Convergence of diffusions and eigenvalues in rough domains

Alexander Teplyaev

joint research with

Michael Hinz (Bielefeld), Masha Gordina (UConn), Marco Carfagnini (UCSD), Anna Rozanova-Pierrat, Gabriel Claret (Paris-Saclay) et al.

Milan 2024
Abstract:

Dirichlet form analysis gives powerful tools to study diffusions in non-smooth settings, and Mosco convergence is a standard approach to study approximations. However, Mosco convergence may not be sufficient to understand finer properties, such as convergence of eigenvalues and small deviations of diffusion processes. The talk will present two recent results that strengthen Mosco convergence of Dirichlet forms. One result deals with Euclidean extension domains with irregular, or fractal, boundaries (joint work with Michael Hinz and Anna Rozanova-Pierrat). The other result deals with small deviations in sub-Riemannian situations (joint work with Marco Carfagnini and Masha Gordina).
M. Hinz, A. Teplyaev. Closability, regularity, and approximation by graphs for separable bilinear forms.


We consider a countably generated and uniformly closed algebra of bounded functions. We assume that there is a lower semicontinuous, with respect to the supremum norm, quadratic form and that normal contractions operate in a certain sense. Then we prove that a subspace of the effective domain of the quadratic form is naturally isomorphic to a core of a regular Dirichlet form on a locally compact separable metric space.

We also show that any Dirichlet form on a countably generated measure space can be approximated by essentially discrete Dirichlet forms, i.e. energy forms on finite weighted graphs, in the sense of Mosco convergence, i.e. strong resolvent convergence.
Analysis on fractals and networks, and applications
18 – 22 March, 2024

Scientific Committee: Simon N. Chandler-Wilde (University of Reading), Marco Marletta (Cardiff University), Katarzyna Pietruska-Paluba (University of Warsaw), Alexander Teplyaev (University of Connecticut), Martina Zähle (Friedrich Schiller University Jena).

Organizing Committee:
Michael Hinz (Bielefeld University),
Maria Rosaria Lancia (Sapienza University of Rome),
Anna Rozanova-Pierrat (Université Paris-Saclay)
In this talk we study first order differential operators on fractals that take functions into functions. These operators generalize first order derivatives on p.c.f. fractals introduced by M. Hino as the derivatives of energy finite functions with respect to a minimal energy-dominant reference function. Here we may also allow minimal energy-dominant differential one-forms as reference elements. In general the domains of such first order differential operators are larger than the domain of the underlying Dirichlet form. We prove an integration by parts formula and well-posedness results for continuity equations on fractals. As a key tool we use recent work of W. Arendt, I. Chalendar, R. Eymard on boundary quadruples.
We study derivations and Fredholm modules on metric spaces with a local regular conservative Dirichlet form. In particular, on finitely ramified fractals, we show that there is a non-trivial Fredholm module if and only if the fractal is not a tree (i.e. not simply connected). This result relates Fredholm modules and topology, and refines and improves known results on p.c.f. fractals. We also discuss weakly summable Fredholm modules and the Dixmier trace in the cases of some finitely and infinitely ramified fractals (including non-self-similar fractals) if the so-called spectral dimension is less than 2. In the finitely ramified self-similar case we relate the p-summability question with estimates of the Lyapunov exponents for harmonic functions and the behavior of the pressure function.
Theorem 5.16 (Non triviality of Fredholm modules for finitely ramified cell structures)

The Fredholm module \((H, F)\) is non trivial, if and only if \(X\) is not a tree.

The result stated in [F. Cipriani and J.-L. Sauvageot, 2009] for p.c.f. fractals omitted the distinction between trees and non-trees; in particular, [CS, Proposition 4.2] does not hold for the unit interval, which is a p.c.f. self-similar set, in the sense of Kigami.
Theorem 3. Suppose that all $n$-harmonic functions are continuous. Then $\mathcal{E}$ is a local regular Dirichlet form (with respect to any measure that charges every nonempty open set).

Proof. The regularity of $\mathcal{E}$ is proved in [J. Kigami, Harmonic analysis for resistance forms. J. Functional Analysis 204 (2003), 399–444.] ... 

Erratum: my theorem proves locality under assumption that $\mathcal{E}$ is regular, which was investigated by Kigami, Kumagai et al.
Plan of the talk:
Mosco convergence, strong and norm resolvent convergence
Introduction and motivation, analysis on “fractafolds”*
  Physics motivation
    Heat Kernel Estimates and Dirichlet Forms
Wave absorption: numerical shape optimization
Wave absorption: theoretical shape optimization
Equations used in architecture
Wentzell Boundary conditions
  Theoretical study
    Discrete approximations
1. Convergence of eigenvalues in fractal domains
2. Discrete spectrum for Dirichlet forms
3. Small deviations
4. Convergence of sub-Riemannian eigenvalues?
7. Local convergence of stochastic flows
New Frontiers: Layer potentials
  Riemann-Hilbert and Poincare variational problems
  Hilbert transform
  Maxwell and other vector equations
Mosco convergence, strong and norm resolvent convergence


Kato, Tosio

[Reed-Simon 1972]: For non-negative closed quadratic forms,
- Mosco convergence is equivalent to the strong resolvent convergence,
- but is weaker than the norm resolvent convergence.
Mosco convergence does not imply convergence of the spectrum

\[ \lim_{n \to \infty} E_n = F \text{ or } E_n \xrightarrow[M]{n \to \infty} F. \]

- \( x_n \in L^2 \) converging weakly to \( x \in L^2 \),
  \( \lim \inf_{n \to \infty} E_n(x_n) \geq F(x) \);

- for each \( x \in L^2 \) there exists an approximating sequence of elements \( x_n \in L^2 \), converging strongly to \( x \), such that
  \( \lim \sup_{n \to \infty} E_n(x_n) \leq F(x) \).

Example:

\( L^2 := \ell^2(\mathbb{Z}_+) \)

\[ E_n((x_k)) := \sum_{k \geq n} |x_k|^2 \xrightarrow[M]{n \to \infty} E = 0 \]

\( \sigma(E_n) = \{0, 1\} \neq \{0\} = \sigma(E) \)
Introduction and motivation, analysis on “fractafoils”*

*Strichartz: A fractafoil, a space that is locally modeled on a specified fractal, is the fractal equivalent of a manifold.

