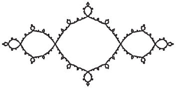
Spectral analysis on self-similar graphs and fractals

Alexander Teplyaev
University of Connecticut



February 2020 $*$ Stony Brook

outline:

- Spectral analysis on fractals:
  - Weak Uncertainty Principle (Okoudjou, Saloff-Coste, Strichartz, T., 2008)
  - Laplacians on fractals with spectral gaps gaps have nicer Fourier series (Strichartz, 2005)
  - Bohr asymptotics on infinite Sierpinski gasket (Chen, Molchanov, T., 2015).
  - Singularly continuous spectrum of a self-similar Laplacian on the half-line (Chen, T., 2016).
  - Spectral zeta function (Derfel-Grabner-Vogl, Steinhurst-T., Chen-T.-Tsougkas, Kajino, 2007–2017)
- **Algebraic applications: spectrum of the Laplacian on the Basilica Julia set** (with Rogers, Brzoska, George, Jarvis arXiv:1908.10505).
- selected technical details (if time permits)

This is a part of the broader program to develop **probabilistic, spectral and vector analysis on singular spaces** by **carefully building approximations by graphs or manifolds**.

7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals: June 9–13, 2020

Home » 7th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals

Welcome!

Planning has begun for Fractals 7 (June 9-13, 2020). The purpose of this conference, held every three years, is to bring together mathematicians who are already working in the area of analysis and probability on fractals with students and researchers from related areas. Information will be posted here as it becomes available.

Financial support will be provided to a limited number of participants to cover the cost of housing in Cornell single dormitory rooms and partially support other travel expenses. Students and junior researchers from underrepresented groups in STEM are particularly encouraged to apply for travel funding. Well-established researchers are encouraged to use their own travel funding; the NSF expects that most funds will be expended on otherwise unfunded mathematicians.

Registration details will be publicized once available.

All general inquiries can be sent to: fractals_math@cornell.edu

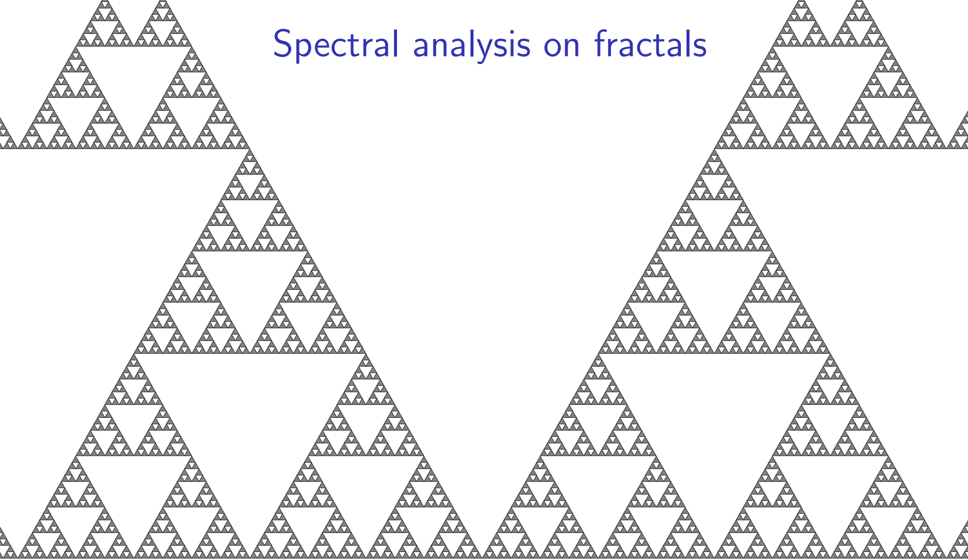
Conference Organizers:

- Robert Strichartz (chair), Cornell University
- Patricia Alonso Ruiz, Texas A&M University
- Michael Hinz, Bielefeld University
- Luke Rogers, University of Connecticut
- Alexander Teplyaev, University of Connecticut

abstract

The talk will describe how spectral theory, geometry of graphs, and dynamical systems are used to analyze spectral properties of the random walk generator on finitely ramified self-similar graphs and fractals. In particular, pure point or singular continuous spectrum appears naturally for such graphs. The standard examples include the Sierpinski triangle, the Vicsek tree, and the Schreier graphs of the Hanoi self-similar group studied by Grigorchuk and Sunic. A more complicated example is related to the Basilica Julia set of the polynomial $z^2 - 1$ and its Iterated Monodromy Group, defined by Nekrashevych. Its spectrum was investigated numerically by Strichartz et al and analytically in a joint work with Luke Rogers and several students at UConn.

Spectral analysis on fractals



A part of an infinite Sierpiński gasket.

