t^{B_1(x)}(x, y)$ exists for all t and for all $x, y \in \mathcal{X}$ and that

$$p_t(x, y) \leqslant c\, t^{-\frac{\alpha}{\beta}}$$

for any t, x, and y. Moreover, assume that there exists a t_0 such that $p_{t_0}^{B_1(x)}(x, y)$ is continuous for $x, y \in \mathcal{X}$. If X_t^x is **self-similar** *then*

$$\lim_{\varepsilon \to 0} e^{\lambda_1 \frac{t}{\varepsilon^\beta}} \mathbb{P}^x \left(\sup_{0 \leqslant s \leqslant t} d(X_s, x) < \varepsilon \right) = c_1 \varphi_1(x),$$

where $\lambda_1 > 0$ is the spectral gap of $A^{B_1(x)}$ with zero boundary conditions outside of the unit ball $B_1(x)$, and φ_1 is the corresponding positive eigenfunction.

11. Mosco convergence, strong and norm resolvent convergence

- Mosco, Umberto *Convergence of convex sets and of solutions of variational inequalities*. Advances in Math. 3 (1969), 510–585.
- Mosco, Umberto *Composite media and asymptotic Dirichlet forms*. J. Funct. Anal. 123 (1994), no. 2, 368–421.

Kato, Tosio
Perturbation theory for linear operators. Springer-Verlag 1966.

[Reed-Simon 1972]: For non-negative closed quadratic forms,

- **Mosco convergence is equivalent to the strong resolvent convergence**,
- but is **weaker than the norm resolvent convergence**.

A sequence $(E^{(n)})_{n=1}^{\infty}$ of (possibly extended real valued) quadratic forms $E^{(n)}$ on $L^2(X, \mathcal{X}, \mu)$ *converges* to a quadratic form E on $L^2(X, \mathcal{X}, \mu)$ *in the sense of Mosco* if

(i) for any sequence $(f_n)_{n=1}^{\infty} \subset L^2(X, \mathcal{X}, \mu)$ converging to some f weakly in $L^2(X, \mathcal{X}, \mu)$ we have

$$E(f) \leq \liminf_n E^{(n)}(f_n)$$

and

(ii) for any $f \in L^2(X, \mathcal{X}, \mu)$ there exists a sequence $(f_n)_{n=1}^{\infty}$ converging to f strongly in $L^2(X, \mathcal{X}, \mu)$ and such that

$$\limsup_n E^{(n)}(f_n) \leq E(f).$$

Theorem

Any separable Dirichlet form $(\mathcal{E}, \mathcal{F})$ can be approximated in the Mosco sense by a sequence of essentially discrete Dirichlet forms (essentially isomorphic to that on finite weighted graphs) and the corresponding generators approximate the generator of $(\mathcal{E}, \mathcal{F})$ in the strong resolvent sense.

M. Hinz, A. Teplyaev, *Closability, regularity, and approximation by graphs for separable bilinear forms*. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 441 (Veroyatnost i Statistika. 22):299-317, 2015. Springer: J. Math. Sci. (2016) 219 807–820 doi:10.1007/s10958-016-3149-7 arXiv:1511.08499

Mosco convergence does not imply convergence of the spectrum

$\operatorname{M-lim}_{n\to\infty} E_n = F$ or $E_n \xrightarrow[n\to\infty]{\mathrm{M}} F$.

- $x_n \in L^2$ converging weakly to $x \in L^2$, $\liminf_{n\to\infty} E_n(x_n) \geq F(x)$;
- for each $x \in L^2$ there exists an approximating sequence of elements $x_n \in L^2$, converging strongly to x, such that $\limsup_{n\to\infty} E_n(x_n) \leq F(x)$.

Example:

$L^2 := \ell^2(\mathbb{Z}_+)$

$E_n((x_k)) := \sum_{k\geqslant n} |x_k|^2 \xrightarrow[n\to\infty]{\mathrm{M}} E = 0$

$\sigma(E_n) = \{0, 1\} \neq \{0\} = \sigma(E)$

12. Convergence of eigenvalues in fractal domains

Theorem (Hinz, Rozanova-Pierrat, T.)