A “fractafoil” is to a fractal what a manifold is to a Euclidean half-space.

This is a part of the broader program to develop probabilistic, spectral and vector analysis on singular spaces by carefully building approximations by graphs or manifolds.
What is the first appearance of fractals in science?

In a sense, the simplest possible fractal appears in the famous Zeno’s paradoxes: Zeno of Elea (c. 495 – c. 430 BC) "Achilles and the Tortoise"

1. Achilles runs to the tortoise’s starting point while the tortoise walks forward.
2. Achilles advances to where the tortoise was at the end of Step 1 while the tortoise goes yet further.
3. Achilles advances to where the tortoise was at the end of Step 2 while the tortoise goes yet further.
   Etc.

Apparently, Achilles never overtakes the tortoise, since however many steps he completes, the tortoise remains ahead of him.
Dichotomy paradox: that which is in locomotion must arrive at the half-way stage before it arrives at the goal. In a race, the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. [Aristotle, Physics VI:9, 239b10, 239b15]

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In 1821, Augustin-Louis Cauchy proved that, for $-1 < x < 1$,

$$a + ax + ax^2 + ax^3 + ... = \frac{a}{1 - x} := S(a, x)$$

This is a weakly-self-similar sum satisfying a re-normalization “fixed-point” functional equation

$$S(a, x) = a + x \cdot S(a, x)$$
MATHEMATICS AND GEOMETRY: Decomposition of pyramids
Red pencil, pen and ink, c. 1515

The sheet shows several diagrams of pyramids broken down into smaller ones. The caption above the major pyramid drawing states that each pyramid with a square base "is resolved into eight pyramids of figures similar to its whole." The same concept is reiterated by the smaller diagram above the larger pyramid. Below, there is another small sketch with a caption explaining how to square a pyramid. Under the base of the main pyramid there is a note that alludes to a German craftsman, while immediately to the side there are grids that could be exercises in perspective.
Cantor, Sierpinski, Julia, Mandelbrot

How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension (Mandelbrot 1967).

The coastline paradox: the measured length of a stretch of coastline depends on the scale of measurement.

Fractal titanium oxide under inverse 10-ns laser deposition in air and water. A. Pan, W. Wang, X. Mei, Q. Lin, J. Cui, K. Wang, Z. Zhai

**Fig. 5** Surface morphology of titanium with the laser energy of 86 mJ, scanning speed of 0.01 mm/s, and scan length of 10 mm. *Inset a* depicts the surface morphology beyond laser irradiation zone. *Inset b* shows a typical fractal structure unit, and *inset c* is size distribution histograms of 50 fractal structure units
Non-quantized penetration of magnetic field in the vortex state of superconductors

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As first pointed out by Bardeen and Ginzburg in the early sixties\(^1,2\), the amount of magnetic flux carried by vortices in superconducting materials depends on their distance from the sample edge, and can be smaller than one flux quantum, \(\phi_0 = h/2e\) (where \(h\) is Planck’s constant and \(e\) is the electronic charge). In bulk superconductors, this reduction of flux becomes negligible at submicrometre distances from the edge, but in thin films the effect may survive much farther into the material\(^3,4\). But the effect has

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Superconducting disk with magnetic coating: Re-entrant Meissner phase, novel critical and vortex phenomena

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PACS 74.25.Op – Mixed state, critical fields, and surface sheaths

Abstract – Within the Ginzburg-Landau formalism, we study the mixed state of a superconducting disk surrounded by a magnetic ring. The stray field of the magnet, concentrated at the rim of the superconducting disk, favors ring-like arrangement of induced vortices, to the point that even a single vortex state exhibits asymmetry. A novel route for the destruction of superconductivity with increasing magnetization of the magnetic coating is found: first all vortices leave the sample, and are replaced by a re-entered Meissner phase with a full depression of the order-parameter at the sample edge; subsequently, superconductivity is then gradually suppressed from the edge inwards, contrary to the well-known surface superconductivity. When exposed to an additional homogeneous magnetic field, we find a field-polarity-dependent vortex structure in our sample —for all vorticities, only giant- or multi-vortex states are found for given polarity of the external field. In large samples, the number of vortex shells and number of flux quanta in each of them can be controlled by the parameters of the magnetic coating.

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Novel vortex phenomena in a superconducting disk with magnetic coating

Fig. 2: The free energy of the states with different vorticity $L$ as a function of the magnetization of the magnetic coating. Insets show the Cooper-pair density contour plots of the corresponding states. (a-c) Superconducting phase and (d-f) $|\psi|^2$-density plots, illustrate simultaneous vortex exit and suppression of superconductivity at the rim of the superconducting disk for high magnetization.
In our theoretical treatment of this system, we use the non-linear Ginzburg-Landau (GL) formalism, combined with Neumann boundary conditions (zero current penetrating the boundary). To investigate the superconducting state of a sample with volume $V$, we minimize, with respect to the order parameter $\psi$, the GL free energy

$$\mathcal{F} = \int \frac{dv}{V} \left( |(-i \vec{\nabla} - \vec{A}_H - \vec{A}_m)\psi|^2 - |\psi|^2 + \frac{1}{2} |\psi|^4 \right),$$

(2)
Minimization of eq. (2) leads to equations for the order parameter and superconducting current

\[ (-i\vec{\nabla} - \vec{A})^2 \psi = (1 - |\psi|^2) \psi, \]

\[ \vec{j} = \Im(\psi^* \nabla \psi) - |\psi|^2 \vec{A}, \]

which we solve following a numerical approach proposed by Schweigert et al. (see ref. [2]) on a uniform Cartesian grid with typically 10 points/ξ in each direction. We then start from randomly generated initial distribution of ψ, increase/decrease the magnetization of the magnet or change the value of the applied external field, and let eq. (3) relax to its steady-state solution. In addition, we always recalculate the vortex structure starting from the pure Meissner state\(^1(\psi = 1)\) or the normal state \((\psi \approx 0)\) as initial condition. All stable states are then collected and their energies are compared to find the ground state configuration.
Fig. 3: Free energy diagram for a large superconducting disk with thin magnetic coating. Insets show the $|\psi|^2$-density plots of distinct vortex states. The first essential difference compared to the case of a smaller sample is the single-vortex ($L=1$) state. The Cooper-pair density of this state is shown as inset (a) of fig. 3, which clearly illustrates the asymmetric placement of the vortex with respect to the sample edge. The origin of this asymmetry is simply the energy minimization, following the competition between the cylindrical confinement and the magnetic field localized away from the center of the disk. Note that such an asymmetric solution to the Ginzburg-Landau equations is enabled by the non-linear term in eq. (3), and is crucially different from the linear case which also relates to semiconductor quantum nanostructures [18]. Therefore, this makes the broken-symmetry state likely unique.