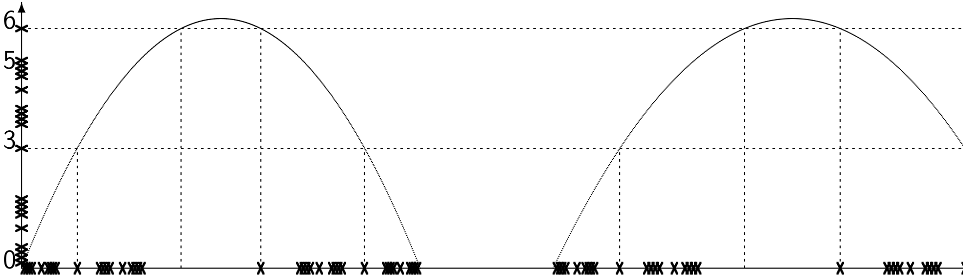


Figure: An illustration to the computation of the spectrum on the infinite Sierpiński gasket. The curved lines show the graph of the function $\mathfrak{R}(\cdot)$.

Theorem (Rammal, Toulouse 1983, Béllissard 1988, Fukushima, Shima 1991, T. 1998, Quint 2009)

On the infinite Sierpiński gasket the spectrum of the Laplacian consists of a **dense set of eigenvalues $\mathfrak{R}^{-1}(\Sigma_0)$ of infinite multiplicity** *and a* **singularly continuous component of spectral multiplicity one supported on $\mathfrak{R}^{-1}(\mathcal{J}_R)$**.

PHYSICAL REVIEW B VOLUME 31, NUMBER 3 1 FEBRUARY 1985

Energy spectrum for a fractal lattice in a magnetic field

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(Received 10 September 1984)

To simulate a kind of magnetic field in a fractal environment we study the tight-binding Schrödinger equation on a Sierpinski gasket. The magnetic field is represented by the introduction of a phase onto each hopping matrix element. The energy levels can then be determined by either direct diagonalization or recursive methods. The introduction of a phase breaks all the degeneracies which exist in and dominate the zero-field solution. The spectrum in the field may be viewed as considerably broader than the spectrum with no field. A novel feature of the recursion relations is that it leads to a power-law behavior of the escape rate. Green's-function arguments suggest that a majority of the eigenstates are truly extended despite the finite order of ramification of the fractal lattice.

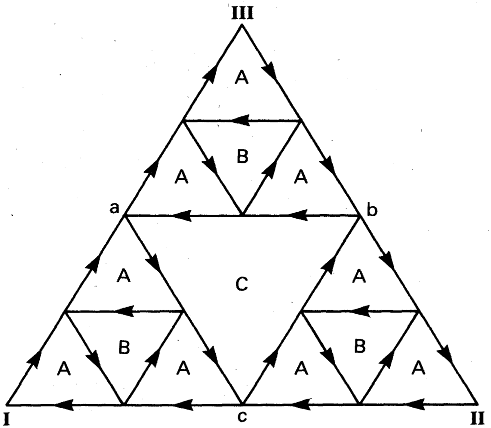

FIG. 1. Fragment of the Sierpinski gasket. The phase of the hopping matrix is equal to ϕ in the direction of the arrow and $-\phi$ otherwise.

Solid State Communications, Vol. 64, No. 10, pp. 1291–1294, 1987.
Printed in Great Britain.

0038–1098/87 $3.00 + .00

BAND SPECTRUM FOR AN ELECTRON ON A SIERPINSKI GASKET IN A MAGNETIC FIELD

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(*Received* 20 *July* 1987 *by S. Alexander*)

We consider a quantum charged particle on a fractal lattice given by a Sierpinski gasket, submitted to a uniform magnetic field, in a tight binding approximation. Its band spectrum is numerically computed and exhibits a fractal structure. The groundstate energy is also compared to the superconductor transition curve measured for a Sierpinski lattice of superconducting material.

choose the gauge in such a way that H depends only upon α and α' in a periodic way with period one. We will denote by $H(\alpha, \alpha')$ this operator from now on.

We also introduce the dilation operator D defined as:

$$D\varphi(m) = \varphi(2m). \tag{2}$$

The scaling properties of this system are expressed in the following Renormalization Group Equation (RGE) [23]:

$$E\{E1 - H(\alpha, \alpha')\}^{-1}D = G\{E^*1 - H(\alpha,^*, \alpha'^*)\}^{-1}, \tag{3}$$

where [7, 16]:

(i) $$G = \{E^3 - 3E - 2(XU + YV)\}/(S^2 + C^2)^{1/2},$$

(ii) $$E^* = \{E^4 - 7E^2 - [2(XU + YV) + 4X]E + 4(1 - U)\}/(S^2 + C^2)^{1/2}, \tag{4}$$

