DIRICHLET METRIC MEASURE SPACES: SPECTRUM, IRREDUCIBILITY AND SMALL DEVIATIONS

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joint research with

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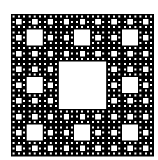
Stochastics and Geometry 24w5314



Plan:

(1) Introduction: Limit laws in metric measure spaces

(2) Discrete spectrum for general DMMS

(3) Spectral convergence $\mathbf{SU}\,(\mathbf{2}) \longrightarrow \mathbb{H}$

(4) Fractals (... if time permits ...)

- Typical Settings: hypoelliptic diffusions on Lie groups, self-similar processes, Brownian motion on fractals

- Applications: limit laws for stochastic processes such as small deviations, Chung's LIL, heat content asymptotics

- Goal: establish spectral properties for the Dirichlet operator on open bounded connected sets with possibly **irregular boundaries** to prove such limit laws

STOCHASTIC PROCESSES IN METRIC SPACES

$(\mathcal{X}, d)$ complete separable locally compact metric space

$(\mathcal{X}, \mathcal{B}(\mathcal{X}), \mu)$ (Radon) metric measure space (with pre-compact balls)

$L^2(\mathcal{X}, \mu)$ square-integrable measurable real-valued functions on $\mathcal{X}$

$\{X_t\}_{t\geqslant 0}$ $\mathcal{X}$-valued (Hunt) stochastic process with $X_0 = x_0 \in \mathcal{X}$ a.s. $\Longleftrightarrow$ regular Dirichlet form $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ $\Longleftrightarrow$ $\{P_t\}_{t\geqslant 0}$ corresponding semi-group $\Longleftrightarrow$ infinitesimal generator $(\mathcal{L}, \mathcal{D}_{\mathcal{L}})$ $\Longleftrightarrow$ spectral properties of $(\mathcal{L}, \mathcal{D}_{\mathcal{L}})$ $\Longleftrightarrow$ properties of the heat kernel $p_t(x, y)$ (if it exists)

$U \subset \mathcal{X}$ bounded open connected

$\{X_t^U\}_{t\geqslant 0}$ killed process $\Longleftrightarrow$ restricted Dirichlet form $\left(\mathcal{E}^U, \mathcal{D}_{\mathcal{E}^U}\right)$ $\Longleftrightarrow$ infinitesimal generator $\left(\mathcal{L}^U, \mathcal{D}_{\mathcal{L}^U}\right)$ $\Longleftrightarrow$ spectral properties of $\left(\mathcal{L}^U, \mathcal{D}_{\mathcal{L}^U}\right)$ $\Longleftrightarrow$ properties of the heat kernel $p_t^U(x, y)$

LIMIT LAWS

$|X_t|$ $\qquad$ $d(X_t, x_0)$

- Small deviations principle: X_t satisfies SDP with rates α and β if there exists a constant $c > 0$ such that

$$\lim_{\varepsilon \to 0} -\varepsilon^{\alpha} |\log \varepsilon|^{\beta} \log \mathbb{P}\left(\max_{0 \leqslant t \leqslant 1} |X_t| < \varepsilon \right) = c$$

- Chung's LIL: X_t satisfies Chung's LIL at *infinity* with rate $a > 0$ if there exists a constant $C > 0$ such that

$$\liminf_{t \to \infty} \left(\frac{\log \log t}{t} \right)^{a} \max_{0 \leqslant s \leqslant t} |X_s| = C \quad \text{a.s.}$$

- Onsager-Machlup functional: find the asymptotics for continuous processes

$$\mathbb{P}\left(\max_{0 \leqslant t \leqslant 1} d(X_t, \varphi(t)) < \varepsilon \right) \qquad \text{as } \varepsilon \to 0, \qquad \varphi \in W_{x_0}(\mathcal{X})$$

SDP AND CHUNG'S LIL IN EXAMPLES

- $\mathbb{R}^n$-valued Brownian motion B_t

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0 \leqslant t \leqslant 1} |B_t| < \varepsilon \right) = \lambda_1$$

$$\liminf_{t \to \infty} \sqrt{\frac{\log \log t}{t}} \max_{0 \leqslant s \leqslant t} |B_s| = \lambda_1^2 \quad \text{a. s.}$$

▶ Lévy's area $\boldsymbol{A_t = \int_0^t B_1(s) dB_2(s) - B_2(s) dB_1(s)}$

$$\lim_{\varepsilon \to 0} -\varepsilon \log \mathbb{P}\left(\max_{0 \leqslant t \leqslant 1} |A_t| < \varepsilon \right) = \frac{\pi}{2}$$

$$\liminf_{t \to \infty} \frac{\log \log t}{t} \max_{0 \leqslant s \leqslant t} |A_s| = \frac{\pi}{2} \text{ a.s.}$$

• B. Rémillard, *On Chung's law of the iterated logarithm for some stochastic integrals*, Ann. Probab., 1994

▶ Onsager-Machlup functional for standard Brownian motion in $\mathbb{R}^d$ and Brownian motions in Riemannian manifolds: Girsanov's transformation

BROWNIAN MOTION IN HEISENBERG GROUP

$\mathbb{H} \cong \mathbb{R}^3$ Heisenberg group

$$g_1 \cdot g_2 := \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right)$$

Geometry horizontal+vertical $\mathfrak{g} = \mathcal{H} \oplus \mathcal{V}$, $\mathcal{H} = \mathbb{R}^2$, $\mathcal{V} = \mathbb{R}$

Only $\mathcal{H}$ is equipped with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$

horizontal path $\gamma : [0,1] \to G$ if γ is absolutely continuous and

$$c_\gamma(t) := (L_{\gamma(t)^{-1}})_* \dot{\gamma}(t) \in \mathcal{H} \quad \text{for a.e. } t$$

$$\ell(\gamma) := \int_0^1 |c_\gamma(t)|_{\mathcal{H}} dt$$

Hörmander $\Longrightarrow$ Chow–Rashevskii: any two points in $\mathbb{H}$ are connected by a horizontal path

ρ_{cc} *Carnot-Carathéodory distance* for any $g_1, g_2 \in \mathbb{H}$

$|\cdot|$ $\rho_{cc}(e, \cdot)$

$$\rho_{cc}(g_1, g_2) := \inf \{\ell(\gamma) : \gamma : [0,1] \to \mathbb{H} \text{ is horizontal}, \gamma(0) = g_1, \gamma(1) = g_2\}$$

g_t hypoelliptic Brownian motion in $\mathbb{H}$

$$g_t = \left(B_1(t), B_2(t), \int_0^t B_1(s)\, dB_2(s) - B_2(s)\, dB_1(s)\right)$$

scaling $\rho_{cc}(g_{\varepsilon t}, e) \stackrel{d}{=} \sqrt{\varepsilon}\rho_{cc}(g_t, e)$

- infinitesimal generator is hypoelliptic

- if a process X_t satisfies a scaling property $|X_{\varepsilon t}| \overset{d}{=} \varepsilon^{\frac{1}{\delta}}|X_t|$, then

$$
\begin{aligned}
&\varepsilon^{\delta} \log \mathbb{P}\left(\max_{0 \leqslant s \leqslant 1} |X_s| < \varepsilon\right) \\
&= \frac{1}{t} \log \mathbb{P}\left(\max_{0 \leqslant s \leqslant t} |X_s| < 1\right) \qquad t = \varepsilon^{-\delta}
\end{aligned}
$$

- $W_{x_0}(\mathbb{H})$ is not a linear space

- g_t is not a Gaussian motion

Theorem (Carfagnini, Gordina '22 TAMS) There exists a $c > 0$ such that

$$\liminf_{t\to\infty} \sqrt{\frac{\log\log t}{t}} \max_{0\leqslant s\leqslant t} |g_s| = c \text{ a.s.}$$

$$\lim_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant s\leqslant 1} |g_s| < \varepsilon\right) = c^2.$$

$$0 < \lambda_1^{(2)} \leqslant \liminf_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant s\leqslant 1} |g_s| < \varepsilon\right)$$
$$\leqslant \limsup_{\varepsilon\to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0\leqslant s\leqslant 1} |g_s| < \varepsilon\right) \leqslant f(\lambda_1^{(1)}, \lambda_1^{(2)}) < \infty$$

HOMOGENEOUS CARNOT GROUPS: SPECTRAL GAP AND SMALL DEVIATIONS

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}\left(\max_{0 \leqslant s \leqslant 1} |g_s| < \varepsilon\right) = c^2$$

G $\mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_r}$ homogeneous Carnot group with Haar measure dx $(G, \rho_{cc}, dx) \xrightarrow{\text{replace by}} (X, d, \mu)$

$\mathcal{L}$ $\sum_{k=1}^{d_1} X_k^2$ sub-Laplacian $\xrightarrow{\text{replace by}}$ infinitesimal generator for a regular Dirichlet form on $L^2(X, \mu)$

$\mathcal{E}(f)$ $\int_G |\nabla_{\mathcal{L}} f|^2_{\mathbb{R}^{d_1}} dx$ regular Dirichlet form on $L^2(G, dx)$

$X_1, \ldots, X_{d_1}$ are left-invariant vector fields satisfying Hörmander condition

$U \subset G$ bounded open connected

$\mathcal{D}_{\mathcal{E}}(U)$ $\overline{\{f \in \mathcal{D}_{\mathcal{E}} : \operatorname{supp} f \subset U\}}^{W_2^1}$ $\Longrightarrow$ $(\mathcal{E}, \mathcal{D}_{\mathcal{E}}(U))$ regular Dirichlet form on $L^2(U, dx)$

X_t $\Longleftrightarrow$ $P_t f(x) = \mathbb{E}^x [f(X_t)]$ $\Longleftrightarrow$ $\mathcal{L}$

X_t^U $\Longleftrightarrow$ $P_t^U f(x) = \mathbb{E}^x \left[f(X_t^U) \right] = \mathbb{E}^x [f(X_t), t < \tau_U]$ $\Longleftrightarrow$ $\mathcal{L}_U$

