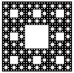
Convergence of diffusions and eigenvalues in rough domains

Alexander Teplyaev

joint research with

Michael Hinz (Bielefeld), Masha Gordina (UConn), Marco Carfagnini (UCSD), Anna Rozanova-Pierrat, Gabriel Claret (Paris-Saclay) et al.



Milan 2024

Abstract:

Dirichlet form analysis gives powerful tools to study diffusions in non-smooth settings, and Mosco convergence is a standard approach to study approximations. However, Mosco convergence may not be sufficient to understand finer properties, such as convergence of eigenvalues and small deviations of diffusion processes. The talk will present two recent results that strengthen Mosco convergence of Dirichlet forms. One result deals with Euclidean extension domains with irregular, or fractal, boundaries (joint work with Michael Hinz and Anna Rozanova-Pierrat). The other result deals with small deviations in sub-Riemannian situations (joint work with Marco Carfagnini and Masha Gordina).

M. Hinz, A. Teplyaev. Closability, regularity, and approximation by graphs for separable bilinear forms.

Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 441 (Veroyatnost i Statistika. 22):299-317, 2015. Springer: J. Math. Sci. (2016) 219 807–820 doi:10.1007/s10958-016-3149-7

We consider a countably generated and uniformly closed algebra of bounded functions. We assume that there is a lower semicontinuous, with respect to the supremum norm, quadratic form and that normal contractions operate in a certain sense. Then we prove that a subspace of the effective domain of the quadratic form is naturally isomorphic to a core of a regular Dirichlet form on a locally compact separable metric space.

We also show that any Dirichlet form on a countably generated measure space can be approximated by essentially discrete Dirichlet forms, i.e. energy forms on finite weighted graphs, in the sense of Mosco convergence, i.e. strong resolvent convergence.

Analysis on fractals and networks, and applications

18 – 22 March, 2024

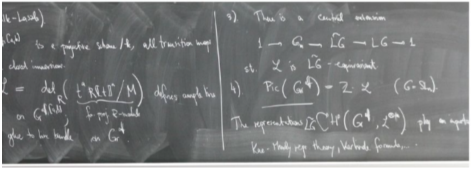
Fractals in Luminy

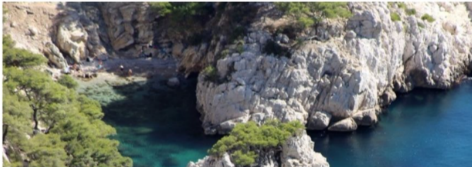
Scientific Committee: Simon N. Chandler-Wilde (University of Reading), Marco Marletta (Cardiff University), Katarzyna Pietruska-Paluba (University of Warsaw), Alexander Teplyaev (University of Connecticut), Martina Zähle (Friedrich Schiller University Jena).

Organizing Committee:
Michael Hinz (Bielefeld University),
Maria Rosaria Lancia (Sapienza University of Rome),
Anna Rozanova-Pierrat (Université Paris-Saclay)

Luminy: Analysis on fractals and networks, and applications March 18-22, 2024

CIRM Luminy – Event N^o 2950





Continuity Equations on Fractals.
Michael Hinz, Waldemar Schefer (Bielefeld University)

In this talk we study first order differential operators on fractals that take func- tions into functions. These operators generalize first order derivatives on p.c.f. fractals introduced by M. Hino as the derivatives of energy finite functions with respect to a minimal energy-dominant reference function. Here we may also allow minimal energy-dominant differential one-forms as reference elements. In general the domains of such first order differential operators are larger than the domain of the underlying Dirichlet form. We prove an integration by parts formula and well-posedness results for continuity equations on fractals. As a key tool we use recent work of W. Arendt, I. Chalendar, R. Eymard on boundary quadruples.

Derivations and Dirichlet forms on fractals.
M. Ionescu, L. G. Rogers, A. Teplyaev, JFA 2012

We study derivations and Fredholm modules on metric spaces with a local regular conservative Dirichlet form. In particular, on finitely ramified fractals, we show that there is a non-trivial Fredholm module if and only if the fractal is not a tree (i.e. not simply connected). This result relates Fredholm modules and topology, and refines and improves known results on p.c.f. fractals. We also discuss weakly summable Fredholm modules and the Dixmier trace in the cases of some finitely and infinitely ramified fractals (including non-self-similar fractals) if the so-called spectral dimension is less than 2. In the finitely ramified self-similar case we relate the p-summability question with estimates of the Lyapunov exponents for harmonic functions and the behavior of the pressure function.

Theorem 5.16 (Non triviality of Fredholm modules for finitely ramified cell structures)

The Fredholm module (H, F) is non trivial, if and only if X is not a tree.

The result stated in [F. Cipriani and J.-L. Sauvageot, 2009] for p.c.f. fractals omitted the distinction between trees and non-trees; in particular, [CS, Proposition 4.2] does not hold for the unit interval, which is a p.c.f. self-similar set, in the sense of Kigami.

Harmonic coordinates on fractals with finitely ramified cell structure. Teplyaev CJM (2008)

Theorem 3. Suppose that all n-harmonic functions are continuous. Then $\mathcal{E}$ is a local regular Dirichlet form (with respect to any measure that charges every nonempty open set).

Proof. The regularity of $\mathcal{E}$ is proved in [J. Kigami, Harmonic analysis for resistance forms. J. Functional Analysis 204 (2003), 399–444.] ...

erratum: my theorem proves locality under assumtion that $\mathcal{E}$ is regular, which was investigated by Kigami, Kumagai et al.

Plan of the talk:

- Mosco convergence, strong and norm resolvent convergence
- Introduction and motivation, analysis on “fractafolds”*
  - Physics motivation
  - Heat Kernel Estimates and Dirichlet Forms
- Wave absorption: numerical shape optimization
- Wave absorption: theoretical shape optimization
- Equations used in architecture
- Wentzell Boundary conditions
  - Theoretical study
  - Discrete approximations
- 1. Convergence of eigenvalues in fractal domains
- 2. Discrete spectrum for Dirichlet forms
- 3. Small deviations
- 4. Convergence of sub-Riemannian eigenvalues?
- 6. Convergence of the Dirichlet heat kernels
- 7. Local convergence of stochastic flows
- New Frontiers: Layer potentials
  - Riemann-Hilbert and Poincare variational problems
  - Hilbert transform
  - Maxwell and other vector equations

Mosco convergence, strong and norm resolvent convergence

- Mosco, Umberto *Convergence of convex sets and of solutions of variational inequalities*. Advances in Math. 3 (1969), 510–585.
- Mosco, Umberto *Composite media and asymptotic Dirichlet forms*. J. Funct. Anal. 123 (1994), no. 2, 368–421.

Kato, Tosio
Perturbation theory for linear operators. Springer-Verlag 1966.

[Reed-Simon 1972]: For non-negative closed quadratic forms,

- **Mosco convergence is equivalent to the strong resolvent convergence**,
- but is **weaker than the norm resolvent convergence**.

Mosco convergence does not imply convergence of the spectrum

$\operatorname*{M-lim}_{n\to\infty} E_n = F$ or $E_n \xrightarrow[n\to\infty]{\mathrm{M}} F$.

- $x_n \in L^2$ converging weakly to $x \in L^2$,
  $\liminf_{n\to\infty} E_n(x_n) \geq F(x)$;
- for each $x \in L^2$ there exists an approximating sequence of elements $x_n \in L^2$, converging strongly to x, such that
  $\limsup_{n\to\infty} E_n(x_n) \leq F(x)$.