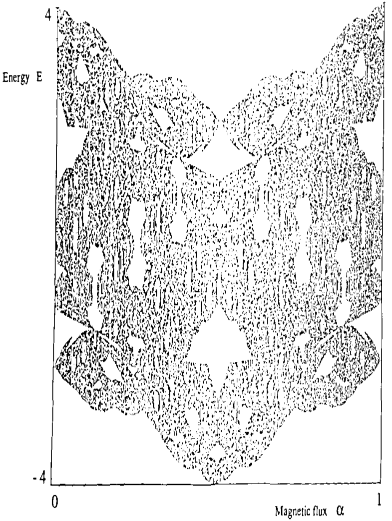

Fig. 2. Spectrum of $H(\alpha)$, computed by 10 iterations of F. α is the horizontal variable, ranging from 0 to 1. E is the vertical variable, ranging from -4 to 4.

These results have been compared with an experiment performed on an array of superconducting Al-wires shaped like a Sierpinski gasket with six levels of hierarchy. A description of this pattern generated by e-beam lithography has been given in [20]. More details will be published in a separate paper [21]. The transition curve in the parameter space (T, B), where observes experimentally the periodicity in the parameter α and also the scaling properties predicted by the RGE (equation 3). The plot in Fig. 4 shows the comparison between the experimental curve in log-log scale together with the theoretical results for the edge

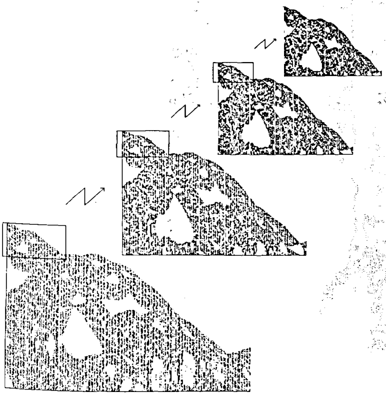

Fig. 3. Four enlargements of the upper left corner of Fig. 2, showing the fractal nature of the spectrum, with the approximate scaling law (7). α is the horizontal variable, ranging from 0 to 2^{-k}, k = 2, 4, 6, 8. E is the vertical variable, ranging from E_0 to 4, E_0 = 2.4, 3.68, 3.936, 3.9872.

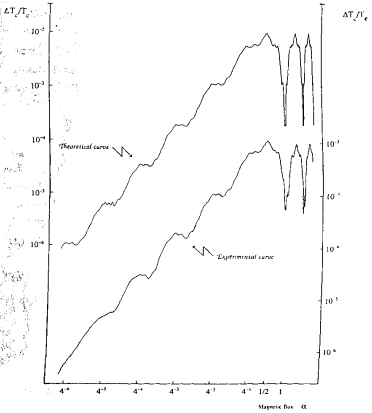

Fig. 4. Comparison between the calculated edge of the spectrum (left scale) with the experimental result (right scale) on the critical temperature of a superconducting gasket: $\Delta T_c/T_c$ vs α in log–log plot, where $\alpha = \Phi/\Phi_0$ is the reduced magnetic flux in the elementary triangle of the gasket: equation 8 has been used to calculate the theoretical curve using the best fit parameters as explained in the text. The two curves have been shifted for clarity.

Renormalization Group Analysis and Quasicrystals

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Table of Content

1.INTRODUCTION

Several quantal systems involving scale invariant properties, have been studied during the last few years by means of a Renormalization Group (RG) method. The most useful type of models is probably the hamiltonian describing the motion of a particle, phonon or

electron, in a quasicrystal. The first quantity to be calculated is the energy spectrum, from which we usually get others like the density of states (DOS), thermodynamical information, like the heat capacity or the magnetic susceptibility, or even various transport coefficients, like the conductivity. Using the spatial macroscopic symmetries, translations and scale invariance, it is possible to get equations satisfied by the model which happen to be sufficient to compute the spectrum in many cases. In particular the scale invariance will produce fractal spectra and scaling laws for the physical quantities.
The main difficulty is that unlike the 1D case for which the calculation can usually be performed by means of the transfer matrix method, the higher dimensional cases are far from being under control yet. In this short paper we want to give an account of a new strategy using operator algebras which should permit to extend the analysis to higher dimension. Eventhough the method is not yet completely developed, it has already given a certain number of convincing results, and we believe it should be the most efficient way of studying these problems. In this paper we compare it with the transfer matrix formulation for 1D chain and we show that both point of view are equivalent. We will only give an insight of what happens for higher dimensional quasicrystals, for this part of the work is still under progress.