Theorem (Carfagnini-Gordina, '24, IMRN)

- The operator $-\mathcal{L}_U$ has a discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$

$\lambda_n \uparrow \infty$ with $\lambda_1 > 0$ (spectral gap)

- There exists an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ of $L^2(U, dx)$ such that

$$\begin{aligned} &-\mathcal{L}_U\varphi_n = \lambda_n\varphi_n, \;\; \varphi_n \in \mathcal{D}(-\mathcal{L}_U) \\ &P_t^U\varphi_n = e^{-\lambda_n t}\varphi_n \end{aligned}$$

- There exists a constant $c(U) > 0$ such that for any $1 \leqslant p \leqslant \infty$

$$\|\varphi_n\|_{L^p(U,dx)} \leqslant c(U)\lambda_n^{\frac{Q}{2}}$$

- $\mathbb{P}^x\left(\tau_U > t\right) = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n \varphi_n(x)$

$c_n := \int_U \varphi_n(y) dy$

- principal eigenvalue: $\dim V_{\lambda_1} = 1$ and $\varphi_1(x) > 0$ for any $x \in U$

No assumptions on ∂U

APPLICATIONS

Small deviations on Carnot groups

$U \subset G$ bounded open connected

$U_\varepsilon := \delta_\varepsilon (U)$ (dilation of U)

Then

$$\lim_{\varepsilon \to 0} e^{\frac{\lambda_1 t}{\varepsilon^2}} \mathbb{P}^e \left(\tau_{U_\varepsilon} > t \right) = c_1 \varphi_1(e),$$

where λ_1 is the spectral gap of $-\mathcal{L}_U$ and

$$\lim_{\varepsilon \to 0} -\varepsilon^2 \log \mathbb{P}^e \left(\tau_{U_\varepsilon} > t \right) = \lambda_1 t$$

- the group of dilations acting on G
- space-time scaling

$$(G, \rho_{cc}, dx) \xrightarrow{\text{replace by}} (X, d, \mu)$$

- estimates for the spectral gap in H-type groups (M. Carfagnini and M. Gordina)

- heat content: if U is a regular domain $\Longrightarrow$ $u(x,t) := \mathbb{P}^{x}\left(\tau_U > t\right)$ is the solution to the heat equation in U

$$Q_U(t) := \int_U u(x,t)dx = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n^2$$

$$\lim_{t\to\infty} e^{\lambda_1 t} Q_U(t) = c_1^2$$

BEYOND CARNOT GROUPS (JOINT WITH M. CARFAGNINI AND M. GORDINA)

Assumptions:

- $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ a regular Dirichlet form
- $P_t, t \geqslant 0$ a strongly continuous contraction semigroup on $L^2(\mathcal{X}, \mu)$
- $p_t(x, y)$ exists for all t and for μ-a.e. $x, y \in \mathcal{X}$
- P_t^U is compact for some $t > 0$
- $M_U(t) = \operatorname*{ess\,sup}_{(x,y) \in U \times U} p_t^U(x, y) < \infty$ for some t

- Carlen-Kusuoka-Stroock '87: Nash inequality $M(t) \leqslant ct^{-\gamma}$

$$\|f\|_{L^2(\mathcal{X},\mu)}^{2+\frac{2\alpha}{\beta}} \leqslant C\mathcal{E}(f,f)\|f\|_{L^1(\mathcal{X},\mu)}^{\frac{2\alpha}{\beta}} \qquad f \in \mathcal{D}_{\mathcal{E}}$$

$$\|P_t f\|_{L^\infty(\mathcal{X},\mu)} \leqslant C\, t^{-\frac{\alpha}{\beta}}\|f\|_{L^1(\mathcal{X},\mu)} \qquad f \in L^1(\mathcal{X},\mu), t > 0$$

- **ultracontractivity**, Davies, Varopoulos et al

- M. Carfagnini, M. Gordina and A. Teplyaev: Riemannian manifolds with non-negative Ricci curvature, self-similar processes, not necessarily continuous; Brownian motion on fractals; Dirichlet forms on mms under Sturm's assumptions (complete closed balls, doubling, weak Poincaré, PHI); group action on metric measure spaces, convergence of spectra

DISCRETE SPECTRUM FOR DIRICHLET FORMS

Theorem [Carfagnini, Gordina, Teplyaev]

- $\mu(U) < \infty \implies$ the spectrum of A^U is discrete and the heat kernel $p_t^U(x,y)$ has the usual eigenfunction expansion

- If there exists a $t_U > 0$ such that

$$M_U(t_U) = \operatorname*{ess\,sup}_{(x,y)\in U\times U} p_{t_U}^U(x,y) < \frac{1}{\mu(U)^2}$$

$\implies \lambda_1 > 0$

GENERALIZED HEAT CONTENT

$$Q_{\mathcal{U}}(t) := \int_{\mathcal{U}} \mathbb{P}^x\left(\tau_{\mathcal{U}} > t\right) dm(x) = \int_{\mathcal{U}} u(t,x)dx$$

Theorem (C-G-T) Under ultracontractivity and other usual assumptions for any open set $\mathcal{U}$ of finite measure

$$\lim_{t\to\infty} e^{\lambda_1 t} Q_{\mathcal{U}}(t) = \sum_{k=1}^{M_1} c_k^2,$$

where $c_k := \int_{\mathcal{U}} \phi_k(x)dm(x)$, and M_1 is the multiplicity of λ_1.

Again, no regularity of the boundary is assumed.

ESTIMATES OF EIGENFUNCTIONS

Theorem (C-G-T) Under the usual assumptions and the Nash inequality, for any open set $\mathcal{U}$ of finite measure, the spectrum is discrete and eigenfunctions satisfy

$$\|\varphi_n\|_{L^\infty} \leqslant c\lambda_n^{\delta},$$

where c is a constant depending on $\mathcal{U}$, α, and β.

$\delta = \frac{\alpha}{\beta} = \frac{2}{\nu}$ where, usually, the space is Alhfors α-regular and β is the time scaling exponent if the process is *(distance-)self-similar*:

$$d(X_{t\varepsilon}^{x}, x) \overset{(d)}{=} \varepsilon^{\frac{1}{\beta}} d(X_t^x, x).$$

Again, no regularity of the boundary is assumed. This inequality was obtained by J.Kigami in case of self-similar p.c.f. fractals.

Our article contains more detailed estimates in more general ultracontractive cases and under more specific heat kernel bounds.

SMALL DEVIATIONS

Theorem [Carfagnini, Gordina, Teplyaev] Assume that

- $P_t^{B_1(x)}$ is irreducible for some $x \in \mathcal{X}$
- the heat kernel $p_t^{B_1(x)}(x, y)$ exists for all t and for all $x, y \in \mathcal{X}$ and that

$$p_t(x, y) \leqslant c\, t^{-\frac{\alpha}{\beta}} \text{ for any } t, x, y$$

- there exists a t_0 such that $p_{t_0}^{B_1(x)}(x, y)$ is continuous for $x, y \in \mathcal{X}$
- X_t^x is self-similar

$$\Longrightarrow$$

- $$\lim_{\varepsilon \to 0} e^{\lambda_1 \frac{t}{\varepsilon^\beta}} \mathbb{P}^x \left(\sup_{0 \leqslant s \leqslant t} d(X_s, x) < \varepsilon \right) = c_1 \varphi_1(x),$$

where $\lambda_1 > 0$ is the spectral gap of $A^{B_1(x)}$ with zero boundary conditions outside of the unit ball $B_1(x)$, and φ_1 is the corresponding positive eigenfunction, $c_n := \int_U \varphi_n(y) \mu\,(dy)$

A Borel set $A \in \mathcal{B}(\mathcal{X})$ is P_t-*invariant* if $P_t(\mathbb{1}_A f) = 0$ μ-a.e. on A for every $t > 0$ and $f \in L^2(\mathcal{X}, \mu)$.

The semigroup $\{P_t\}_{t \geqslant 0}$ is called *irreducible* if for any P_t-invariant set A either $\mu(A) = 0$ or $\mu(A^c) = 0$.

- If a diffusion has a positive heat kernel then this diffusion is irreducible in each path-connected open set (killed at exiting this open set).

No regularity of the boundary is assumed.