The remaining insets in fig. 3 depict two more distinct vortex states found in the larger sample. Namely, with increasing magnetization of the coating, new vortices are added to the multi-vortex shell closest to the disk edge — up to $L=12$, when one vortex enters the central part of the disk (as shown in inset (b)). However, this tendency does not continue for higher magnetization, as additional vortices are placed at the outer shell up to $L=14$. For $M > 70 \, H_{c2}$, the vortex shell starts to leave the disk in the same fashion as in figs. 2(a-f), but with one difference — the central vortex remains in the sample (see inset (c)), until the very transition to the normal state.

Field-polarity-dependent vortex structure. In the remainder of the article, we will consider the influence of the magnetic coating on the properties of the sample in an applied homogeneous magnetic field. We fix the magnetization of the coating to $M = 4.0 \, H_{c2}$ (not sufficient for vortex nucleation), and sweep up/down the applied field $H$. The outcome is shown in fig. 4(a), for the same parameters of the sample as in fig. 2. Figure 4(b) shows the same diagram but for a demagnetized coating, i.e. $M = 0$, for comparison.

The first striking effect of the coating is the very pronounced asymmetry of fig. 4(a) with respect to the polarity of the applied field. For $\vec{H} \parallel -\vec{M}$, the upper critical field is $\sim 60\%$ lower than in the case of $M = 0$, and maximal vorticity is $L_{\text{max}} = 5$ (in (b), $L_{\text{max}} = 8$). On the other hand, for $\vec{H} \parallel \vec{M}$, the critical field becomes $\sim 70\%$ higher than in the case of $M = 0$, with $L_{\text{max}} = 13$. Similar enhancement of the critical field due to the compensation of the applied and the magnet's stray field has been found experimentally in ref. [9]. Note that there the magnetic dot was placed on top and approximately in the center of the superconducting disk, so that the stray field had the opposite polarity to the one in our case, and was maximal in the central part of the sample (i.e. under the dot). Consequently, they observed enhancement of the critical field for $\vec{H} \parallel -\vec{M}$, and vortex matter in the sample showed different behavior. As we mentioned earlier, vortex...
Novel vortex phenomena in a superconducting disk with magnetic coating.

Fig. 4: (a) Free energy of a superconducting disk with magnetic coating as a function of applied homogeneous magnetic field. Insets show the Cooper-pair density plots for indicated states. (b) Same as (a), but for demagnetized coating. In (b), dashed lines denote multi-vortex and solid lines giant-vortex configurations.

Fig. 5: The $|\psi|^2$-density plots illustrating the arrangement of vortex shells in a large superconducting disk for $L = 53$ and $L = 60$, with magnetic coating with (a,c) negative ($M = -8H_{c2}$), or (b,d) positive ($M = 8H_{c2}$) magnetization.
GEOMETRICAL DESCRIPTION OF VORTICES IN GINZBURG-LANDAU BILLIARDS

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4.2 The Bogomol'nyi identities

For the special value $\kappa = \frac{1}{\sqrt{2}}$, the equations for $\psi$ and $\vec{A}$ can be reduced to first order differential equations. This special point was first used by Sarma [41] in his discussion of type-I vs. type-II superconductors and then identified by Bogomol'nyi [40] in the more general context of stability and integrability of classical solutions of some quantum field theories. This special point is also called a duality point. We first review some properties of the Ginzburg-Landau free energy at the duality point. We use the following identity true for two dimensional systems

$$|(\vec{\nabla} - i\vec{A})\psi|^2 = |D\psi|^2 + \vec{\nabla} \times \vec{j} + B|\psi|^2$$  \hspace{1cm} (64)

where $\vec{j} = \text{Im}(\psi^* \vec{\nabla} \psi) - |\psi|^2 \vec{A}$ is the current density and the operator $D$ is defined as $D = \partial_x + i\partial_y - i(A_x + iA_y)$. This relation is a relative of the Weitzenböck formula (61). At the duality point $\kappa = \frac{1}{\sqrt{2}}$ the expression (63) for $\mathcal{F}$ can be rewritten using (64) as

$$\mathcal{F} = \int_{\Omega} \left( \frac{1}{2} |B - 1 + |\psi|^2|^2 + |D\psi|^2 \right) + \oint_{\partial\Omega} (\vec{j} + \vec{A}).\vec{dl}$$  \hspace{1cm} (65)
Physics motivation

François Englert
From Wikipedia, the free encyclopedia

François Baron Englert (French: [ɑ̃ɡlek]; born 6 November 1932) is a Belgian theoretical physicist and 2013 Nobel prize laureate (shared with Peter Higgs). He is Professor emeritus at the Université libre de Bruxelles (ULB) where he is member of the Service de Physique Théorique. He is also a Sackler Professor by Special Appointment in the School of Physics and Astronomy at Tel Aviv University and a member of the Institute for Quantum Studies at Chapman University in California. He was awarded the 2010 J. J. Sakurai Prize for Theoretical Particle Physics (with Gerry Guralnik, C. R. Hagen, Tom Kibble, Peter Higgs, and Robert Brout), the Wolf Prize in Physics in 2004 (with Brout and Higgs) and the High Energy and Particle Prize of the European Physical Society (with Brout and Higgs) in 1997 for the mechanism which unifies short and long range interactions by generating massive gauge vector bosons. He has made contributions in statistical physics, quantum field theory, cosmology, string theory and supergravity.[4] He is the recipient of the 2013 Prince of Asturias Award in technical and scientific research, together with Peter Higgs and the CERN.
We take a model of foamy space-time structure described by self-similar fractals. We study the propagation of a scalar field on such a background and we show that for almost any initial conditions the renormalization group equations lead to an effective highly symmetric metric at large scale.
Fig. 1. The first two iterations of a 2-dimensional 3-fractal.
Fig. 5. The plane of 2-parameter homogeneous metrics on the Sierpinski gasket. The hyperbole $\alpha = -\beta/(\beta + 1)$ separates the domain of euclidean metrics from minkowskian metrics and corresponds – except at the origin – to 1-dimensional metrics. $M_1, M_2, M_3$ denote unstable minkowskian fixed geometries while $E$ corresponds to the stable euclidean fixed point. The unstable fixed points $0_1, 0_2$ and $0_3$ associated to 0-dimensional geometries are located at the origin and at infinity on the $(\alpha, \beta)$ coordinates axis. The six straight lines are subsets invariant with respect to the recursion relation but repulsive in the region where they are dashed. The first points of two sequences of iterations are drawn. Note that for one of them the 10th point ($\alpha = -56.4, \beta = -52.5$) is outside the frame of the figure.
Fig. 10. A metrical representation of the two first iterations of a 2-dimensional 2-fractal corresponding to the euclidean fixed point. Vertices are labelled according to fig. 4.
Figure 6.4. Geometric interpretation of Proposition 6.1.