Example:
$L^2 := \ell^2(\mathbb{Z}_+)$
$E_n((x_k)) := \sum_{k \geqslant n} |x_k|^2 \xrightarrow[n\to\infty]{\mathrm{M}} E = 0$
$\sigma(E_n) = \{0, 1\} \neq \{0\} = \sigma(E)$

Introduction and motivation, analysis on “fractafolds”*

- *Strichartz: *A fractafold, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold.*
  - *A “fractafold” is to a fractal what a manifold is to a Euclidean half-space.*

This is a part of the broader program to develop probabilistic, spectral and vector analysis on singular spaces by carefully building approximations by graphs or manifolds.

What is the first appearance of fractals is science?

In a sense, the simplest possible fractal appears in the famous Zeno's paradoxes: Zeno of Elea (c. 495 – c. 430 BC) "Achilles and the Tortoise"

1. Achilles runs to the tortoise's starting point while the tortoise walks forward.
2. Achilles advances to where the tortoise was at the end of Step 1 while the tortoise goes yet further.
3. Achilles advances to where the tortoise was at the end of Step 2 while the tortoise goes yet further.
   Etc.

Apparently, Achilles never overtakes the tortoise, since however many steps he completes, the tortoise remains ahead of him.

Dichotomy paradox: that which is in locomotion must arrive at the half-way stage before it arrives at the goal. In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. [Aristotle, Physics VI:9, 239b10, 239b15]

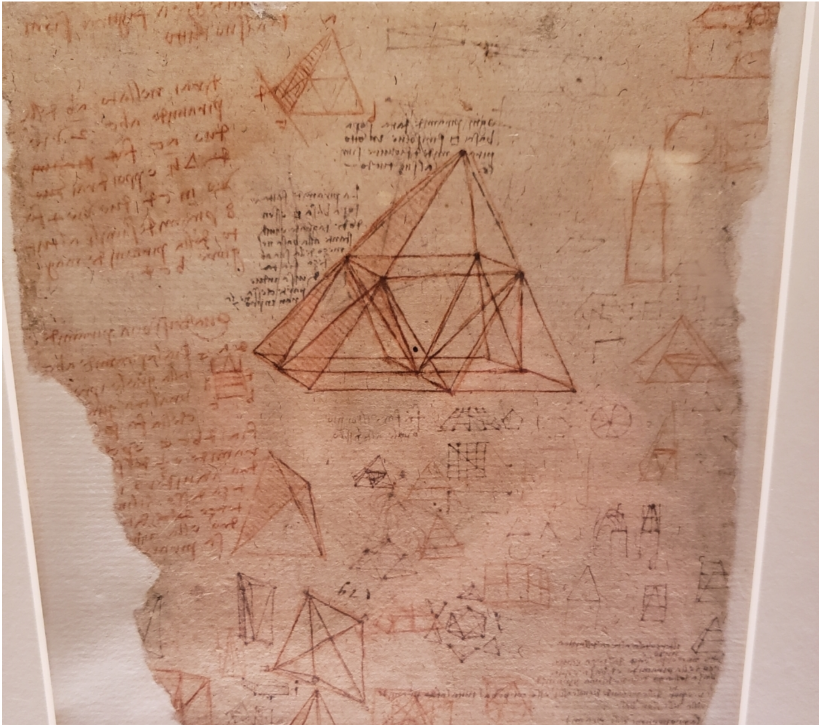
***

In 1821, Augustin-Louis Cauchy proved that, for $-1 < x < 1$,

$$a + ax + ax^2 + ax^3 + \dots = \frac{a}{1-x} := S(a, x)$$

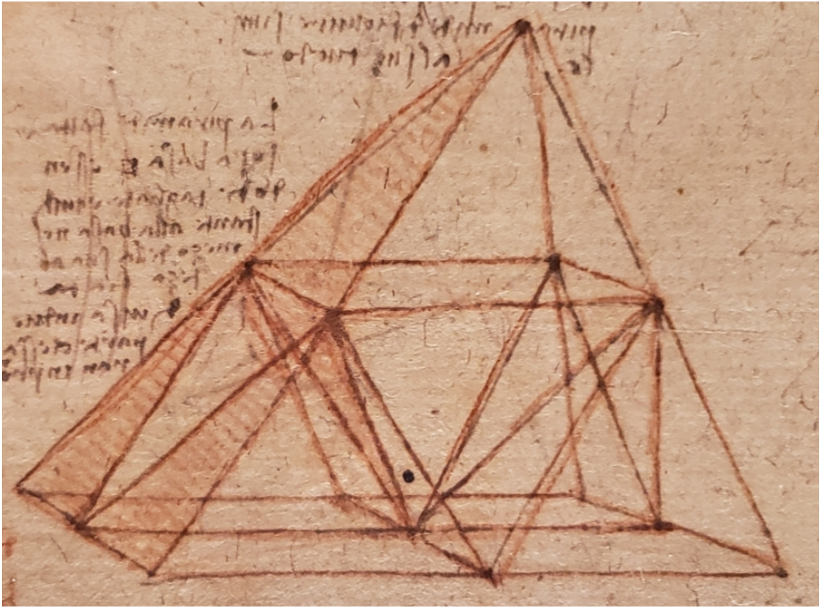
This is a weakly-self-similar sum satisfying a re-normalization “fixed-point” functional equation

$$S(a, x) = a + x \cdot S(a, x)$$



MATHEMATICS AND GEOMETRY: *Decomposition of pyramids*
Red pencil, pen and ink, c. 1515

The sheet shows several diagrams of pyramids broken down into smaller ones. The caption above the major pyramid drawing states that each pyramid with a square base "is resolved into eight pyramids of figures similar to its whole." The same concept is reiterated by the smaller diagram above the larger pyramid. Below, there is another small sketch with a caption explaining how to square a pyramid. Under the base of the main pyramid there is a note that alludes to a German craftsman, while immediately to the side there are grids that could be exercises in perspective.



Cantor, Sierpinski, Julia, Mandelbrot

- How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension (Mandelbrot 1967).

The coastline paradox: the measured length of a stretch of coastline depends on the scale of measurement.

Fractal titanium oxide under inverse 10-ns laser deposition in air and water. A. Pan, W. Wang, X. Mei, Q. Lin, J. Cui, K. Wang, Z. Zhai Applied Physics A volume 123, Article number: 253 (2017)

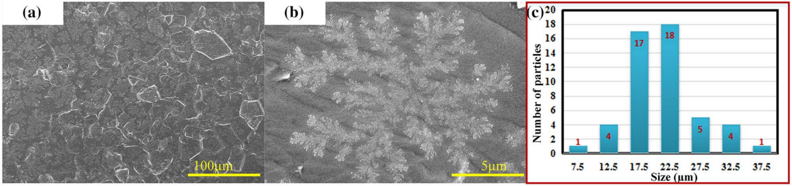


Fig. 5 Surface morphology of titanium with the laser energy of 86 mJ, scanning speed of 0.01 mm/s, and scan length of 10 mm. *Inset* **a** depicts the surface morphology beyond laser irradiation zone. *Inset* **b** shows a typical fractal structure unit, and *inset* **c** is size distribution histograms of 50 fractal structure units

Non-quantized penetration of magnetic field in the vortex state of superconductors

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As first pointed out by Bardeen and Ginzburg in the early sixties[1,2], the amount of magnetic flux carried by vortices in superconducting materials depends on their distance from the sample edge, and can be smaller than one flux quantum, $\phi_0 = h/2e$ (where h is Planck's constant and e is the electronic charge). In bulk superconductors, this reduction of flux becomes negligible at submicrometre distances from the edge, but in thin films the effect may survive much farther into the material[3,4]. But the effect has

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Superconducting disk with magnetic coating: Re-entrant Meissner phase, novel critical and vortex phenomena

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PACS 74.78.Na – Mesoscopic and nanoscale systems
PACS 74.25.Op – Mixed state, critical fields, and surface sheaths

Abstract – Within the Ginzburg-Landau formalism, we study the mixed state of a superconducting disk surrounded by a magnetic ring. The stray field of the magnet, concentrated at the rim of the superconducting disk, favors ring-like arrangement of induced vortices, to the point that even a *single vortex state exhibits asymmetry.* A novel route for the destruction of superconductivity with increasing magnetization of the magnetic coating is found: first all vortices leave the sample, and are replaced by a *re-entered Meissner phase* with a full depression of the order-parameter at the sample edge; subsequently, superconductivity is then gradually suppressed from the edge inwards, *contrary to the well-known surface superconductivity.* When exposed to an additional homogeneous magnetic field, we find a *field-polarity–dependent vortex structure* in our sample —for all vorticities, only giant- *or* multi-vortex states are found for given polarity of the external field. In large samples, the *number of vortex shells and number of flux quanta in each of them can be controlled* by the parameters of the magnetic coating.