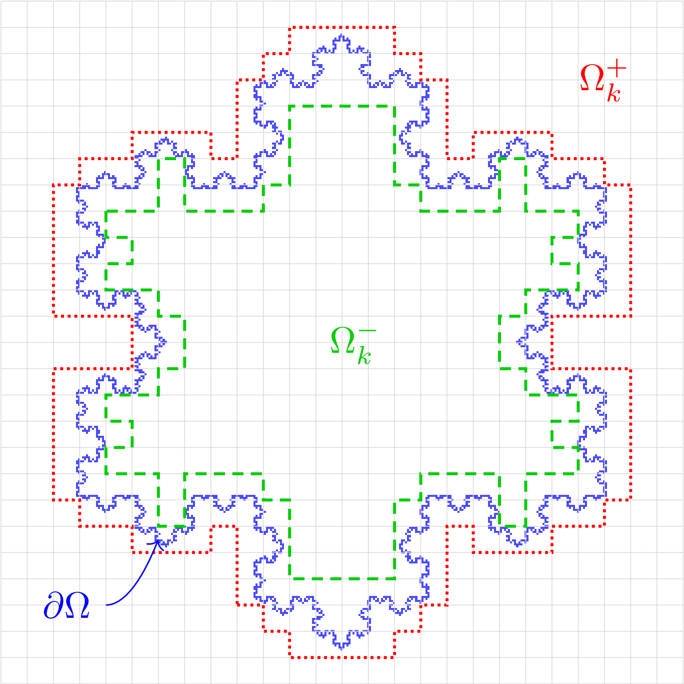
Let n, D, α, γ, ε, d, $(\Omega_m)_m$ *and* $(\mu_m)_m$ *be a* **a sequence of admissible domains**. *Suppose that* $\lim_m \Omega_m = \Omega$ *in the Hausdorff sense and in the sense of characteristic functions and* $\lim_m \mu_m = \mu$ *weakly. There is a sequence* $(m_k)_{k=1}^{\infty}$ *with* $m_k \uparrow +\infty$ *such that the following hold.*

(i) *We have* $\lim_{k\to\infty} P_{\Omega_{m_k}} \circ \hat{G}_{\alpha,\gamma}^{\Omega_{m_k},\mu_{m_k},*} = P_\Omega \circ \hat{G}_{\alpha,\gamma}^{\Omega,\mu,*}$ *in operator norm.*

(ii) *If* $0 < a < b$ *are in the resolvent set of* $-\mathcal{L}_\gamma^{\Omega,\mu,*}$, *then* $\lim_{k\to\infty} \pi_{(a,b)}(\Omega_{m_k}, \mu_{m_k}, *) = \pi_{(a,b)}(\Omega, \mu, *)$ *in operator norm.*

(iii) *The spectra of the operators* $-\mathcal{L}_\gamma^{\Omega_{m_k},\mu_{m_k},*}$ *converge to the spectrum of* $-\mathcal{L}_\gamma^{\Omega,\mu,*}$ *in the Hausdorff sense. The eigenvalues* $\lambda_n(\Omega,\mu,*)$ *of the operator* $-\mathcal{L}_\gamma^{\Omega,\mu,*}$ *are exactly the limits as* $k \to \infty$ *of sequences of the eigenvalues of the operators* $-\mathcal{L}_\gamma^{\Omega_{m_k},\mu_{m_k},*}$,

$$\lambda_n(\Omega,\mu,*) = \lim_{k\to\infty} \lambda_n(\Omega_{m_k}, \mu_{m_k}, *).$$


Ω_k^+
Ω_k^-
$\partial\Omega$

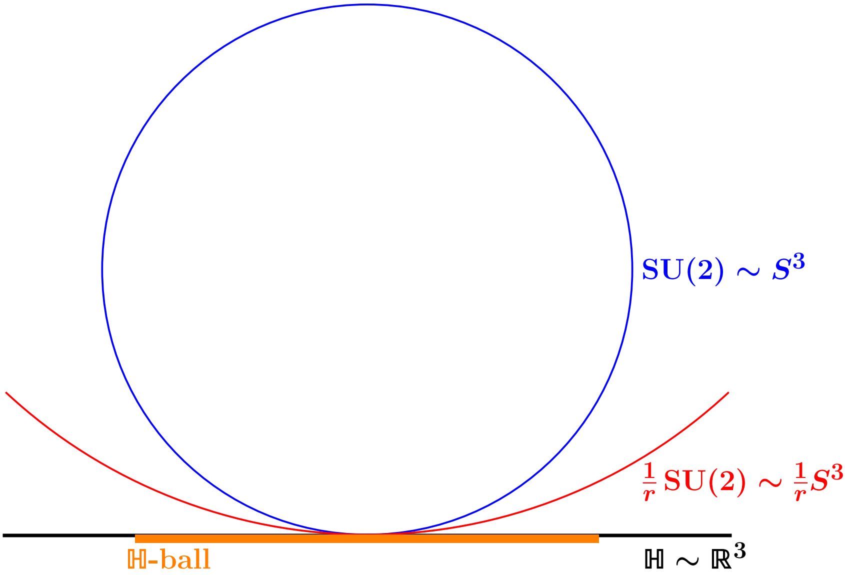
Convergence of the re-normalized eigenvalues of small balls in $\mathbf{SU(2)}$ to corresponding eigenvalues in the unit ball of $\mathbb{H}$

Here $\mathbb{H}$ is the Heisenberg group, which is a re-scaled limit of $\mathbf{SU(2)}$ near the identity.