**Theorem 7.1.** Every point of $V^*$ has positive capacity.

Proof. Let $x \in V^*$. Then $x \in V_n$ for some $n$. The capacity of $\{x\}$ with respect to $E$ is the same as that with respect to $E_n$ because of the compatibility of the sequence of networks. The latter capacity is positive because $V_n$ is a finite set. □

The effective resistance is defined for any $x, y \in V^*$ by

$$R(x, y) = \left(\min_{u} \{E(u, u) : u(x) = 1, u(y) = 0\}\right)^{-1}.$$  

Here the minimum is taken over all functions on $V^*$. Note that $x, y \in V_n$ for large enough $n$ and that (7.1) does not change if $E$ is replaced by $E_n$, because of the compatibility condition (see $K_4$, Proposition 2.1.11). By Theorem 2.1.14 in $K_4$, $R(x, y)$ is a metric on $V^*$. In what follows we will write $R$-continuity, $R$-closure etc. for continuity, closure etc. with respect to the effective resistance metric $R$. It is known that if $E(u, u) < \infty$ then $u$ is $R$-continuous ($K_4$, Theorem 2.2.6(1)). The main ingredient in the proof of this fact is the inequality

$$|u(x) - u(y)|^2 \leq R(x, y) E(u, u).$$  

(7.2)

Let $\Omega$ be the $R$-completion of $V^*$. We can conclude from (7.2) that if $u$ is a function on $V^*$ such that $E(u, u) < \infty$ then $u$ has a unique continuation in $\Omega$.
The Spectral Dimension of the Universe is Scale Dependent

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We measure the spectral dimension of universes emerging from nonperturbative quantum gravity, defined through state sums of causal triangulated geometries. While four dimensional on large scales, the quantum universe appears two dimensional at short distances. We conclude that quantum gravity may be “self-renormalizing” at the Planck scale, by virtue of a mechanism of dynamical dimensional reduction.

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Quantum gravity as an ultraviolet regulator? — A shared hope of researchers in otherwise disparate approaches to quantum gravity is that the microstructure of space and time may provide a physical regulator for the ultraviolet infinities encountered in perturbative quantum field theory.
other hand, the “short-distance spectral dimension,” obtained by extrapolating Eq. (12) to $\sigma \to 0$ is given by

$$D_S(\sigma = 0) = 1.80 \pm 0.25,$$

(15)

and thus is compatible with the integer value two.

Random Geometry and Quantum Gravity
A thematic semestre at Institut Henri Poincaré
14 April, 2020 - 10 July, 2020
Organizers : John BARRETT, Nicolas CURIEN, Razvan GURAU, Renate LOLL, Gregory MIERMONT, Adrian TANASA
Fractal space-times under the microscope: 
a renormalization group view on Monte Carlo data

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Abstract: The emergence of fractal features in the microscopic structure of space-time is a common theme in many approaches to quantum gravity. In this work we carry out a detailed renormalization group study of the spectral dimension $d_s$ and walk dimension $d_w$ associated with the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG). We discover three scaling regimes where these generalized dimensions are approximately constant for an extended range of length scales: a classical regime where $d_s = d, d_w = 2$, a semi-classical regime where $d_s = 2d/(2+d), d_w = 2 + d$, and the UV-fixed point regime where $d_s = d/2, d_w = 4$. On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is shown to be in very good agreement with the data. This comparison also provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

Keywords: Models of Quantum Gravity, Renormalization Group, Lattice Models of Gravity, Nonperturbative Effects
Fractal space-times under the microscope: A Renormalization Group view on Monte Carlo data (Martin Reuter, Frank Saueressig):

Three scaling regimes of the effective space-times of asymptotically safe Quantum Einstein Gravity (QEG):

1. a classical regime $d_s = d, \ d_w = 2$,
2. a semi-classical regime $d_s = 2d/(2 + d), \ d_w = 2 + d$,
3. the UV-fixed point regime $d_s = d/2, \ d_w = 4$.

On the length scales covered by three-dimensional Monte Carlo simulations, the resulting spectral dimension is in very good agreement with the data and provides a natural explanation for the apparent puzzle between the short distance behavior of the spectral dimension reported from Causal Dynamical Triangulations (CDT), Euclidean Dynamical Triangulations (EDT), and Asymptotic Safety.

▶ Mathav Murugan: $d_w = d_f$ consistent with $d_s = 2d_f/d_w = 2$
▶ Growth and percolation on the uniform infinite planar triangulation by Omer Angel (GAFA 2003)
▶ Anomalous diffusion of random walk on random planar maps by Ewain Gwynne and Tom Hutchcroft (PTRF 2020)
Heat Kernel Estimates and Dirichlet Forms

\[ p_t(x, y) \sim \frac{1}{t^{d_f/d_w}} \exp \left( -c \frac{d(x, y)^{d_w}}{t^{1/d_w-1}} \right) \]

**distance** \( \sim \) **(time)** \( \frac{1}{d_w} \)

\[ d_f = \text{Hausdorff dimension} \]
\[ \frac{1}{\gamma} = d_w = \text{“walk dimension” (}\gamma=\text{diffusion index)} \]
\[ \frac{2d_f}{d_w} = d_s = \text{“spectral dimension” (diffusion dimension)} \]

First example: Sierpiński gasket; Kusuoka, Fukushima, Kigami, Barlow, Bass, Perkins (mid 1980’—)
Stability Theorem (Barlow, Bass, Kumagai (2006))

Under natural assumptions on the MMD (geodesic Metric Measure space with a regular symmetric conservative Dirichlet form), the sub-Gaussian heat kernel estimates are stable under rough isometries, i.e. under maps that preserve distance and energy up to scalar factors.