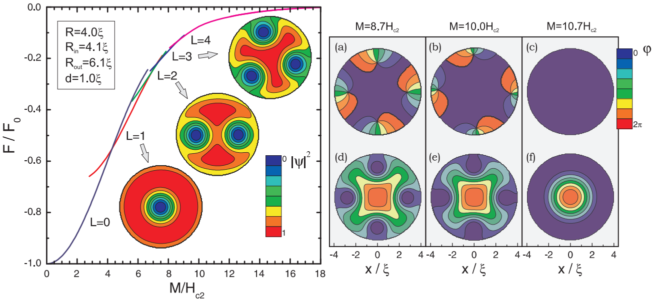

Fig. 2: The free energy of the states with different vorticity L as a function of the magnetization of the magnetic coating. Insets show the Cooper-pair density contourplots of the corresponding states. (a-c) Superconducting phase and (d-f) $|\psi|^2$-density plots, illustrate simultaneous vortex exit and suppression of superconductivity at the rim of the superconducting disk for high magnetization.

=

In our theoretical treatment of this system, we use the non-linear Ginzburg-Landau (GL) formalism, combined with Neumann boundary conditions (zero current perpetrating the boundary). To investigate the superconducting state of a sample with volume V, we minimize, with respect to the order parameter ψ, the GL free energy

$$\mathcal{F} = \int \frac{\mathrm{d}v}{V} \left(|(-i\vec{\nabla} - \vec{A}_H - \vec{A}_m)\psi|^2 - |\psi|^2 + \frac{1}{2}|\psi|^4 \right), \tag{2}$$

Minimization of eq. (2) leads to equations for the order parameter and superconducting current

$$(-i\vec{\nabla} - \vec{A})^2\psi = (1 - |\psi|^2)\psi, \tag{3}$$

$$\vec{j} = \Im(\psi^*\vec{\nabla}\psi) - |\psi|^2\vec{A}, \tag{4}$$

which we solve following a numerical approach proposed by Schweigert *et al.* (see ref. [2]) on a uniform Cartesian grid with typically 10 points/ξ in each direction. We then start from randomly generated initial distribution of ψ, increase/decrease the magnetization of the magnet or change the value of the applied external field, and let eq. (3) relax to its steady-state solution. In addition, we always recalculate the vortex structure starting from the pure Meissner state[1] ($\psi = 1$) or the normal state ($\psi \approx 0$) as initial condition. All stable states are then collected and their energies are compared to find the ground state configuration.

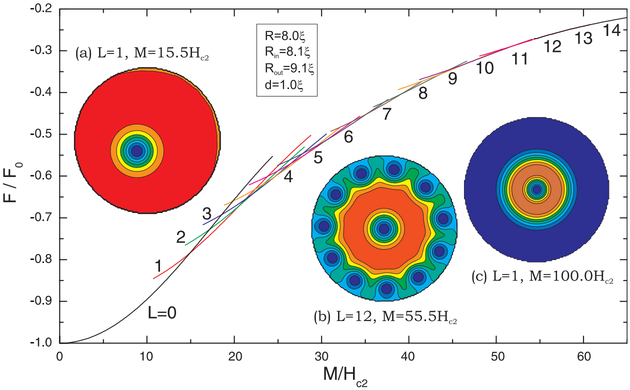

Fig. 3: Free energy diagram for a large superconducting disk with thin magnetic coating. Insets show the $|\psi|^2$-density plots of distinct vortex states.

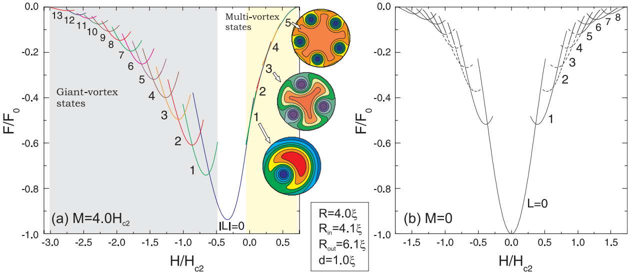

Fig. 4: (a) Free energy of a superconducting disk with magnetic coating as a function of applied homogeneous magnetic field. Insets show the Cooper-pair density plots for indicated states. (b) Same as (a), but for demagnetized coating. In (b), dashed lines denote multi-vortex and solid lines giant-vortex configurations.

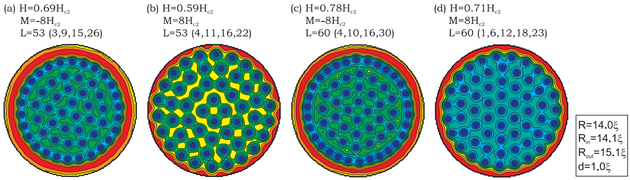

Fig. 5: The $|\psi|^2$-density plots illustrating the arrangement of vortex shells in a large superconducting disk for $L = 53$ and $L = 60$, with magnetic coating with (a,c) negative ($M = -8H_{c2}$), or (b,d) positive ($M = 8H_{c2}$) magnetization.

GEOMETRICAL DESCRIPTION OF VORTICES IN GINZBURG-LANDAU BILLIARDS

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4.2 The Bogomol'nyi identities

For the special value $\kappa = \frac{1}{\sqrt{2}}$, the equations for ψ and $\vec{A}$ can be reduced to first order differential equations. This special point was first used by Sarma [41] in his discussion of type-I *vs.* type-II superconductors and then identified by Bogomol'nyi [40] in the more general context of stability and integrability of classical solutions of some quantum field theories. This special point is also called a duality point. We first review some properties of the Ginzburg-Landau free energy at the duality point. We use the following identity true for two dimensional systems

$$|(\vec{\nabla} - i\vec{A})\psi|^2 = |\mathcal{D}\psi|^2 + \vec{\nabla} \times \vec{\jmath} + B|\psi|^2 \tag{64}$$

where $\vec{\jmath} = \mathrm{Im}(\psi^* \vec{\nabla}\psi) - |\psi|^2 \vec{A}$ is the current density and the operator $\mathcal{D}$ is defined as $\mathcal{D} = \partial_x + i\partial_y - i(A_x + iA_y)$. This relation is a relative of the Weitzenböck formula (61). At the duality point $\kappa = \frac{1}{\sqrt{2}}$ the expression (63) for $\mathcal{F}$ can be rewritten using (64) as

$$\mathcal{F} = \int_\Omega \left(\frac{1}{2}|B - 1 + |\psi|^2|^2 + |\mathcal{D}\psi|^2 \right) + \oint_{\partial\Omega} (\vec{\jmath} + \vec{A}).\vec{\mathrm{d}l} \tag{65}$$

Physics motivation

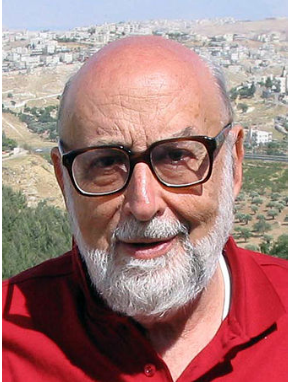
- R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983)
- R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984)
- E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984)
- Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983)**17** (1984)

François Englert

From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃glɛʁ]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research,

François Englert



François Englert in Israel, 2007

Nuclear Physics B280 [FS 18] (1987) 147–180
North-Holland, Amsterdam

METRIC SPACE-TIME AS FIXED POINT OF THE RENORMALIZATION GROUP EQUATIONS ON FRACTAL STRUCTURES

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Ph. SPINDEL

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Received 19 February 1986

We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.