ASYMPTOTIC DILATIONS

Contraction $\Phi : \mathbf{SU}\left(2\right) \longrightarrow \mathbb{H}$

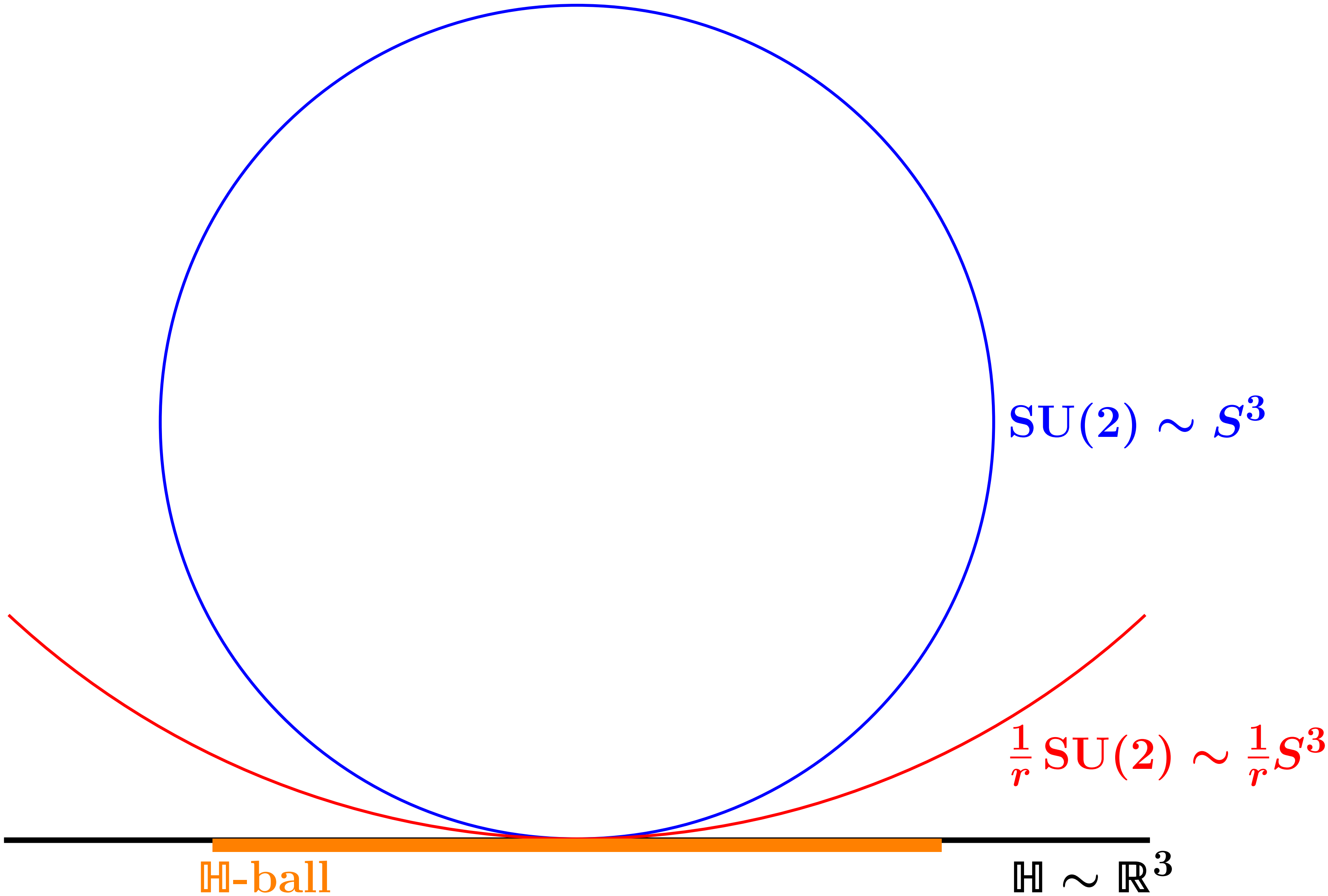
- Both groups are equipped with a sub-Riemannian structure
- Heisenberg group $\mathbb{H}$ viewed as a re-scaled limit of $\mathbf{SU}\left(2\right)$ near the identity

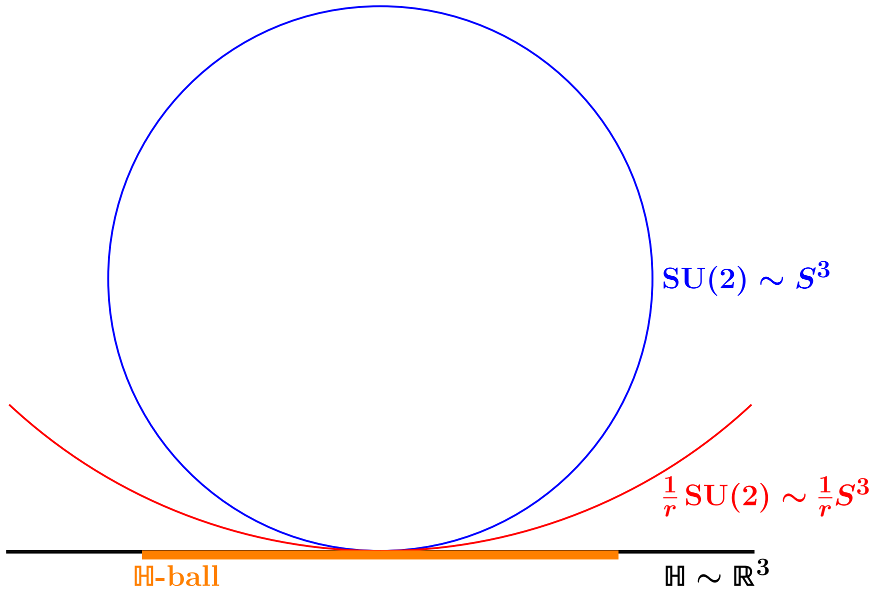
Convergence of the re-normalized spectrum in $\mathbf{SU}(2)$ to the spectrum in the unit ball of $\mathbb{H}$

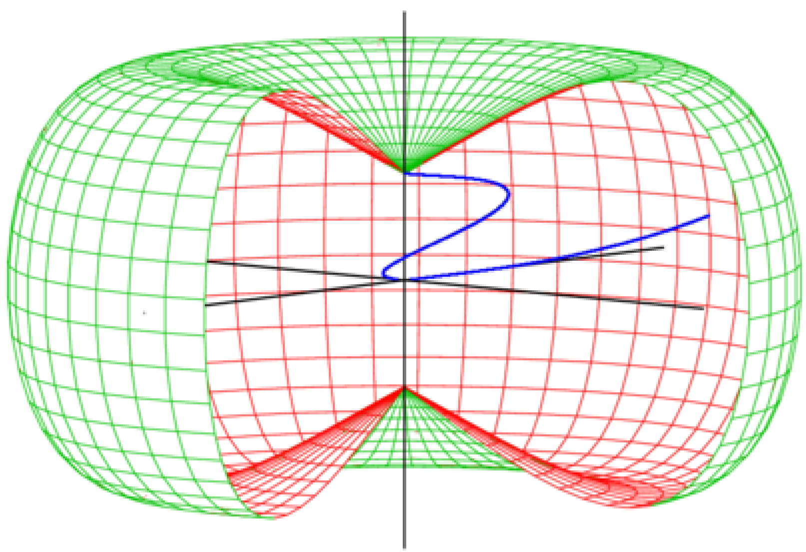
Theorem (Carfagnini, Gordina, Teplyaev)

- $0 < \lambda_1^{\mathbb{H}} < \lambda_2^{\mathbb{H}} \leqslant \lambda_3^{\mathbb{H}} \leqslant \ldots$ Dirichlet eigenvalues in the unit ball $B_1^{\mathbb{H}}$ in $\mathbb{H}$, counted with multiplicity
- $0 < \lambda_1^{r} < \lambda_2^{r} \leqslant \lambda_3^{r} \leqslant \ldots$ Dirichlet eigenvalues in the r-ball $B_r^{\mathrm{SU}(2)}$ in $\mathbf{SU}(2)$, counted with multiplicity

$$\Longrightarrow \quad \lim_{r \to 0} r^2 \lambda_n^r = \lambda_n^{\mathbb{H}} \qquad n \geqslant 1$$


$\mathbf{SU(2)} \sim S^3$
$\frac{1}{r}\mathbf{SU(2)} \sim \frac{1}{r}S^3$
$\mathbb{H}$-ball
$\mathbb{H} \sim \mathbb{R}^3$





the Heisenberg ball [picture made by Nate Eldredge]

MOSCO CONVERGENCE, STRONG AND NORM RESOLVENT CONVERGENCE

- U.Mosco *Convergence of convex sets and of solutions of variational inequalities* Adv. Math. (1969), *Composite media and asymptotic Dirichlet forms* JFA (1994)
- T.Kato *Perturbation theory for linear operators*. Springer 1966.
- Reed-Simon 1972, non-negative closed quadratic forms:
  - ► Mosco convergence is equivalent to the strong resolvent convergence.
  - ► The norm resolvent convergence is stronger than the strong resolvent convergence.

► ► We aim at even stronger uniform convergence of resolvent and heat kernels and eigenfunctions using Dynkin-Hunt formula:

$$p_t^{\mathcal{U}}(x,y) := p_t(x,y) - \mathbb{E}^x\left[\mathbb{1}_{\{\tau_{\mathcal{U}}<t\}}\, p_{t-\tau_{\mathcal{U}}}\left(X_{\tau_{\mathcal{U}}},y\right)\right]$$

Mosco convergence does not imply eigenvalues convergence:

$$\underset{n\to\infty}{\text{M-lim}}\, E_n = F \quad \text{or} \quad E_n \xrightarrow[n\to\infty]{\mathrm{M}} F \quad \text{if:}$$

- $x_n \in L^2$ converging weakly to $x \in L^2$, $\liminf_{n\to\infty} E_n(x_n) \geq F(x)$;
- for each $x \in L^2$ there exists an approximating sequence of elements $x_n \in L^2$, converging strongly to x $\limsup_{n\to\infty} E_n(x_n) \leq F(x)$.

Example: $L^2 := \ell^2(\mathbb{Z}_+)$

$$E_n((x_k)) := |x_n|^2 \xrightarrow[n\to\infty]{\mathrm{M}} E = 0$$

$$\sigma(E_n) = \{0, 1\} \neq \{0\} = \sigma(E)$$

CONVERGENCE OF THE DIRICHLET HEAT KERNELS

- $p_t^{B_1^{\mathbb{H}}}(\cdot,\cdot)$ Dirichlet heat kernel in the unit ball $B_1^{\mathbb{H}}$ in $\mathbb{H}$
- $p_t^{B_r^{\mathrm{SU}(2)}}(\cdot,\cdot)$ Dirichlet heat kernel in the r-ball $B_r^{\mathrm{SU}(2)}$ in $\mathbf{SU}(2)$

Theorem (Carfagnini, Gordina, Teplyaev) For each $t>0$

$$\lim_{r\to 0} r^{4} p_{r^2 t}^{B_r^{\mathrm{SU}(2)}}\left(\Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right),\Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right)\right)=p_t^{B_1^{\mathbb{H}}}(x,y)$$

uniformly for $x,y\in B_1^{\mathbb{H}}$

LOCAL CONVERGENCE OF STOCHASTIC FLOWS

- g_s hypoelliptic Brownian motion on $\mathbf{SU}(2)$
- X_s hypoelliptic Brownian motion on $\mathbb{H}$