Gromov-Hausdorff + energy
Theorem. (Barlow, Bass, Kumagai, T. (1989–2010).) On any generalized Sierpiński carpet there exists a unique, up to a scalar multiple, local regular Dirichlet form that is invariant under the local isometries.

Therefore there is a unique symmetric Markov process and a unique Laplacian.

Moreover, the Markov process is strong Feller and its transition density satisfies sub-Gaussian heat kernel estimates.

Main difficulties: If it is not a cube in $\mathbb{R}^n$, then

- $d_S < d_f, d_w > 2$
- the energy measure and the Hausdorff measure are mutually singular;
- the domain of the Laplacian is not an algebra;
- if $d(x, y)$ is the shortest path metric, then $d(x, \cdot)$ is not in the domain of the Dirichlet form (not of finite energy) and so methods of Differential geometry seem to be not applicable;
- Lipschitz functions are not of finite energy;
- in fact, we can not compute any functions of finite energy;
- Fourier and complex analysis methods seem to be not applicable.
Wave absorption: numerical shape optimization


Fig. 1. An illustration for the Open Set Condition in the case of the square Koch curve, also called the Minkowski fractal. The thick dotted line outlines the set $O$, which is called the 0-cell. The thin dotted lines outlines the open sets in $\Phi_1(O)$, which are called 1-cells. The bottom picture illustrates the stronger form of the Open Set Condition used in Theorem 5.1: the thin solid lines outline the open sets $O'$ and $\Phi_1(O')$.

Note that, by the standard decompositions into different scales, it is essentially enough to consider the case when all contraction factors $d_{i,m}$ are equal, that is $d = d_{i,m}$ for all $i$ and $m$. To verify the Harnack chain condition, assume that $x,y \in \Omega_m$ such that distance to the boundary of each $x$ and $y$ is comparable to $\delta_1 \sim d_{m,1}$ and $|x - y| = \delta_2 \sim d_{m,2}$, where $m \geq m_1 \geq m_2$.

We proceed by considering different cases. To begin with, assume that $y$ is in a 0-cell but not in any 1-cell. In this case we can apply the following strategy: connect $x$ to the outer boundary of its 1-cell by the Harnack chains of balls lying in this 1-cell, and connect this Harnack chain to the outer boundary of its 1-cell.
Wave absorption: theoretical shape optimization


Equations used in architecture


Given a domain $\Omega \subset \mathbb{R}^N$ and a vector field $v \in W^{1,2}(\Omega)^N$ we denote the symmetric part of its gradient by

$$e(v) = \frac{1}{2} (\nabla v + (\nabla v)^t).$$

Let $A \in L^\infty(\Omega, \mathcal{M}_N^s(\alpha, \beta))$ and write $\sigma(v) = Ae(v)$, $v \in W^{1,2}(\Omega)^N$. We are interested in solutions $u \in W^{1,2}(\Omega)^N$ of BVP:

$$\begin{cases}
-\text{div} \sigma(u) &= f \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \Gamma_{\text{Dir}}, \\
\sigma(u) \cdot n &= g \quad \text{on} \ \Gamma_{\text{Neu}}.
\end{cases} \quad (1)$$
Wentzell Boundary conditions


\[ \mathcal{E}(u) = \int_{\Omega} \| \nabla u \|^2 \, dx + \mathcal{E}_{\partial \Omega}(u) \]
Theoretical study


\[ E(u) = \int_{\Omega} \| \nabla u \|^2 \, dx + E_{\partial \Omega}(u) \]

\[ \mathcal{E}(u) = \int_{\Omega} \| \nabla u \|^2 dx + \mathcal{E}_{\partial\Omega}(u) \]
where $c_n(p_1, p_2)$ is the conductance between points $p_1$ and $p_2$. Unless stated otherwise we take
\begin{align}
(c_n(p_1, p_2)) = \begin{cases}
1 & \text{if } p_1 \text{ and } p_2 \text{ are connected by an interior edge} \\
4n & \text{if } p_1 \text{ and } p_2 \text{ are connected by an outer boundary edge} \\
0 & \text{if } p_1 \text{ and } p_2 \text{ are not connected by an edge}.
\end{cases}
\end{align}

We introduce a measure on the vertices $V_n$:
\begin{align}
m_n(p) = \begin{cases}
1 & \text{if } p \text{ is an interior vertex} \\
4n & \text{if } p \text{ is an outer boundary vertex}.
\end{cases}
\end{align}

The sequence of graph energies $\{E_n(u)\}_{n \geq 1}$ serves as an approximation of $E(u)$ as justified in the following proposition.

**Theorem 3.1.**

Let $u$ be a core function as described in Corollary 2.4. Then $E(u) = \lim_{n \to \infty} E_n(u)$.

**Remark 3.2.**

$E_n(u)$ is understood as the graph energy of the restriction of $u$ on $V_n$. 
Figure 2. Algorithm to generate the vertices of the graph $\Box_n$. After the construction of $\Box_n$ we generate the corresponding discrete Laplacian $L_n$ as well as the Dirichlet Laplacian. To generate the discrete Laplacian $L_n$ we use KD Trees to determine efficiently which vertices are boundary points and which are interior. We then create a weighted adjacency matrix with this information where the weights are defined as in (3.2). From the constructed discrete Laplacian $L_n$ we are able to generate...
Figure 3. Eigenvectors of $L_n$ (left) compared with Dirichlet eigenvectors (right). (a) 1st eigenvalue of $L_n$, eigenvalue 0. (b) 1st Dirichlet eigenvector, eigenvalue 118.8. (c) 2nd eigenvector of $L_n$, eigenvalue 15.1. (d) 2nd Dirichlet eigenvector, eigenvalue 294.5. (e) 4th eigenvector of $L_n$, eigenvalue 48.1. (f) 4th Dirichlet eigenvector, eigenvalue 499.8.
Figure 4. Eigenvector of $L_n$ (left) compared with Dirichlet eigenvectors (right). (a) 34th eigenvector of $L_n$, eigenvalue 1098.6. (b) 13th Dirichlet eigenvector, eigenvalue 1084.6. (Level 4 graph approximation)

to the 13th Dirichlet eigenvector but also correspond to the eigenvalue 1098.6, which is very close to the 13th Dirichlet eigenvalue (1084.6).