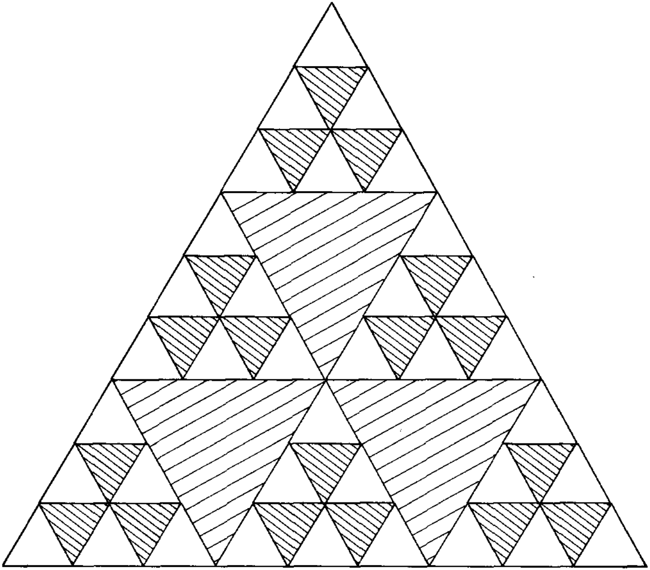

Fig. 1. The first two iterations of a 2-dimensional 3-fractal.

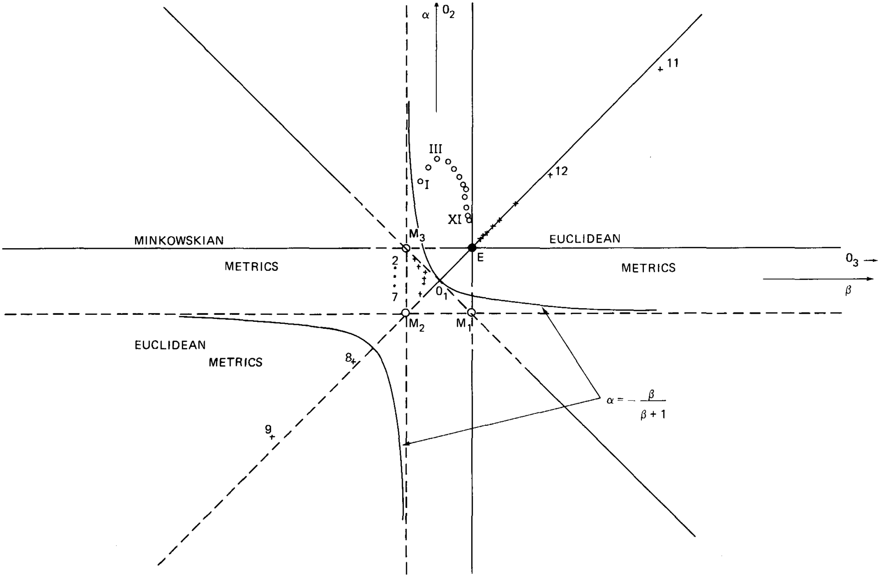

Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta+1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. M_1, M_2, M_3 denote unstable minkowskian fixed geometries while E corresponds to the stable euclidean fixed point. The unstable fixed points 0_1, 0_2 and 0_3 associated to 0-dimensional geometries are located at the origin and at infinity on the (α, β) coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4$, $\beta = -52.5$) is outside the frame of the figure.

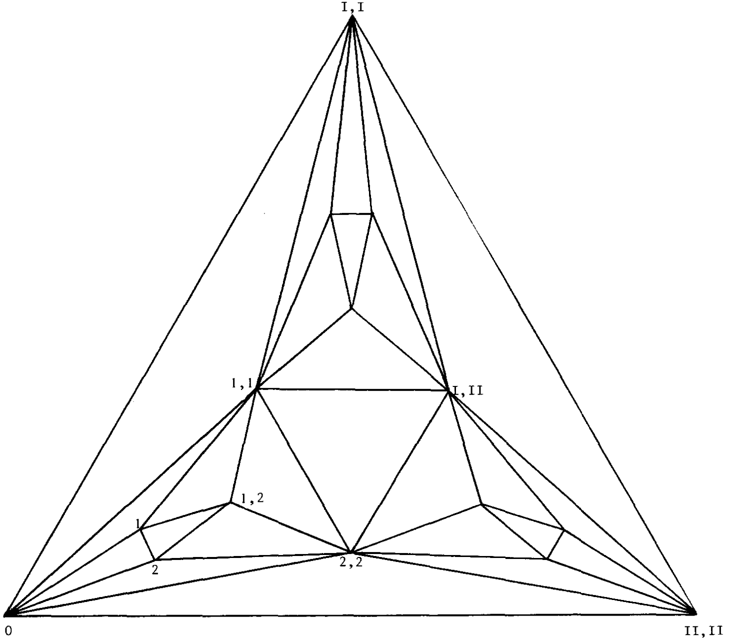

Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.

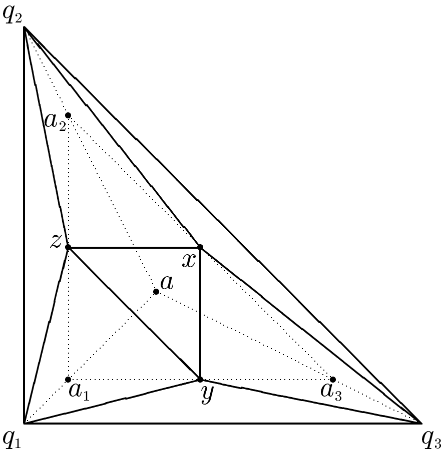

Figure 6.4. Geometric interpretation of Proposition 6.1.

PRL **95,** 171301 (2005) PHYSICAL REVIEW LETTERS week ending 21 OCTOBER 2005

The Spectral Dimension of the Universe is Scale Dependent

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(Received 13 May 2005; published 20 October 2005)

We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be "self-renormalizing" at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

DOI: 10.1103/PhysRevLett.95.171301 PACS numbers: 04.60.Gw, 04.60.Nc, 98.80.Qc

Quantum gravity as an ultraviolet regulator?—A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.

tral dimension, a diffeomorphism-invariant quantity obtained from studying diffusion on the quantum ensemble of geometries. On large scales and within measuring accuracy, it is equal to four, in agreement with earlier measurements of the large-scale dimensionality based on the other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \rightarrow 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25, \qquad (15)$$

and thus is compatible with the integer value two.

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Fractal space-times under the microscope: a renormalization group view on Monte Carlo data

JHEP12(2011)012

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension d_s and walk dimension d_w associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2+d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects

Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data (Martin Reuter, Frank Saueressig):

Three scaling regimes of the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG):

1. a classical regime $d_s = d, d_w = 2$,
2. a semi-classical regime $d_s = 2d/(2+d),\ d_w = 2+d$,
3. the UV-fixed point regime $d_s = d/2,\ d_w = 4$.