Theorem (Carfagnini, Gordina, Teplyaev) For small enough r there is a continuous stochastic process Y_s^r in $\mathbb{H}$ such that

$$Y_s^r :=: \delta^{\mathbb{H}}_{1/r}\Phi\left(g_{r^2 s}\right) \qquad s < \inf\{t : d_{\mathbb{H}}(I, Y_s^r) \geqslant 1\}$$

in the sense of distributions and

$$\lim_{r\to 0} \sup_{0\leqslant s\leqslant T} |Y_s^r - X_s| = 0$$

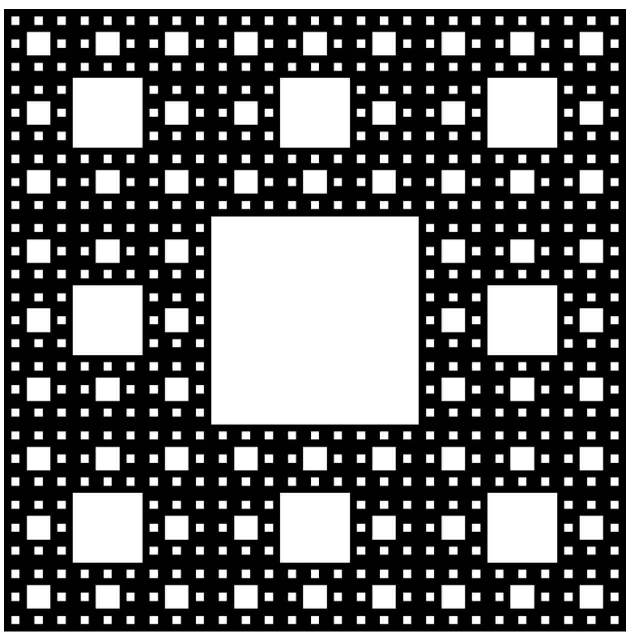
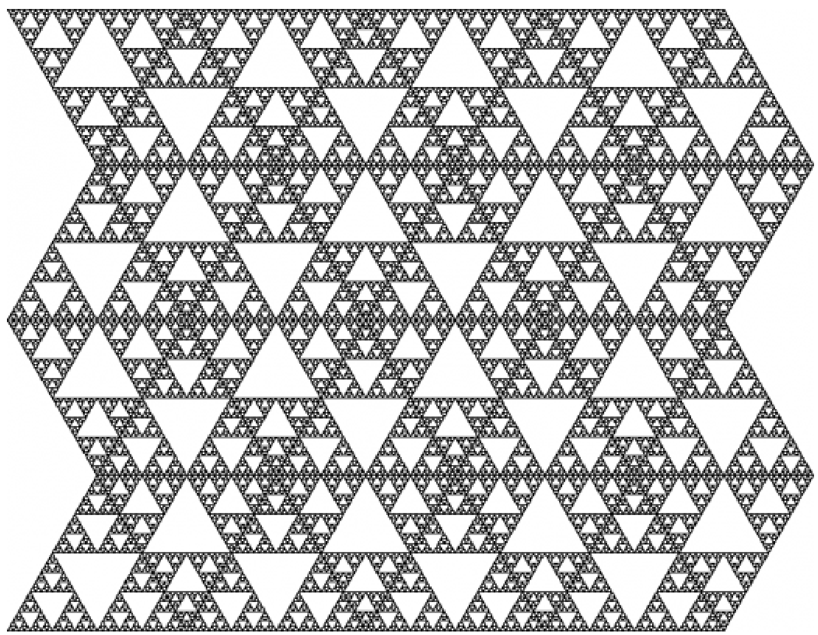
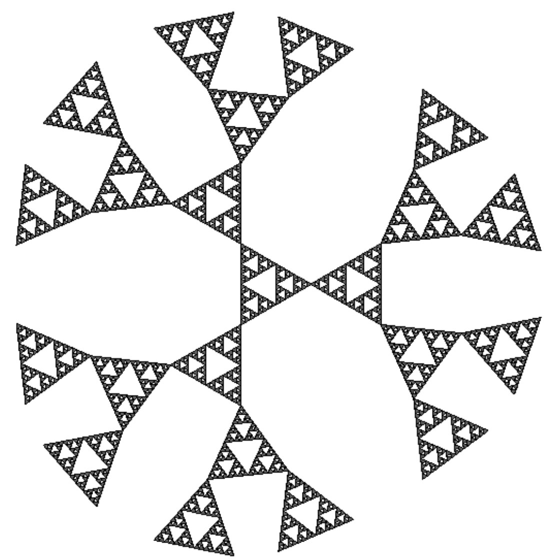
in probability.

Proof. ... Kunita 1986 *Lectures on stochastic flows and applications*, ... plus geometric localization arguments. □

"Elliptic results" + pointed Gromov-Hausdorff convergence:
Hui-Chun Zhang and Xi-Ping Zhu. Weyl's law on RCD(K,N) metric measure spaces. Comm. Anal. Geom. 2019.

FRACTALS (OR FRACTAFOLDS*)

- *Strichartz: *A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold.*
  - *A "fractafold" is to a fractal what a manifold is to a Euclidean half-space.*
    - There is no generally agreed upon definition of "fractal", other than "I know one when I see one":

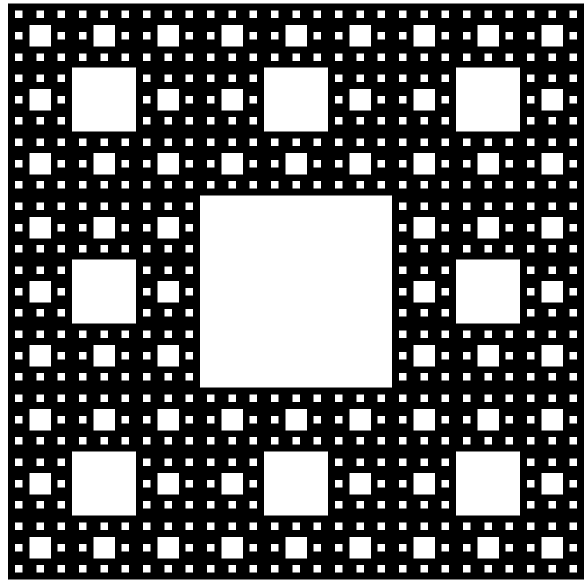
sub-Gaussian heat kernel estimates (sGHKE)

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right) \tag{1}$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$$d_f = \text{Hausdorff dimension}$$
$$\frac{1}{\gamma} = d_w = \text{“walk dimension” } (\gamma\text{=diffusion index})$$
$$\frac{2d_f}{d_w} = d_S = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980'—)



$$1 = d_t = d_{mart} < d_{tH} = \frac{\ln 2}{\ln 3} + 1 < d_S < d_f = \frac{\ln 8}{\ln 3} < 2 < d_w$$

For Sierpinski carpets there exists a unique Dirichlet form and diffusion process due to [Barlow and Bass 1998, 1999] (see also [Barlow-Bass-Kumagai-T 2010]).

$d_{mart} = 1$ is a deep result of Kusuoka-Hino, see also Kajino-Murugan.

Here $\boldsymbol{d_{tH}} = \dfrac{\mathbf{\ln 2}}{\mathbf{\ln 3}} + \mathbf{1}$ is the ***topological-Hausdorff dimension*** of the Sierpinski carpet defined in Theorem 5.4 in:

[**R.Balka, Z.Buczolich, M.Elekes.** ***A new fractal dimension: the topological Hausdorff dimension***. Adv. Math. 2015.]

Roughly speaking:

$$\boldsymbol{d_{tH}} := \mathbf{1} + \mathbf{inf}\{\text{Hausdorff dim. of boundaries of a base of open sets}\}$$

Barlow (Proceedings of SMS Montreal, 2011):

Given a regular fractal F, since L and M are given by the construction, one can calculate d_f easily. The constant ρ which gives d_w is somehow deeper, and seems to require some analysis on the set or its approximations. Loosely one can say that d_f is a 'geometric' constant, while d_w is an 'analytic' constant. One may guess that in some sense ρ or β are in general inaccessible by any purely geometric argument. (An exception is for trees, where one has $d_w = 1 + d_f$.)

Open questions:

On the Sierpinski carpet,

$$\boldsymbol{\kappa = d_W - d_f + d_{tH} - 1 = d_W - d_f + \frac{\log 2}{\log 3}}$$

would give the best Hölder exponent for harmonic functions?
[*Strongly supported by numerical results: L.Rogers et al*]

Note that $(\boldsymbol{d_W - d_f})$ –Hölder continuity is known: Martin Barlow. Diffusions on fractals. In Lectures on probability theory and statistics (Saint-Flour, 1995), volume 1690 of Lecture Notes in Math. Springer, 1998. Heat kernels and sets with fractal structure. In Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), volume 338 of Contemp. Math., pages 11–40. Amer. Math. Soc., Providence, RI, 2003.

BV and weak Bakry-Émery non-negative curvature [P.Alonso-Ruiz, F.Baudoin, L.Chen, L.Rogers, N.Shanmugalingam, A.T.]