In general we observe that the Dirichlet eigenvectors exhibit more complex pattern than the corresponding eigenvectors of $L_n$. Note that from a physics point of view, this is expected as the Dirichlet eigenvectors correspond in our computations to a higher eigenvalues (energy) than their corresponding eigenvectors of $L_n$. Moreover, eigenvectors correspond to higher eigenvalues show increasing oscillatory behavior which limits the graphical representation. For a better view of such eigenvectors, in particular regarding their symmetries, we display in Figure 5 contour plots for a selection of eigenvectors of $L_n$. The blue regions
Figure 5. Contour Plots of the Eigenvectors of $L_n$ corresponding to eigenvalues $\lambda$: (a) 4th eigenvector, $\lambda = 48.1$. (b) 5th eigenvector, $\lambda = 48.1$. (c) 6th eigenvector, $\lambda = 85.1$. (d) 8th eigenvector $\lambda = 125.4$. (e) 1153rd eigenvector $\lambda = 49965.7$. (f) 1157th eigenvector $\lambda = 50156.6$. (g) 1161st eigenvector, $\lambda = 50188.8$ and (h) 1162nd eigenvector, $\lambda = 50188.83$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in $(\epsilon, \infty)$ and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$. (Level 4 graph approximation)
Figure 6. (Upper) Eigenvalue counting functions of Dirichlet Laplacian (orange) and $L_n$ (blue). (Lower) Log-Log plot of the eigenvalue counting functions of Dirichlet Laplacian (orange) and $L_n$ (blue) (Level 4 graph approximation).
The oval-shaped graph is due to a high oscillation of both eigenvectors bouncing ball modes and the existence of such modes is known mostly for a class of convex domains. For details see [NG13] where also an example is provided, for which the high-frequency localized modes exist in a non-convex domain (an elliptical annulus).

Our numerical observations show that taking the boundary energy into account while defining the discrete Laplacian $L_n$ induces eigenvectors corresponding to eigenvalues in the higher part of the spectrum that demonstrate features of whispering gallery type modes. The geometrical information of the boundary is encoded in $L_n$ via the edge weights (conductance) and the vertices measure. Hence we are confronted with the question whether it is possible to predict such modes by directly studying the Laplacian $L_n$. We will give a partial answer in the next section by using ideas of Filoche and Mayboroda introduced in [FM].
Figure 8. The last $L_n$ eigenvector, $\lambda = 524999.69$. The graph splits into two parts, above and below the Koch snowflake domain due to a high oscillation (Level 4 graph approximation).
Figure 9. $L_n$ eigenvectors localization with eigenvalues $\lambda$: (a) 5030th eigenvector, $\lambda = 118048.66$. (b) 5031th eigenvector, $\lambda = 119678.65$. (c) 5032th eigenvector, $\lambda = 119678.65$. (d) 5033th eigenvector, $\lambda = 121460.72$. (e) 5100th eigenvector, $\lambda = 185367.41$. (f) 5200th eigenvector, $\lambda = 291364.38$. (g) 5300th eigenvector, $\lambda = 392584.97$. (h) 5557th eigenvector, $\lambda = 524999.69$. Blue regions indicate the values of an eigenvector in $(-\epsilon, \epsilon)$, red regions in $(\epsilon, \infty)$ and green regions in $(-\infty, -\epsilon)$, where $\epsilon = 0.01$ (Level 4 graph approximation).
Computations on the Koch Snowflake
Carlos D. Lima, Malcolm H. Gabbard, Gamal Mograby, Luke Rogers, Alexander Teplyaev

The theory gives rise to what they call a landscape Mapping, which we explain briefly with an example of disordered systems and weak localizations (irregular geometries) originate from the same universal mechanism. The counting function (a) gives a characterizing feature of energy localization to the boundary of \( \bar{\Omega} \) (left) and a continued "zeroing out" of the inner region (middle) and the shape of one of the eigenvectors (right) for the Koch Snowflake. Numerical approximations are shown in the sequence of finite graphs,\( K \), and for the sake of brevity they impose boundary and interior energies. Results from [3] were reproduced by imposing Dirichlet B.C. on an open bounded set \( \Omega \).

The eigenvalues and eigenvectors of the Koch Snowflake. Numerical approximations are shown in the sequence of finite graphs, \( K \), and for the sake of brevity they impose boundary and interior energies.

\[ L \phi \bigg|_{\partial \Omega} = 0 \]
\[ \phi \bigg|_{\partial \Omega} = 0 \]

\[ 2 \]
\[ 1 \]

\[ \int_{\Omega} G(x,y) \phi(x) \phi(y) \mathrm{d}x \mathrm{d}y \]
The high frequency landscape vector attains just the following two values the boundary vertices 527360 and 524288. It is constant on the interior vertices with the value 157464. (Level 4 graph approximation)
1. Convergence of eigenvalues in fractal domains

Theorem (Hinz, Rozanov-Piéerrat, T.)

Let \( n, D, \alpha, \gamma, \varepsilon, d, (\Omega_m)_m \) and \((\mu_m)_m\) be a sequence of admissible domains. Suppose that \( \lim_m \Omega_m = \Omega \) in the Hausdorff sense and in the sense of characteristic functions and \( \lim_m \mu_m = \mu \) weakly. There is a sequence \((m_k)_{k=1}^{\infty}\) with \( m_k \uparrow +\infty \) such that the following hold.

(i) We have \( \lim_{k \to \infty} P_{\Omega_{m_k}} \circ \hat{G}_{\Omega_{m_k},\mu_{m_k},*} = P_{\Omega} \circ \hat{G}_{\Omega,\mu,*} \) in operator norm.

(ii) If \( 0 < a < b \) are in the resolvent set of \( -\mathcal{L}_\gamma^{\Omega,\mu,*} \), then \( \lim_{k \to \infty} \pi(a,b)(\Omega_{m_k}, \mu_{m_k},*) = \pi(a,b)(\Omega, \mu,*) \) in operator norm.