On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is in very good agreement with the data and provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

- Mathav Murugan: $d_w = d_f$ consistent with $d_s = 2d_f/d_w = 2$
- Growth and percolation on the uniform infinite planar triangulation by Omer Angel (GAFA 2003)
- Anomalous diffusion of random walk on random planar maps by Ewain Gwynne and Tom Hutchcroft (PTRF 2020)

Heat Kernel Estimates and Dirichlet Forms

$$p_t(x,y) \sim \frac{1}{t^{d_f/d_w}} \exp\left(-c\frac{d(x,y)^{\frac{d_w}{d_w-1}}}{t^{\frac{1}{d_w-1}}}\right)$$

$$distance \sim (time)^{\frac{1}{d_w}}$$

$$d_f = \text{Hausdorff dimension}$$

$$\frac{1}{\gamma} = d_w = \text{“walk dimension” } (\gamma\text{=diffusion index})$$

$$\frac{2d_f}{d_w} = d_s = \text{“spectral dimension” (diffusion dimension)}$$

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980’—)

Stability Theorem (Barlow, Bass, Kumagai (2006))

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian ***heat kernel estimates are stable under rough isometries***, *i.e. under maps that preserve distance and energy up to scalar factors.*

Gromov-Hausdorff + energy

Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any generalized Sierpiński carpet there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there is a unique symmetric Markov process and **a unique Laplacian**.

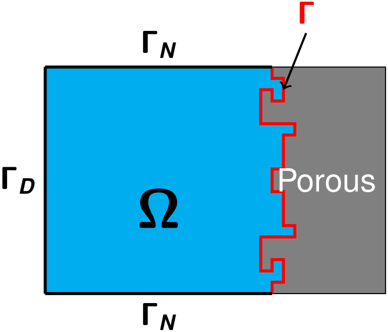
Moreover, the Markov process is strong Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties: If it is not a cube in $\mathbb{R}^n$, then

- $d_S < d_f$, $d_w > 2$
- the energy measure and the Hausdorff measure are mutually singular;
- the domain of the Laplacian is not an algebra;
- if $d(x, y)$ is the shortest path metric, then $d(x, \cdot)$ is not in the domain of the Dirichlet form (not of finite energy) and so methods of Differential geometry seem to be not applicable;
- Lipschitz functions are not of finite energy;
- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.

Wave absorption: numerical shape optimization

- F. Magoulès, T.P. Kieu Nguyen, P. Omnes, A. Rozanova-Pierrat, Optimal absorption of acoustic waves by a boundary. SIAM J. Control Optimization 59 (2021)
  + more numerical results
- C. Bardos, D. Grebenkov, A. Rozanova-Pierrat, Short-time heat diffusion in compact domains with discontinuous transmission boundary conditions. Math. Mod. Meth. Appl. Sci. 26 (2016)
- A. Rozanova-Pierrat, D. S. Grebenkov, and B. Sapoval, Faster diffusion across an irregular boundary. Phys. Rev. Lett. 108 (2012)


Γ
Γ_N
Γ_D
Ω
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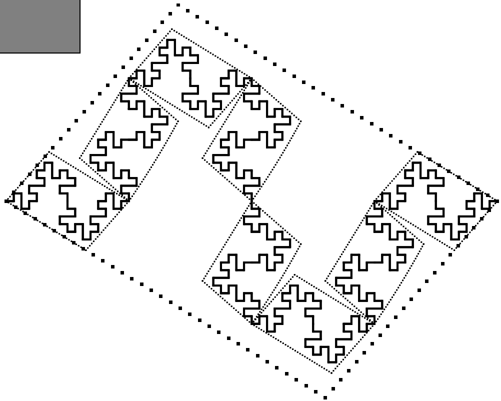
Γ_N



Wave absorption: theoretical shape optimization

- M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, *Non-Lipschitz uniform domain shape optimization in linear acoustics*. SIAM J. Control Optim. 59 (2021)

- M. Hinz, A. Rozanova-Pierrat, A. Teplyaev, *Boundary value problems on non-Lipschitz uniform domains: Stability, compactness and the existence of optimal shapes*. Asymptotic Analysis (2023)

Equations used in architecture

- M. Hinz, F. Magoulès, A. Rozanova-Pierrat, M. Rynkovskaya, A. Teplyaev, *On the existence of optimal shapes in architecture*. Applied Mathematical Modelling 94 (2021)

Given a domain $\Omega \subset \mathbb{R}^N$ and a vector field $\mathbf{v} \in W^{1,2}(\Omega)^N$ we denote the symmetric part of its gradient by

$$e(\mathbf{v}) = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^t\right).$$

Let $A \in L^\infty(\Omega, \mathcal{M}_N^s(\alpha, \beta))$ and write $\sigma(\mathbf{v}) = Ae(\mathbf{v})$, $\mathbf{v} \in W^{1,2}(\Omega)^N$.
We are interested in solutions $\mathbf{u} \in W^{1,2}(\Omega)^N$ of BVP:

$$\begin{cases} -\mathrm{div}\,\sigma(\mathbf{u}) & = \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} & = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{Dir}}, \\ \sigma(\mathbf{u}) \cdot n & = \mathbf{g} \quad \text{on } \Gamma_{\mathrm{Neu}}. \end{cases} \tag{1}$$

Wentzell Boundary conditions

- A. Wentzell. On boundary conditions for multi-dimensional diffusion processes. Theor. Probability Appl. (1959)

$$\mathcal{E}(u) = \int_{\Omega} \|\nabla u\|^2 dx + \mathcal{E}_{\partial\Omega}(u)$$

Theoretical study

- M. R. Lancia, P. Vernole,
  Venttsel' problems in fractal domains
  J. Evol. Equ. 14 (2014), no. 3, 681–712.

  ...

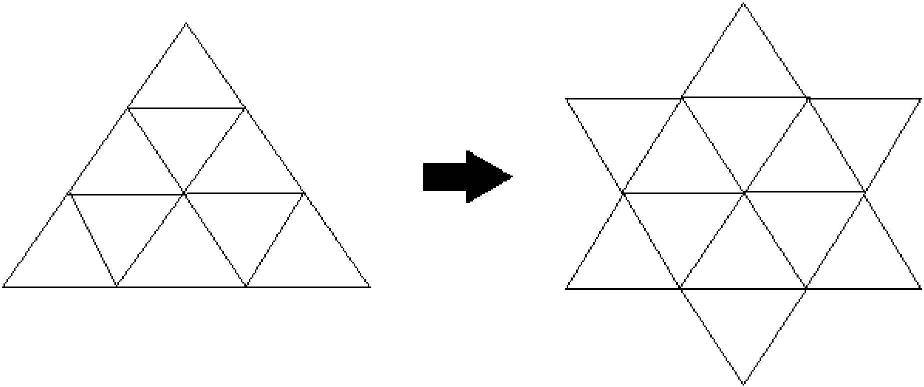
- M. Hinz, M. R. Lancia, A. Teplyaev, P. Vernole, Fractal snowflake domain diffusion with boundary and interior drifts, J. Math. Anal. Appl. 457 (2018)

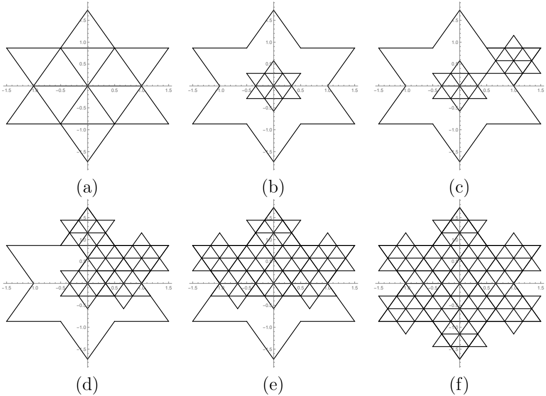
$$\mathcal{E}(u) = \int_{\Omega} \|\nabla u\|^2 dx + \mathcal{E}_{\partial\Omega}(u)$$