Definition. $BV(X) := KS^{\lambda_1^\#,1}(X) = \mathrm{B}^{1,\alpha_1^\#}(X)$ with $\alpha_1^\# = \frac{\lambda_1^\#}{d_W}$ the L^1–Besov critical exponent, and for $f \in BV(X)$

$$\mathrm{Var}(f) := \liminf_{r \to 0^+} \iint_{\Delta_r} \frac{|f(y) - f(x)|}{r^{\lambda_1^\#} \mu(B(x,r))} \, d\mu(y) \, d\mu(x).$$

Definition. We say that $(X, \mu, \mathcal{E}, \mathcal{F})$ satisfies the weak-Bakry-Émery non-negative curvature condition $wBE(\kappa)$ if there exist a constant $C > 0$ and a parameter $0 < \kappa < d_W$ such that for every $t > 0$, $g \in L^\infty(X, \mu)$ and $x, y \in X$,

$$|P_t g(x) - P_t g(y)| \leq C \frac{d(x,y)^\kappa}{t^{\kappa/d_W}} \|g\|_{L^\infty(X,\mu)}. \tag{2}$$

- If (X, d, μ) satisfies $wBE(\kappa)$ with $\kappa = \frac{d_W}{2}$, then the form $\mathcal{E}$ admits a carré du champ operator, which means that $d_w = 2$ by [**Kajino-Murugan 2019** Ann. Probab. 48, 2020]

- $\kappa \leqslant 1$ because $d(x, y)$ has to be essentially equivalent to a geodesic metric [Corollary 1.8, Theorem 2.11 **Mathav Murugan** JFA 2020]

- For nested fractals, p.c.f. with sGHKE (1)

$$\lambda_1^{\#} = \lambda_1^{*} = d_W \alpha_1^{*} = d_f$$

- For the Sierpinski carpet we <u>conjecture</u>

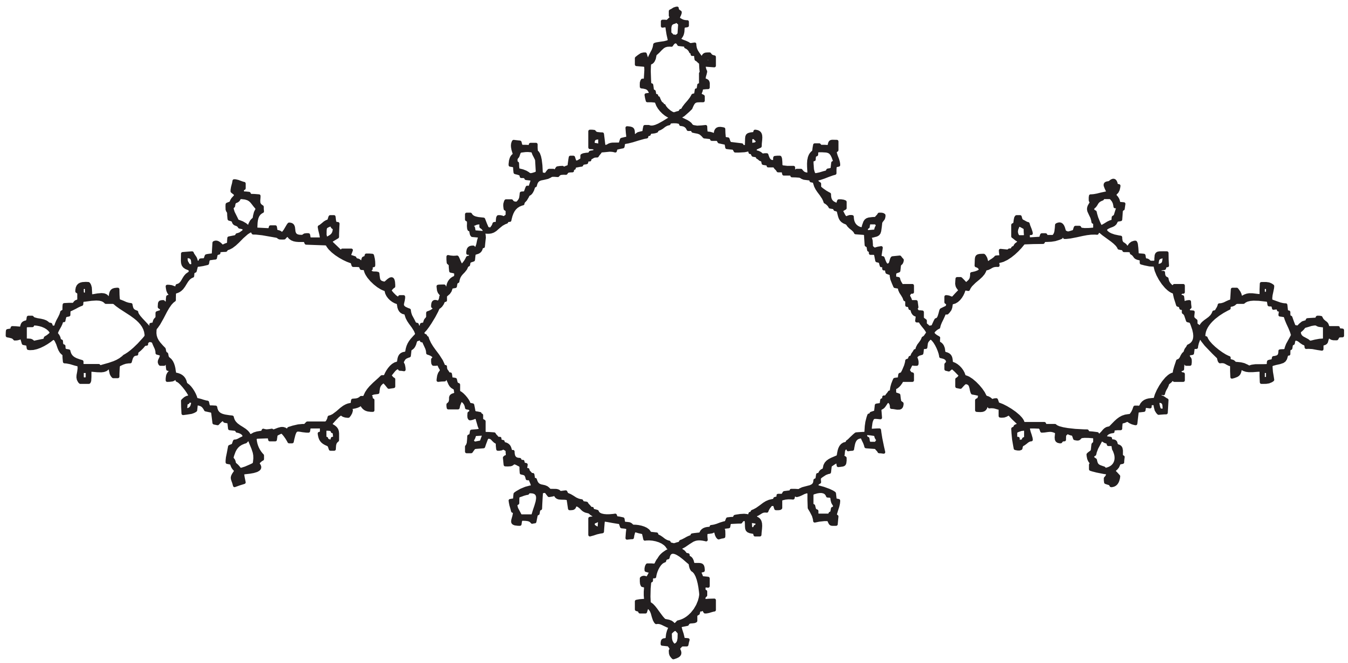
$$\lambda_1^{\#} = \lambda_1^{*} = d_f - d_{tH} + 1$$

where $d_{tH} = \dfrac{\ln 2}{\ln 3} + 1$ is the topological-Hausdorff dimension of the Sierpinski carpet

Connections to other areas

- **Martin Barlow, Thierry Coulhon, Alexander Grigor'yan. Manifolds and graphs with slow heat kernel decay**. Invent. Math. 144 (2001), no. 3, 609–649.
- Joint Spectra and related Topics in Complex Dynamics and Representation Theory: BIRS Banff 23w5033 May 21–26, 2023
- Quantum gravity and other topics in physics
- Applied mathematics

Group Theory and Complex Dynamics



The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda, Lyubich et al.).
The unique graph-directed self-similar Dirichlet form has $d_s = 4/3$
[L.Rogers and T., CPAA 2010]
Computing spectral dimension of Julia sets is an unresolved [Nekrashevych, T, 2008]

Asymptotic aspects of Schreier graphs and Hanoi Towers groups

Rostislav Grigorchuk [1], Zoran Šunić

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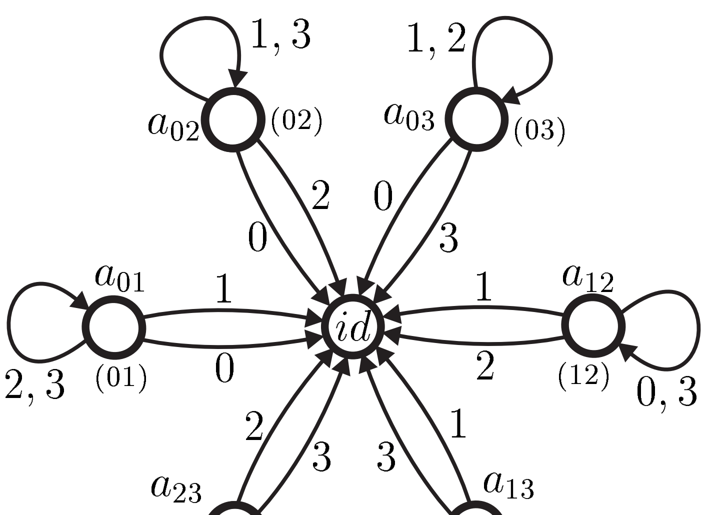
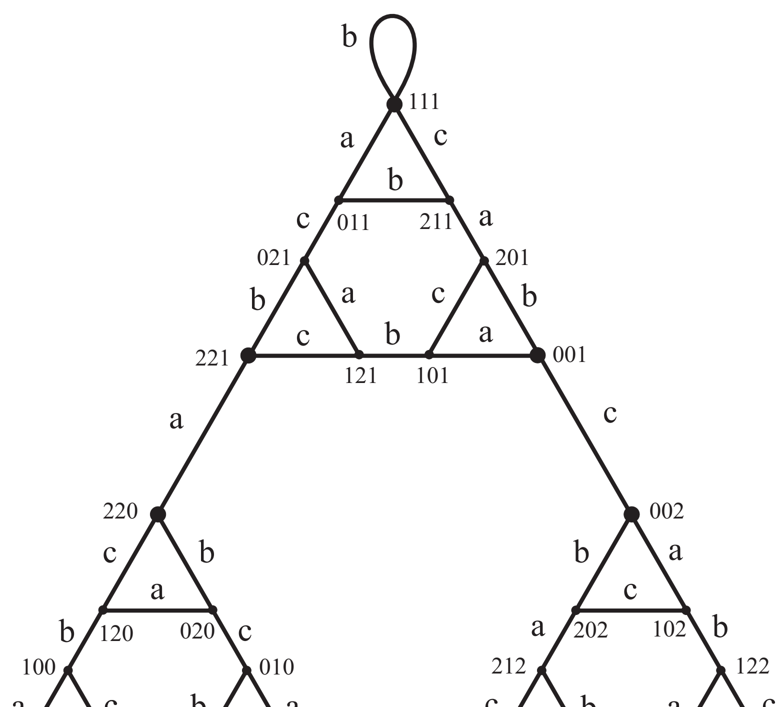
Received 23 January, 2006; accepted after revision +++++