(iii) The spectra of the operators \( -\mathcal{L}_\gamma^{\Omega_{m_k},\mu_{m_k},*} \) converge to the spectrum of \( -\mathcal{L}_\gamma^{\Omega,\mu,*} \) in the Hausdorff sense. The eigenvalues \( \lambda_n(\Omega, \mu,*) \) of the operator \( -\mathcal{L}_\gamma^{\Omega,\mu,*} \) are exactly the limits as \( k \to \infty \) of sequences of the eigenvalues of the operators \( -\mathcal{L}_\gamma^{\Omega_{m_k},\mu_{m_k},*} \),

\[
\lambda_n(\Omega, \mu,*) = \lim_{k \to \infty} \lambda_n(\Omega_{m_k}, \mu_{m_k},*). \quad (2)
\]
Figure 1: Dyadic approximations of a Von Koch snowflake ($\partial \Omega$, in blue) in $\mathbb{R}^2$. The interior approximation $\Omega^k_-$ lies inside the green dashed line; the exterior approximation $\Omega^k_+$ lies outside the red dotted line.

Proposition 3.3. Let $\Omega$ be an arbitrary bounded domain of $\mathbb{R}^n$ such that $\Omega^c$ is connected too. Then $\Omega^k_- \cup \Omega^k_+ \rightarrow \mathbb{R}^n \setminus \partial \Omega$.

Proof. From Remark 3.2, let $k_0 \in \mathbb{N}$ be such that $(\Omega^k_-)^{k \geq k_0}$ is non-decreasing. Then it holds:

$$\Omega^k_- \rightarrow k \rightarrow \infty \Omega^- = \bigcup_{k \geq k_0} \Omega^k_- \subset \Omega.$$

Let $x \in \Omega$. Then $d(x, \partial \Omega) > 0$. In the same way as for (5), there exists $\tilde{k} \geq k_0$ so that:

$$\{x\} \cup \Omega^k_0 \subset \Omega^-_k.$$

Hence $x \in \Omega^\infty_-$, and $\Omega^\infty_- = \Omega$.

To show $\Omega^k_+ \rightarrow \Omega^c$, we proceed in the same way considering $x \in \Omega^c$ a path on $\Omega^c$ linking $x$ and a large square containing $\Omega$.

Remark 3.4. Lemma 3.1, Remark 3.2 and Proposition 3.3 can easily be generalized in the case of an arbitrary (potentially unbounded) open set $\Omega$ such that $\Omega$ and $\Omega^c$ have a finite number of connected components.

3.1 Spaces of functions on $\mathbb{R}^n$
Let us define the following spaces:

$$H_1^{\partial \Omega}(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid \nabla u \mid_{\mathbb{R}^n \setminus \partial \Omega} \in L^2(\mathbb{R}^n \setminus \partial \Omega)\},$$

$$H_1^{\Delta \Omega_k}(\mathbb{R}^n) := \{u \in L^2(\mathbb{R}^n) \mid \nabla u \mid_{\Omega^k_- \cup \Omega^k_+} \in L^2(\Omega^k_- \cup \Omega^k_+)\}.$$
2. Discrete spectrum for Dirichlet forms

Theorem (Carfagnini, Gordina, T.)

Let \( \mathcal{U} \) be an open bounded subset of \( \mathcal{X} \), and \( \mathcal{P}_t^\mathcal{U} \) be the semigroup associated to \((\mathcal{E}, \mathcal{D}_\mathcal{E})\) with the infinitesimal generator \( A_\mathcal{U} \). Assume that \( p_t(x, y) \) exists for all \( t \) and for \( m \)-a.e. \( x, y \in \mathcal{X} \). If there exists a \( t_\mathcal{U} > 0 \) such that

\[
M_\mathcal{U}(t_\mathcal{U}) = \text{ess sup}_{(x,y) \in \mathcal{U} \times \mathcal{U}} p_{t_\mathcal{U}}(x, y) < \frac{1}{m(\mathcal{U})^2},
\]

(3)

then the spectrum of \(-A_\mathcal{U}\) is discrete and \( \lambda_1 > 0 \).
3. Small deviations

Theorem (Carfagnini, Gordina, T.)

Let \( \{P_t\}_{t \geq 0} \) be a strongly continuous contraction semigroup on \( L^2(X, m) \). Let \( x \in X \) and assume that \( P_t^{B_1(x)} \) is irreducible. Assume that the heat kernel \( p_t(x, y) \) exists for all \( t \) and for all \( x, y \in X \) and that

\[
p_t(x, y) \leq c \, t^{-\frac{\alpha}{\beta}}
\]

for any \( t, x, \) and \( y \). Moreover, assume that there exists a \( t_0 \) such that \( p_{t_0}(x, y) \) is continuous for \( x, y \in X \). If \( X_t^x \) is self-similar then

\[
\lim_{\varepsilon \to 0} e^{\lambda_1 \frac{t}{\varepsilon^\beta}} P^x \left( \sup_{0 \leq s \leq t} d(X_s, x) < \varepsilon \right) = c_1 \varphi_1(x),
\]

(4)

where \( \lambda_1 > 0 \) is the spectral gap of \( A \) restricted to the unit ball \( B_1(x) \), and \( \varphi_1 \) is the corresponding positive eigenfunction.
4. Convergence of sub-Riemannian eigenvalues?


- N. Eldredge, M. Gordina, L. Saloff-Coste, *Left-invariant geometries on SU(2) are uniformly doubling*, GAFA, 2018, 28,
5. Convergence of the re-normalized eigenvalues of small balls in $\text{SU}(2)$ to corresponding eigenvalues in the unit ball of $\mathbb{H}$

Here $\mathbb{H}$ is the Heisenberg group, which is a re-scaled limit of $\text{SU}(2)$ near the identity.