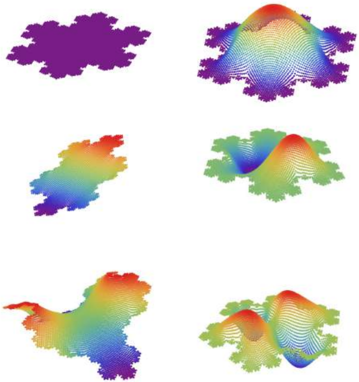
Discrete approximations

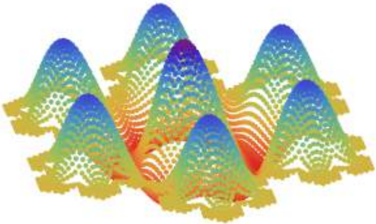
- M. Gabbard, C. Lima, G. Mograby, L. G. Rogers, A. Teplyaev, Discretization of the Koch Snowflake Domain with Boundary and Interior Energies, SEMA SIMAI Springer Series ICIAM2019 Fractals in engineering: Theoretical aspects and Numerical approximations (2021)

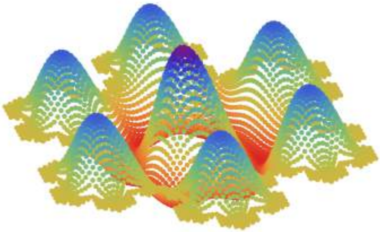
$$\mathcal{E}(u) = \int_{\Omega} ||\nabla u||^2 dx + \mathcal{E}_{\partial\Omega}(u)$$




(a)
(b)
(c)
(d)
(e)
(f)







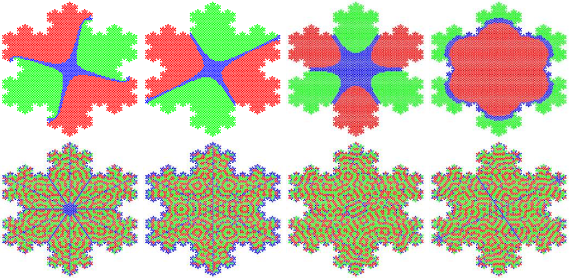


FIGURE 5. Contour Plots of the Eigenvectors of L_n corresponding to eigenvalues λ: (a) 4th eigenvector, $\lambda = 48.1$. (b) 5th eigenvector, $\lambda = 48.1$. (c) 6th eigenvector, $\lambda = 85.1$. (d) 8th eigenvector $\lambda = 125.4$. (e) 1153rd eigenvector $\lambda = 49965.7$. (f) 1157th eigenvector $\lambda = 50156.6$. (g) 1161st eigenvector, $\lambda = 50188.8$ and (h) 1162nd eigenvector, $\lambda = 50188.83$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in (ϵ, ∞) and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$. (Level 4 graph approximation)

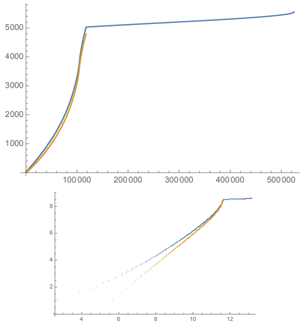


FIGURE 6. (Upper) Eigenvalue counting functions of Dirichlet Laplacian (orange) and L_n (blue). (Lower) Log-Log plot of the eigenvalue counting functions of Dirichlet Laplacian (orange) and L_n (blue) (Level 4 graph approximation).

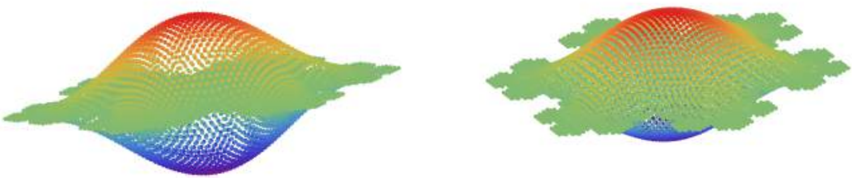


FIGURE 7. (a) The 5,028th eigenvector of L_n, $\lambda = 118038.02$. (b) The last Dirichlet eigenvector, $\lambda = 118039.37$. The oval-shaped graph is due to a high oscillation of both eigenvectors

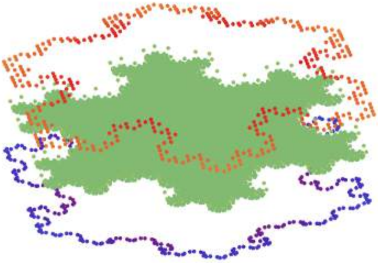


FIGURE 8. The last L_n eigenvector, $\lambda = 524999.69$. The graph splits into two parts, above and below the Koch snowflake domain due to a high oscillation (Level 4 graph approximation).

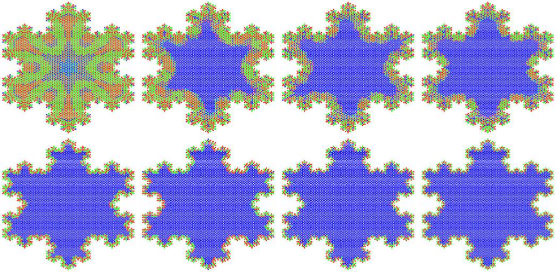
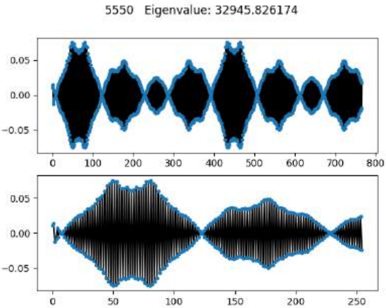
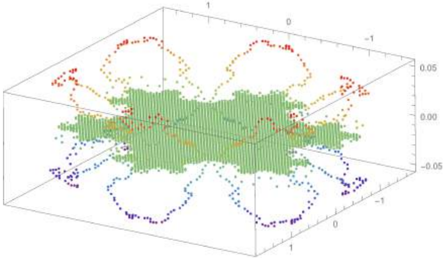


FIGURE 9. L_n eigenvectors localization with eigenvalues λ: (a) 5030th eigenvector, $\lambda = 118048.66$. (b) 5031th eigenvector, $\lambda = 119678.65$. (c) 5032th eigenvector, $\lambda = 119678.65$. (d) 5033th eigenvector, $\lambda = 121460.72$. (e) 5100th eigenvector, $\lambda = 185367.41$. (f) 5200th eigenvector, $\lambda = 291364.38$. (g) 5300th eigenvector, $\lambda = 392584.97$. (h) 5557th eigenvector, $\lambda = 524999.69$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in (ϵ, ∞) and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$ (Level 4 graph approximation).



5550 Eigenvalue: 32945.826174
0.05
0.00
−0.05
0
100
200
300
400
500
600
700
800
0.05
0.00
−0.05
0
50
100
150
200
250



1
0
−1
0.05
0.00
−0.05
−1
0
1

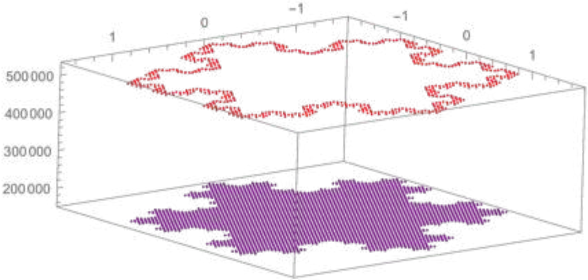


FIGURE 10. The high frequency landscape vector attains just the following two values the boundary vertices 527360 and 524288. It is constant on the interior vertices with the value 157464. (Level 4 graph approximation)

1. Convergence of eigenvalues in fractal domains

Theorem (Hinz, Rozanova-Pierrat, T.)