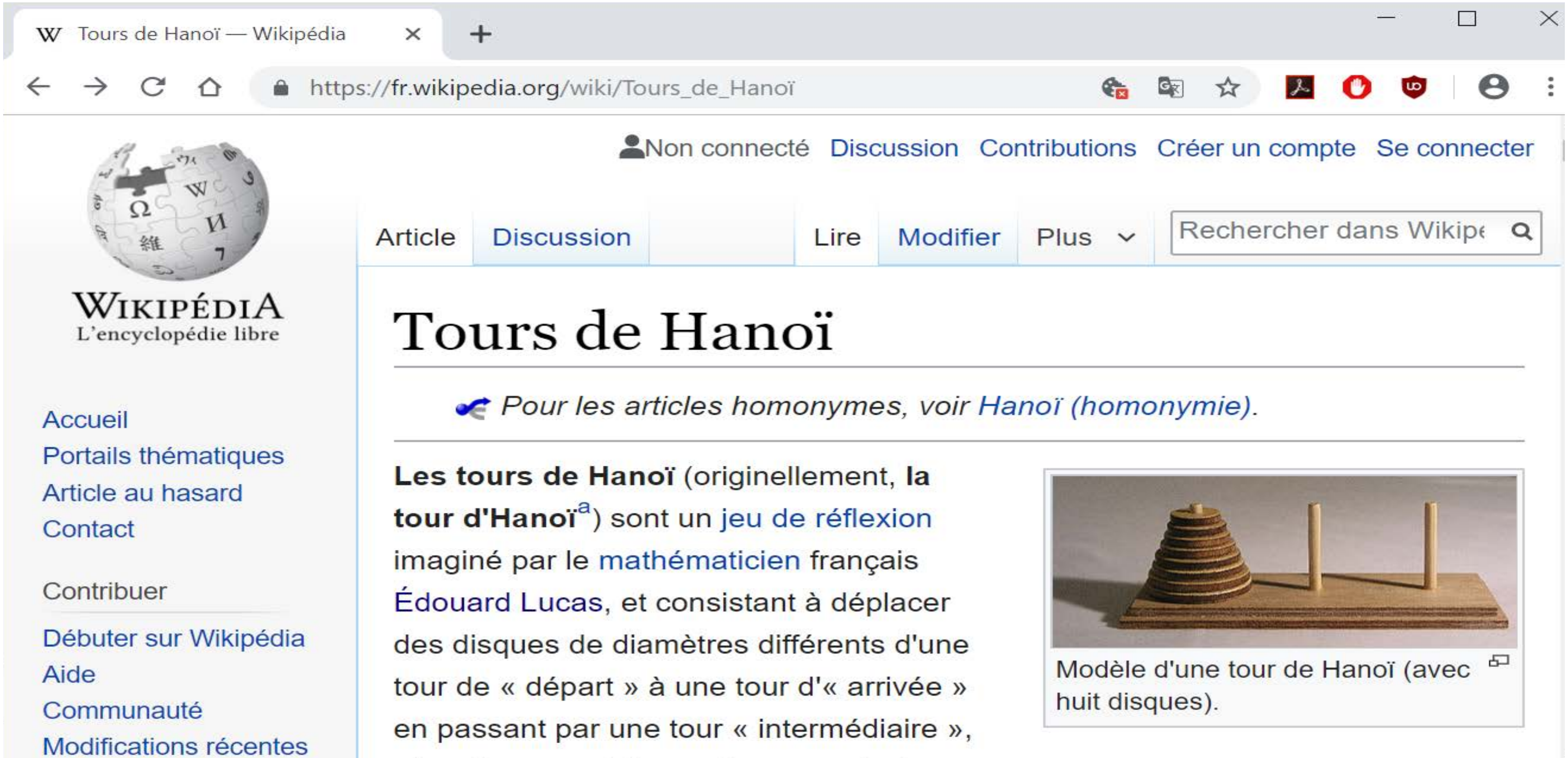
Presented by Étienne Ghys

Abstract

We present relations between growth, growth of diameters and the rate of vanishing of the spectral gap in Schreier graphs of automaton groups. In particular, we introduce a series of examples, called Hanoi Towers groups since they model the well known Hanoi Towers Problem, that illustrate some of the possible types of behavior. *To cite this article: R. Grigorchuk, Z. Šunić, C. R. Acad. Sci. Paris, Ser. I 344 (2006).*







This is a "canonical" example of a recursive algorithm in computer science. The optimal time to solve the puzzle is $2^n - 1$ while the random time to solve the puzzle is roughly 5^n, which reflect the underlying self-similar fractal structure of the Sierpinski gasket and the sub-Gaussian HKEs.

Early physics motivation

- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

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Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at

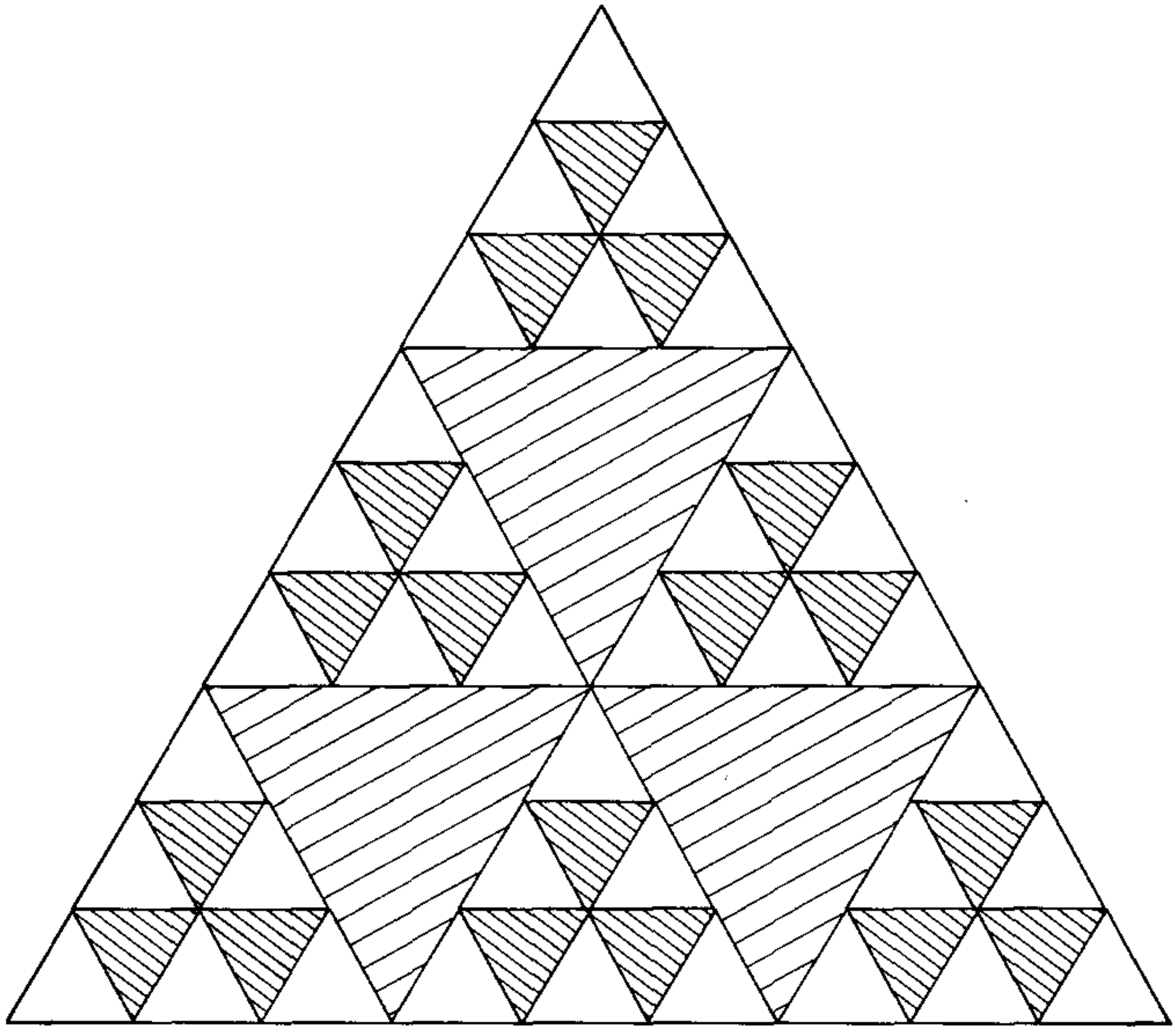

Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

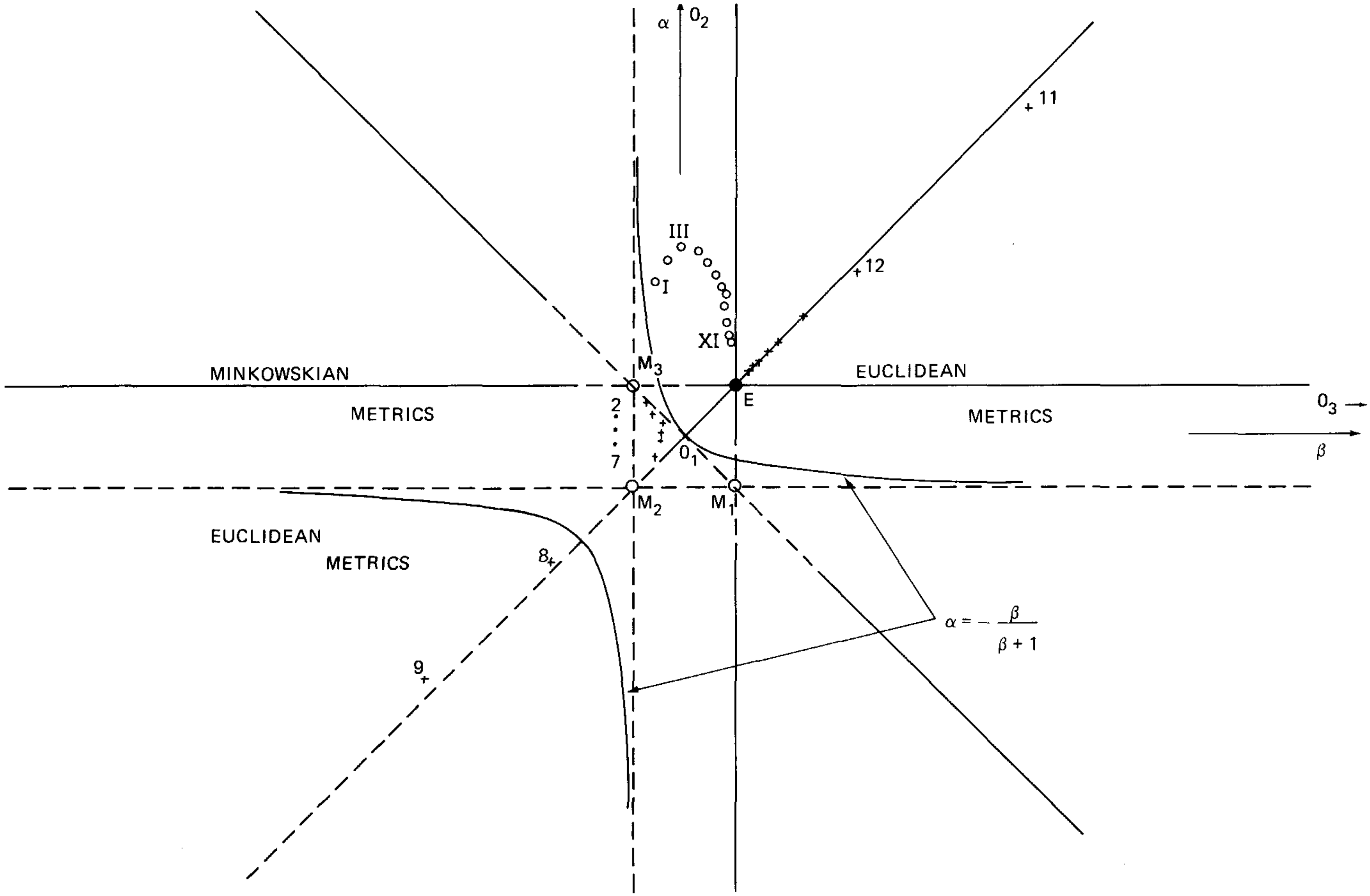

Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta + 1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