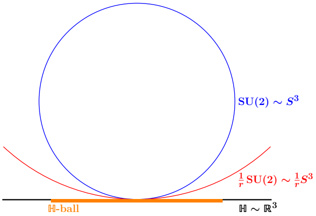
Theorem (Carfagnini, Gordina, T.)

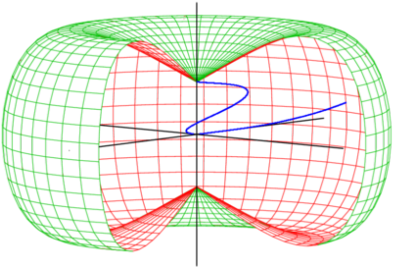
Let $0 < \lambda_1^{\mathbb{H}} < \lambda_2^{\mathbb{H}} \leqslant \lambda_3^{\mathbb{H}} \leqslant \dots$ be the Dirichlet eigenvalues in the unit ball $B_1^{\mathbb{H}}$ of $\mathbb{H}$, counted with multiplicity. Let $0 < \lambda_1^r < \lambda_2^r \leqslant \lambda_3^r \leqslant \dots$ be the Dirichlet eigenvalues in the r-ball $B_r^{\mathbf{SU(2)}}$ of $\mathbf{SU(2)}$, counted with multiplicity. Then for each $n \geqslant 1$ we have

$$\lim_{r \to 0} r^2 \lambda_n^r = \lambda_n^{\mathbb{H}}.$$



$\mathbf{SU(2)} \sim S^3$
$\frac{1}{r}\,\mathbf{SU(2)} \sim \frac{1}{r}S^3$
$\mathbb{H}$-ball
$\mathbb{H} \sim \mathbb{R}^3$





the Heisenberg ball [picture made by Nate Eldredge]

Convergence of the Dirichlet heat kernels

Let $p_t^{B_1^{\mathbb{H}}}(\cdot,\cdot)$ be the Dirichlet heat kernel in the unit ball $B_1^{\mathbb{H}}$ of $\mathbb{H}$, and $p_t^{B_r^{SU(2)}}(\cdot,\cdot)$ be the Dirichlet heat kernel in the r-ball $B_r^{SU(2)}$ of $SU(2)$, where the balls are centered at the identity of the groups.

Theorem (Carfagnini, Gordina, T.)

For each $t > 0$

$$\lim_{r\to 0} r^4\, p_{r^2 t}^{B_r^{SU(2)}}\left(\Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right), \Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right)\right) = p_t^{B_1^{\mathbb{H}}}(x,y).$$

uniformly for $x, y \in B_1^{\mathbb{H}}$.

Local convergence of stochastic flows

Let

$$g_s^{B_r^{\mathbf{SU(2)}}} := \begin{cases} g_s & s < \tau_{B_r^{\mathbf{SU(2)}}} \\ \partial & s \geqslant \tau_{B_r^{\mathbf{SU(2)}}} \end{cases} \tag{13}$$

where g_s denotes a hypoelliptic Brownian motion on $\mathbf{SU(2)}$, and

$$\tau_{B_r^{\mathbf{SU(2)}}} := \inf\left\{ s > 0,\ g_s \notin B_r^{\mathbf{SU(2)}} \right\}. \tag{14}$$

Similarly, let

$$X_s^{B_r^{\mathbb{H}}} := \begin{cases} X_s & s < \tau_{B_r^{\mathbb{H}}} \\ \partial & s \geqslant \tau_{B_r^{\mathbb{H}}} \end{cases} \tag{15}$$

where X_s denotes a hypoelliptic Brownian motion on $\mathbb{H}$, and

$$\tau_{B_r^{\mathbb{H}}} := \inf\left\{ s > 0,\ X_s \notin B_r^{\mathbb{H}} \right\}. \tag{16}$$

Theorem (Carfagnini, Gordina, T.)

For any $0 < r < \frac{1}{7} r_{1/7}$ *there is a continuous stochastic process* Y_s^r *in* $\mathbb{H}$ *such that*

$$Y_s^r :=: \delta_{1/r}^{\mathbb{H}} \Phi \left(g_{r^2 s}^{B_{3r}^{SU(2)}} \right)$$

in the sense of conditional probability distributions on the event $A_{3r} := \{ s < \tau_{B_{3r}^{\mathbb{H}}} \}$ *and*

$$\lim_{r \to 0} \mathbb{1}_{A_{3r}} \sup_{0 \leqslant s \leqslant T} |Y_s^r - X_s| = 0$$

in probability.

We use Theorem 3.3.1, page 76, in Kunita 1986 Lectures on stochastic flows and applications, Tata Institute of Fundamental Research Lectures on Mathematics and Physics.

Thank you for your attention!