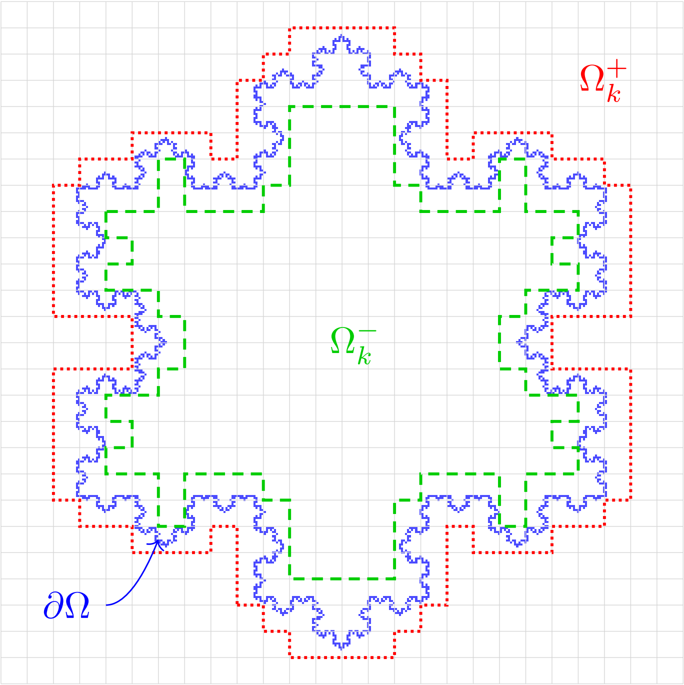
Let n, D, α, γ, ε, d, $(\Omega_m)_m$ *and* $(\mu_m)_m$ *be a* **a sequence of admissible domains**. *Suppose that* $\lim_m \Omega_m = \Omega$ *in the Hausdorff sense and in the sense of characteristic functions and* $\lim_m \mu_m = \mu$ *weakly. There is a sequence* $(m_k)_{k=1}^{\infty}$ *with* $m_k \uparrow +\infty$ *such that the following hold.*

(i) *We have* $\lim_{k\to\infty} P_{\Omega_{m_k}} \circ \hat{G}_{\alpha,\gamma}^{\Omega_{m_k},\mu_{m_k},*} = P_\Omega \circ \hat{G}_{\alpha,\gamma}^{\Omega,\mu,*}$ *in operator norm.*

(ii) *If* $0 < a < b$ *are in the resolvent set of* $-\mathcal{L}_\gamma^{\Omega,\mu,*}$, *then* $\lim_{k\to\infty} \pi_{(a,b)}(\Omega_{m_k}, \mu_{m_k}, *) = \pi_{(a,b)}(\Omega, \mu, *)$ *in operator norm.*

(iii) *The spectra of the operators* $-\mathcal{L}_\gamma^{\Omega_{m_k},\mu_{m_k},*}$ *converge to the spectrum of* $-\mathcal{L}_\gamma^{\Omega,\mu,*}$ *in the Hausdorff sense. The eigenvalues* $\lambda_n(\Omega, \mu, *)$ *of the operator* $-\mathcal{L}_\gamma^{\Omega,\mu,*}$ *are exactly the limits as* $k \to \infty$ *of sequences of the eigenvalues of the operators* $-\mathcal{L}_\gamma^{\Omega_{m_k},\mu_{m_k},*}$,

$$\lambda_n(\Omega, \mu, *) = \lim_{k\to\infty} \lambda_n(\Omega_{m_k}, \mu_{m_k}, *). \tag{2}$$



Ω_k^+
Ω_k^-
$\partial\Omega$

2. Discrete spectrum for Dirichlet forms

Theorem (Carfagnini, Gordina, T.)

Let $\mathcal{U}$ be an open bounded subset of $\mathcal{X}$, and $P_t^{\mathcal{U}}$ be the semigroup associated to $(\mathcal{E}, \mathcal{D}_{\mathcal{E}})$ with the infinitesimal generator $A_{\mathcal{U}}$. Assume that $p_t(x, y)$ exists for all t and for m-a.e. $x, y \in \mathcal{X}$. If there exists a $t_{\mathcal{U}} > 0$ such that

$$M_{\mathcal{U}}(t_{\mathcal{U}}) = \operatorname*{ess\,sup}_{(x,y)\in\mathcal{U}\times\mathcal{U}} p_{t_{\mathcal{U}}}(x, y) < \frac{1}{m(\mathcal{U})^2}, \tag{3}$$

then the spectrum of $-A_{\mathcal{U}}$ is discrete and $\lambda_1 > 0$.

3. Small deviations

Theorem (Carfagnini, Gordina, T.)

Let $\{P_t\}_{t\geqslant 0}$ be a strongly continuous contraction semigroup on $L^2(\mathcal{X}, m)$. Let $x \in \mathcal{X}$ and assume that $P_t^{B_1(x)}$ is irreducible. Assume that the heat kernel $p_t(x, y)$ exists for all t and for all $x, y \in \mathcal{X}$ and that

$$p_t(x, y) \leqslant c\, t^{-\frac{\alpha}{\beta}}$$

for any t, x, and y. Moreover, assume that there exists a t_0 such that $p_{t_0}(x, y)$ is continuous for $x, y \in \mathcal{X}$. If X_t^x is **self-similar** *then*

$$\lim_{\varepsilon \to 0} e^{\lambda_1 \frac{t}{\varepsilon^\beta}} \mathbb{P}^x \left(\sup_{0 \leqslant s \leqslant t} d(X_s, x) < \varepsilon \right) = c_1 \varphi_1(x), \tag{4}$$

where $\lambda_1 > 0$ is the spectral gap of A restricted to the unit ball $B_1(x)$, and φ_1 is the corresponding positive eigenfunction.

4. Convergence of sub-Riemannian eigenvalues?

- M. Carfagnini, M. Gordina: * *Spectral gap bounds on H-type groups*, 14 pp. * *On the Onsager-Machlup functional for the Brownian motion on the Heisenberg group*, 24 pp. * *Dirichlet sub-Laplacians on homogeneous Carnot groups: spectral properties, asymptotics, and heat content*, 30 pp., IMRN, 2023. * *Small deviations and Chung's law of iterated logarithm for a hypoelliptic Brownian motion on the Heisenberg group*, 24 pp., Trans. AMS, 2022.
- N. Eldredge, M. Gordina, E. Le Donne, S. Li, *Notions of null sets in infinite-dimensional Carnot groups*, 37 pp.
- M. Gordina, T. Melcher, J. Wang, *Large deviations principle for sub-Riemannian random walks*, 40 pp.
- L. Gao, M. Gordina, *Complete modified logarithmic Sobolev inequality for sub-Laplacian on* $\boldsymbol{SU(2)}$, 23 pp.
- M. Gordina, L. Luo, Logarithmic *Sobolev inequalities on non-isotropic Heisenberg groups*, 30 pp., JFA, 2022.
- M. Gordina, M. Röckner, A. Teplyaev, *Singular perturbations of Ornstein-Uhlenbeck processes: integral estimates and Girsanov densities*, 24 pp., PTRF, 2020.
- N. Eldredge, M. Gordina, L. Saloff-Coste, *Left-invariant geometries on* $\boldsymbol{SU(2)}$ *are uniformly doubling*, GAFA, 2018, 28,

5. Convergence of the re-normalized eigenvalues of small balls in $\mathbf{SU(2)}$ to corresponding eigenvalues in the unit ball of $\mathbb{H}$

Here $\mathbb{H}$ is the Heisenberg group, which is a re-scaled limit of $\mathbf{SU(2)}$ near the identity.

Theorem (Carfagnini, Gordina, T.)