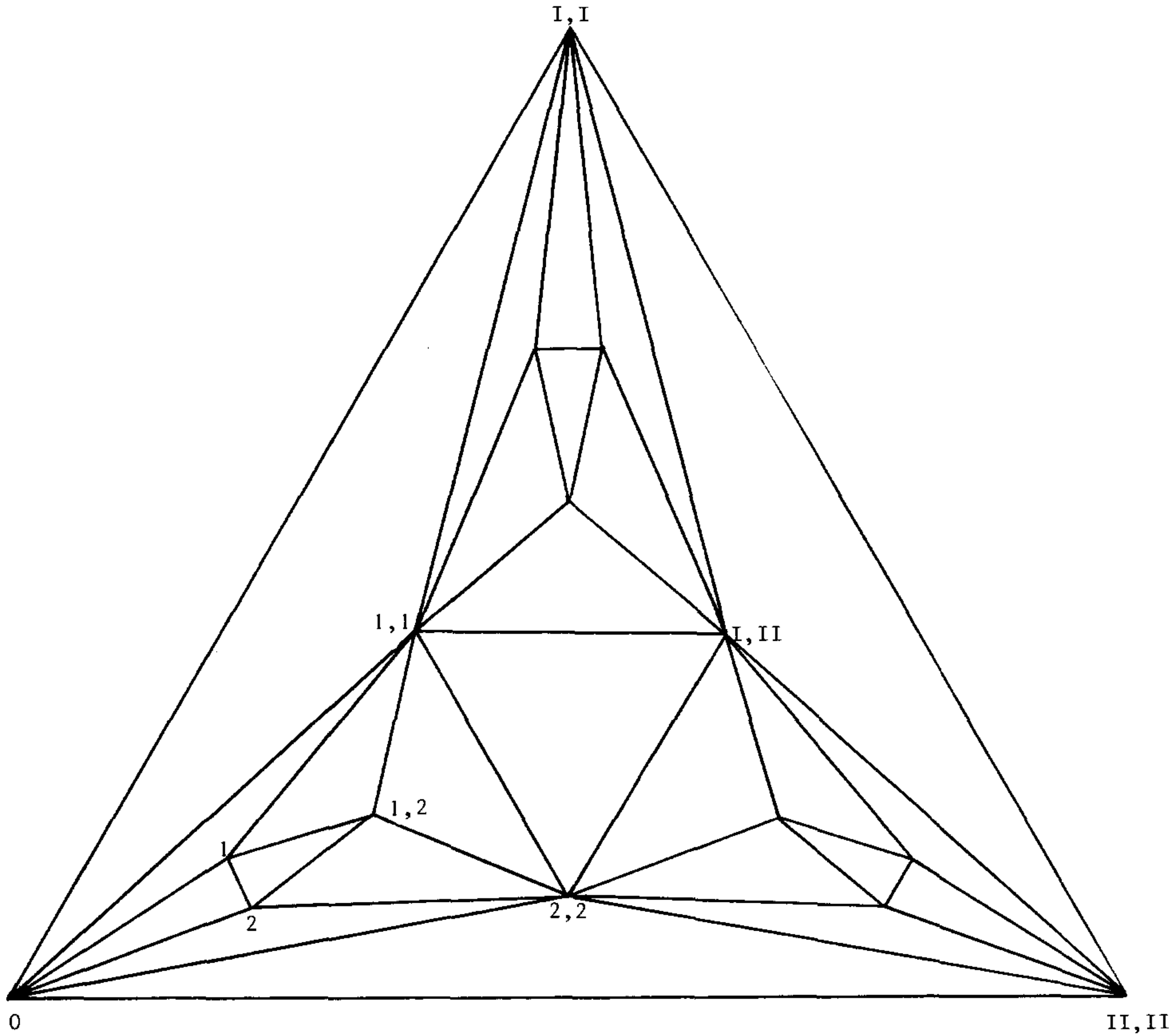

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

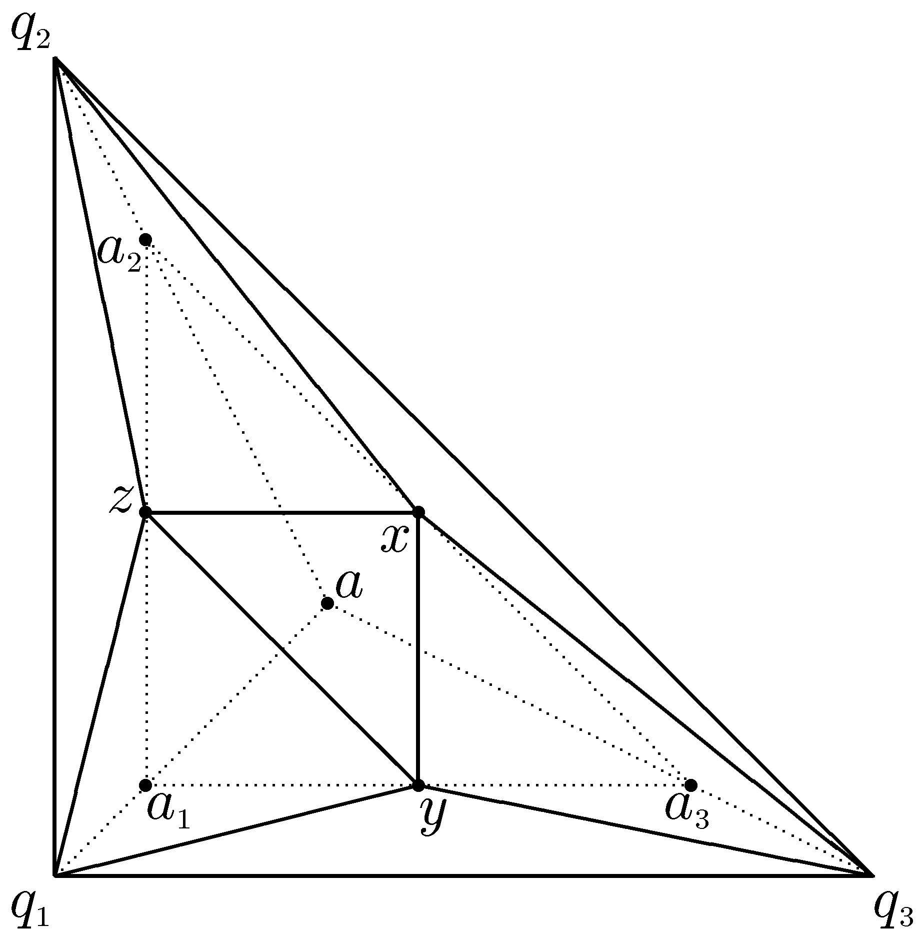

Figure 6.4. Geometric interpretation of Proposition 6.1.

The Spectral Dimension of the Universe is Scale Dependent

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[2]*Mark Kac Complex Systems Research Centre, Marian Smoluchowski Institute of Physics, Jagellonian University, Reymonta 4, PL 30-059 Krakow, Poland*
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(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

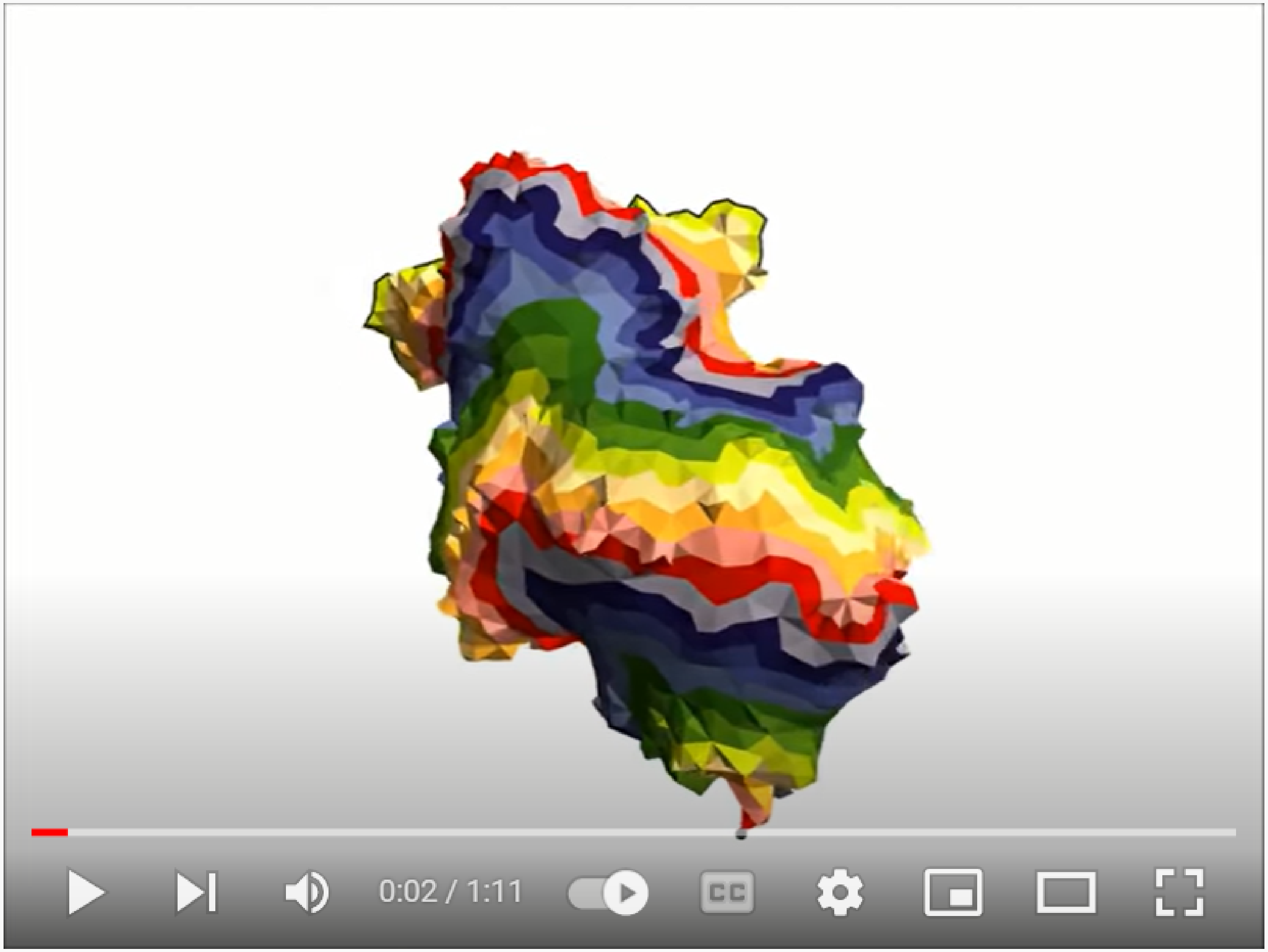
tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data [Martin Reuter, Frank Saueressig]:

Three scaling regimes of the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG):
(1) a classical regime $d_s = d, d_w = 2$,
(2) a semi-classical regime $d_s = 2d/(2+d),\ d_w = 2+d$,
(3) the UV-fixed point regime $d_s = d/2,\ d_w = 4$.
On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is in very good agreement with the data and provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

- *Quasisymmetric uniformization and heat kernel estimates* by Mathav Murugan: $d_w = d_f$ which is consistent with $d_s = 2d_f/d_w = 2$

Timothy Budd, Radboud University Nijmegen



Causal dynamical triangulations

25,971 views Jan 26, 2013 Causal dynamical triangulation (CDT) is a lattice model of quantum gravity. In two space-time dimensions (instead of the four we live in) it

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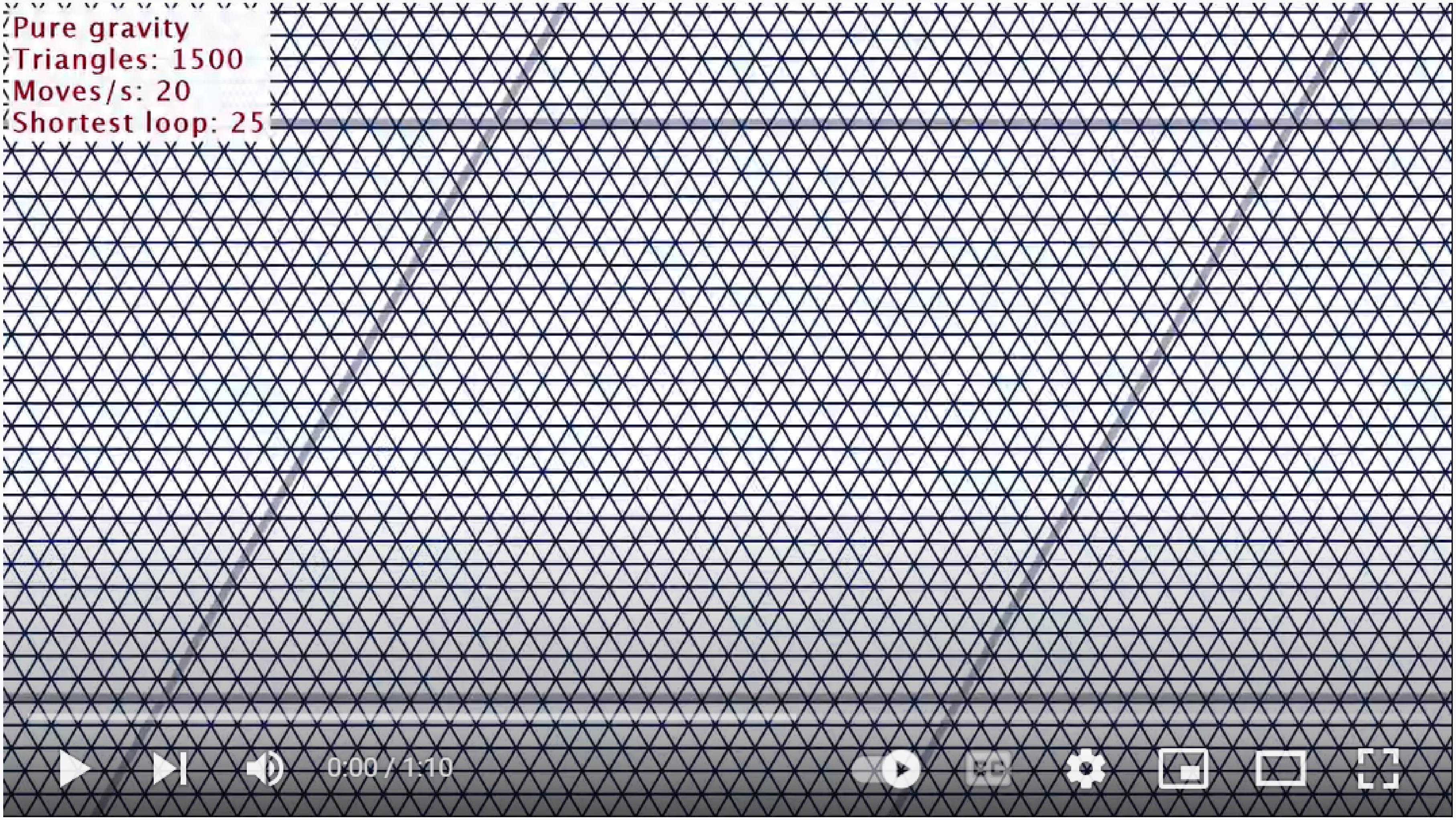

Dynamical triangulation of the 2-torus

1,435 views Sep 7, 2013 This video illustrates a Monte Carlo simulation for two-dimensional quantum gravity on a torus. Starting with a regular triangulation of the torus repeatedly a so-called flip move is performed on a randomly chosen edge. The triangulations obtained after a large

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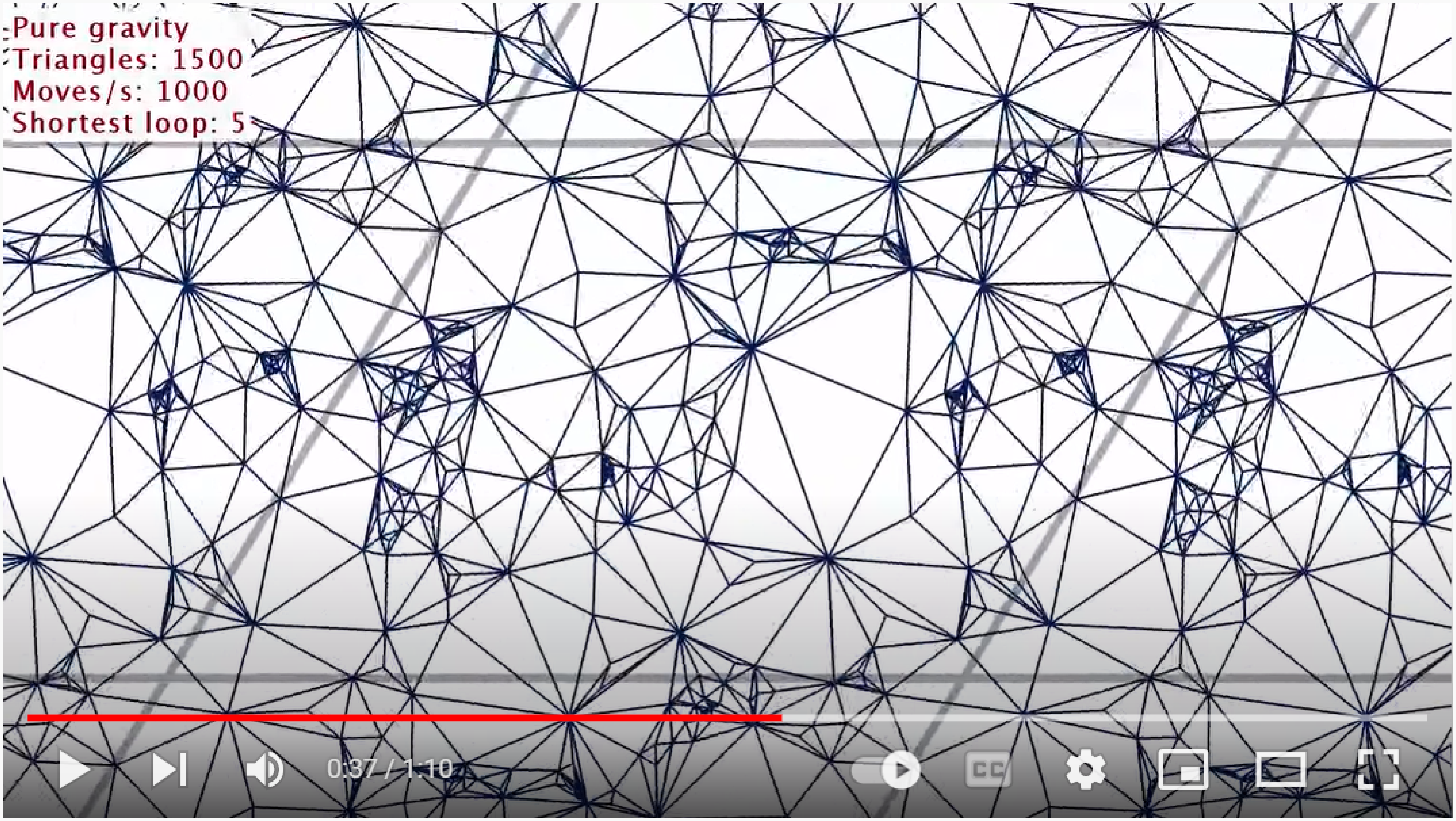

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