**Theorem (Carfagnini, Gordina, T.)**

Let $0 < \lambda_1^\mathbb{H} < \lambda_2^\mathbb{H} \leq \lambda_3^\mathbb{H} \leq \ldots$ be the Dirichlet eigenvalues in the unit ball $B_1^\mathbb{H}$ of $\mathbb{H}$, counted with multiplicity. Let $0 < \lambda_1^r < \lambda_2^r \leq \lambda_3^r \leq \ldots$ be the Dirichlet eigenvalues in the $r$-ball $B_r^{\text{SU}(2)}$ of $\text{SU}(2)$, counted with multiplicity. Then for each $n \geq 1$ we have

$$\lim_{r \to 0} r^2 \lambda_n^r = \lambda_n^\mathbb{H}. \quad (5)$$
Let $p_{t}^{B_{1}^{\mathbb{H}}} (\cdot, \cdot)$ be the Dirichlet heat kernel in the unit ball $B_{1}^{\mathbb{H}}$ of $\mathbb{H}$, and $p_{t}^{B_{r}^{SU(2)}} (\cdot, \cdot)$ be the Dirichlet heat kernel in the $r$-ball $B_{r}^{SU(2)}$ of $SU(2)$, where the balls are centered at the identity of the groups.

Theorem (Carfagnini, Gordina, T.)

For each $t > 0$

$$\lim_{r \to 0} r^{4} p_{r^{2} t}^{B_{r}^{SU(2)}} (\Phi^{-1} (\delta_{r}^{\mathbb{H}} (x)) , \Phi^{-1} (\delta_{r}^{\mathbb{H}} (x))) = p_{t}^{B_{1}^{\mathbb{H}}} (x, y). \quad (6)$$

uniformly for $x, y \in B_{1}^{\mathbb{H}}$.
7. Local convergence of stochastic flows

Let

\[ g_s^{B_{SU(2)}} := \begin{cases} g_s & s < \tau_{B_{SU(2)}} \\ \partial & s \geq \tau_{B_{SU(2)}} \end{cases} \]  

(7)

where \( g_s \) denotes a hypoelliptic Brownian motion on \( SU(2) \), and

\[ \tau_{B_{SU(2)}} := \inf \left\{ s > 0, \ g_s \notin B_{SU(2)} \right\} . \]  

(8)

Similarly, let

\[ X_s^{B_{\mathbb{H}}} := \begin{cases} X_s & s < \tau_{B_{\mathbb{H}}} \\ \partial & s \geq \tau_{B_{\mathbb{H}}} \end{cases} \]  

(9)

where \( X_s \) denotes a hypoelliptic Brownian motion on \( \mathbb{H} \), and

\[ \tau_{B_{\mathbb{H}}} := \inf \left\{ s > 0, \ X_s \notin B_{\mathbb{H}} \right\} . \]  

(10)
Theorem (Carfagnini, Gordina, T.)

For any $0 < r < \frac{1}{7} r_{1/7}$ there is a continuous stochastic process $Y'_s$ in $H$ such that

\[ Y'_s := \delta_{1/r}^H \Phi \left( g_{r^2 s}^{B_{3r}^{SU(2)}} \right) \quad (11) \]

in the sense of conditional probability distributions on the event $A_{3r} := \{ s < \tau_{B_{3r}^H} \}$ and

\[ \lim_{r \to 0} \mathbb{1}_{A_{3r}} \sup_{0 \leq s \leq T} | Y'_s - X_s | = 0 \quad (12) \]

in probability.

We use Theorem 3.3.1, page 76, in Kunita 1986 Lectures on stochastic flows and applications, Tata Institute of Fundamental Research Lectures on Mathematics and Physics.
New Frontiers: Layer potentials

\[ u(x) = \int_S \rho(y) \frac{\partial}{\partial \nu} P(x, y) d\sigma(y) \]

\[ v(x) = G * f = \int_{\mathbb{R}^n} g(x, y) d\mu(y) \]
Riemann-Hilbert and Poincare variational problems

Find a function in $\mathbb{C}$, unanalytic outside of a curve, with prescribed values and jumps on the curve. Research in progress: Anna Rozanova-Pierrat, Gabriel Claret (CentraleSupélec), Michael Hinz (Bielefeld).

Classical applications:

- Integrable models, inverse scattering or inverse spectral problem
- the inverse monodromy problem for Painlevé equations
- Orthogonal polynomials, Random matrices
- Combinatorial probability
- Algebraic geometry, Donaldson–Thomas theory
Hilbert transform

\[ H(u)(t) = \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{u(\tau)}{(t - \tau)} d\tau \]

Research in progress: Anna Rozanova-Pierrat, Gabriel Claret (CentraleSupélec), Michael Hinz (Bielefeld).

Closely connected to the Riemann-Hilbert and Poincare variational problems and is extensively used in analysis and in signal processing.
Maxwell and other vector equations

We develop new mathematical tools in the vector case in order to study and solve Maxwell’s equations in non-Lipschitz, possibly fractal domains. To that extent, we would like to show here one use of those tools with the time-harmonic Maxwell problem completed with a homogeneous Dirichlet boundary condition, which becomes with our notations:

\[
\begin{align*}
\text{curl}(\mu^{-1}\text{curl } E) - \omega^2 \varepsilon E &= f \quad \text{on } \Omega \\
\text{Tr}_T(E) &= 0 \quad \text{on } \partial\Omega
\end{align*}
\]

where \( f \in L^2(\Omega) \) and we look for \( E \in H(\text{curl}, \Omega) \).

This problem is equivalent to the following variational formulation:

Find \( E \in H_0(\text{curl}, \Omega) \) such that \( \forall F \in H_0(\text{curl}, \Omega) \):

\[
(\mu^{-1}\text{curl } E, \text{curl } F) - \omega^2 (\varepsilon E, F) = (f, F).
\]

Research in progress: Anna Rozanova-Pierrat (CentraleSupélec), Patrick Ciarlet (ENSTA Paris) et al.
In Memory of Professor Robert Strichartz

We will be dedicating the entire conference to Professor Strichartz. A special session will be scheduled during the conference for all to attend and reflect on their thoughts and memories of Bob. Bob is appreciated and recognized for his organizing of the Fractals Conference Community in 2002. He will be profoundly missed by family, friends, colleagues, and most of all, the students he mentored and influenced throughout his career.

A message from the Cornell Department of Mathematics Chair, Tara Holm:

Dear friends,

I am sad to share that our colleague and friend Professor Robert Strichartz died yesterday, 19 December 2021, after a long illness. He was 78.
8th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 2025

Everybody is invited!
Thank you for your attention!

Picture: the *Sierpinski-flower*

UConn REU 2023: Fractal Eigenmaps