Let $0 < \lambda_1^{\mathbb{H}} < \lambda_2^{\mathbb{H}} \leqslant \lambda_3^{\mathbb{H}} \leqslant \dots$ be the Dirichlet eigenvalues in the unit ball $B_1^{\mathbb{H}}$ of $\mathbb{H}$, counted with multiplicity. Let $0 < \lambda_1^r < \lambda_2^r \leqslant \lambda_3^r \leqslant \dots$ be the Dirichlet eigenvalues in the r-ball $B_r^{\mathbf{SU(2)}}$ of $\mathbf{SU(2)}$, counted with multiplicity. Then for each $n \geqslant 1$ we have

$$\lim_{r \to 0} r^2 \lambda_n^r = \lambda_n^{\mathbb{H}}. \tag{5}$$

6. Convergence of the Dirichlet heat kernels

Let $p_t^{B_1^{\mathbb{H}}}(\cdot,\cdot)$ be the Dirichlet heat kernel in the unit ball $B_1^{\mathbb{H}}$ of $\mathbb{H}$, and $p_t^{B_r^{\mathrm{SU}(2)}}(\cdot,\cdot)$ be the Dirichlet heat kernel in the r-ball $B_r^{\mathrm{SU}(2)}$ of $\mathrm{SU}(2)$, where the balls are centered at the identity of the groups.

Theorem (Carfagnini, Gordina, T.)

For each $t > 0$

$$\lim_{r\to 0} r^4\, p_{r^2 t}^{B_r^{\mathrm{SU}(2)}}\left(\Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right), \Phi^{-1}\left(\delta_r^{\mathbb{H}}(x)\right)\right) = p_t^{B_1^{\mathbb{H}}}(x,y). \tag{6}$$

uniformly for $x, y \in B_1^{\mathbb{H}}$.

7. Local convergence of stochastic flows

Let

$$g_s^{B_r^{\mathrm{SU}(2)}} := \begin{cases} g_s & s < \tau_{B_r^{\mathrm{SU}(2)}} \\ \partial & s \geqslant \tau_{B_r^{\mathrm{SU}(2)}} \end{cases} \tag{7}$$

where g_s denotes a hypoelliptic Brownian motion on $\mathbf{SU(2)}$, and

$$\tau_{B_r^{\mathrm{SU}(2)}} := \inf\left\{ s > 0,\ g_s \notin B_r^{\mathrm{SU}(2)} \right\}. \tag{8}$$

Similarly, let

$$X_s^{B_r^{\mathbb{H}}} := \begin{cases} X_s & s < \tau_{B_r^{\mathbb{H}}} \\ \partial & s \geqslant \tau_{B_r^{\mathbb{H}}} \end{cases} \tag{9}$$

where X_s denotes a hypoelliptic Brownian motion on $\mathbb{H}$, and

$$\tau_{B_r^{\mathbb{H}}} := \inf\left\{ s > 0,\ X_s \notin B_r^{\mathbb{H}} \right\}. \tag{10}$$

Theorem (Carfagnini, Gordina, T.)

For any $0 < r < \frac{1}{7} r_{1/7}$ *there is a continuous stochastic process* Y_s^r *in* $\mathbb{H}$ *such that*

$$Y_s^r :=: \delta^{\mathbb{H}}_{1/r} \Phi \left(g^{B_{3r}^{\mathrm{SU}(2)}}_{r^2 s} \right) \tag{11}$$

in the sense of conditional probability distributions on the event $A_{3r} := \{ s < \tau_{B_{3r}^{\mathbb{H}}} \}$ *and*

$$\lim_{r \to 0} \mathbb{1}_{A_{3r}} \sup_{0 \leqslant s \leqslant T} |Y_s^r - X_s| = 0 \tag{12}$$

in probability.

We use Theorem 3.3.1, page 76, in Kunita 1986 Lectures on stochastic flows and applications, Tata Institute of Fundamental Research Lectures on Mathematics and Physics.

New Frontiers: Layer potentials

$$u(x) = \int_S \rho(y) \frac{\partial}{\partial \nu} P(x, y) d\sigma(y)$$

$$v(x) = G * f = \int_{\mathbb{R}^n} g(x, y) d\mu(y)$$

Riemann-Hilbert and Poincare variational problems

Find a function in $\mathbb{C}$, unanlytic outside of a curve, with prescribed values and jumps on the curve.
Research in progress: Anna Rozanova-Pierrat, Gabriel Claret (CentraleSupélec), Michael Hinz (Bielefeld).

Calssical applications:

- Integrable models, inverse scattering or inverse spectral problem
- the inverse monodromy problem for Painlevé equations
- Orthogonal polynomials, Random matrices
- Combinatorial probability
- Algebraic geometry, Donaldson–Thomas theory

Hilbert transform

$$H(u)(t) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{u(\tau)}{(t-\tau)} d\tau$$

Research in progress: Anna Rozanova-Pierrat, Gabriel Claret (CentraleSupélec), Michael Hinz (Bielefeld).

Closely connected to the Riemann-Hilbert and Poincare variational problems and is extensively used in analysis and in sygnal processing.

Maxwell and other vector equations

We develop new mathematical tools in the vector case in order to study and solve Maxwell's equations in non-Lipschitz, possibly fractal domains. To that extent, we would like to show here one use of those tools with the time-harmonic Maxwell problem completed with a homogeneous Dirichlet boundary condition, which becomes with our notations:

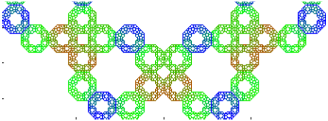
$$\begin{cases} \mathbf{curl}(\mu^{-1}\mathbf{curl}\ \mathbf{E}) - \omega^2\varepsilon\mathbf{E} = \mathbf{f} & \text{on } \Omega \\ \mathbf{Tr}_T(\mathbf{E}) = 0 & \text{on } \partial\Omega \end{cases}$$

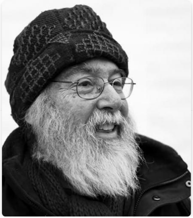
where $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and we look for $\mathbf{E} \in \mathbf{H}(\mathbf{curl}, \Omega)$.
This problem is equivalent to the following variational formulation:
Find $\boldsymbol{E} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$ *such that* $\forall \boldsymbol{F} \in \boldsymbol{H}_0(\boldsymbol{curl}, \Omega)$*:*

$$(\mu^{-1}\mathbf{curl}\ \mathbf{E}, \mathbf{curl}\ \mathbf{F}) - \omega^2(\varepsilon\mathbf{E}, \mathbf{F}) = (\mathbf{f}, \mathbf{F}).$$

Research in progress: Anna Rozanova-Pierrat (CentraleSupélec), Patrick Ciarlet (ENSTA Paris) et al.

7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 4–8, 2022





In Memory of Professor Robert Strichartz

We will be dedicating the entire conference to Professor Strichartz. A special session will be scheduled during the conference for all to attend and reflect on their thoughts and memories of Bob. Bob is appreciated and recognized for his organizing of the Fractals Conference Community in 2002. He will be profoundly missed by family, friends, colleagues, and most of all, the students he mentored and influenced throughout his career.

A message from the Cornell Department of Mathematics Chair, Tara Holm:

Dear friends,

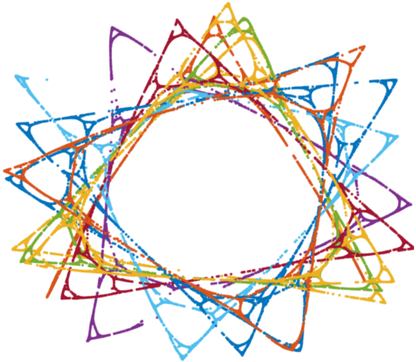
I am sad to share that our colleague and friend Professor Robert Strichartz died yesterday, 19 December 2021, after a long illness. He was 78.

UCONN
UNIVERSITY OF CONNECTICUT

8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 2025

Everybody is invited !

Thank you for your attention!



Picture: the ***Sierpinski-flower***

UConn REU 2023: Fractal Eigenmaps

Bernard Akwei, Rachel Bailey, Luke Rogers et al.