2.JACOBI MATRIX OF A JULIA SET

2.1 The Julia Set of a Polynomial

The simplest model was designed in 1982 [Bellissard(82)], to get a new class of hamiltonians with Cantor spectra. It is the Jacobi matrix associated to a Julia set. Let $P(z) = z^N+p_1z^{N-1}+\ldots+p_{N-1}z+p_N$ be a polynomial with real coefficients. We then consider the dynamical system on the complex plane defined by $z_{n+1}=P(z_n)$. Clearly the point at infinity is fixed by P, and it is attractive, for there is $R>0$ big enough, such that whenever $|z|\geq R$, then $|P(z)|\geq R^N/2$. Let ζ be a fixpoint, namely a solution of $P(\zeta)=\zeta$, and let $D(\zeta)$ be the "domain of attraction of ζ", namely the open set of points z_0 such that $z_n\to\zeta$ as $n\to\infty$. The Julia set $J(P)$ of P is the complement of the union of the attraction domain of all fixpoints. Since the point at infinity is always attractive, $J(P)$ is always compact. A famous theorem by Julia and Fatou [Julia(18), Fatou(19), Douady(82)] asserts that $J(P)$ is completely disconnected whenever all critical points of P are attracted by the point at infinity.

3.SIERPINSKY LATTICE IN A MAGNETIC FIELD

3.1 The 2D Sierpinsky Lattice [Alexander(83,84), Rammal(84)]

The Sierpinsky lattice S in 2D is usually constructed according to the fig.1 below. Namely, let e_1, e_2 be two unit vectors making an angle of 60°. Then S is contained in the set Ne_1+Ne_2 . Let then S_k be the subset of points $x \in S$ with $x = me_1+ne_2$ and $0 < m+n \leq 2^k$. S_k is recursively constructed as $S_1 = \{me_1+ne_2\ ;\ 0 \leq m+n \leq 2\}$, $S_{k+1} = S_k \cup \{S_k+2^k e_1\} \cup \{S_k+2^k e_2\}$ for $k \geq 1$, and $S = \cup_{k \geq 1} S_k$.

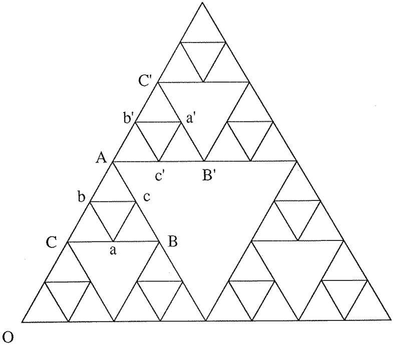

Fig.1- The subset S_3 of the Sierpinsky lattice in 2D -

From this construction it follows that $2S$ is included in S . A site in $2S$ is called "even", the others "odd". Any odd site admits the decomposition $2x+y$ where $x \in S$ and $y \in T = \{e_1, e_2, e_1+e_2\}$. The subsets $T(x) = T+2x$ are called "blocks". If $x \in S$, its nearest neigbours are all points in S within a distance 1 of x .

3.2 The Laplacean on S

The Laplace operators Δ_+ and Δ_- are defined on the Hilbert spaces $l^2(S)$ and $l^2(S\backslash\{0\})$ respectively by:

$$\Delta_+\phi\,(0) = \sqrt{2}\sum_{x';|x'|=1} \phi(x')\ , \qquad \Delta_+\phi\,(x) = \sqrt{2}\psi(0)+\sum_{x'\neq 0;|x'-x|=1} \phi(x')\ ,$$

if $|x| = 1$, and

$$\Delta_+\phi\,(x) = \sum_{x';|x'-x|=1} \phi(x')\ , \qquad \text{if } |x|>1,\ \phi \in l^2(S)\ , \qquad (11a)$$

$$\Delta_-\psi\,(x) = \sum_{x';|x'-x|=1} \psi(x')\ , \qquad x \in S\backslash\{0\}\ ,\ \psi \in l^2(S\backslash\{0\})\ . \qquad (11b)$$

Our goal is to compute the spectrum of Δ . In order to do so we will use the scale invariance of the Sierpinsky lattice. The main result is the following [Rammal(84), Bellissard(85)]

Theorem 3: The spectrum of $\Delta_\pm$ is made of two infinite sequences of eigenvalues of infinite multiplicity accumulating on the Julia set of the polynomial $P(z) = z(z-3)$. The first sequence consists of one isolated eigenvalue in each gap of $J(P)$, whereas the other consists of one edge of each gap of $J(P)$. ◊

Proof: Let us introduce the dilation operator D defined by

$$D\psi\,(x) = \psi(2x) \qquad x \in S, \qquad \psi \in l^2(S) \qquad (12)$$

It is a partial isometry such that $DD^* = 1$. Then we claim that $\Delta_\pm$ are solutions of the following RG equation [Bellissard(85)]:

$$D\{z\mathbf{1}-\Delta\}^{-1}D^* = (z-2)(z+1)/(z+2)\ \{P(z)\mathbf{1}-\Delta\}^{-1}\ , \qquad P(z) = z(z-3) \qquad (13)$$

$$E = \{z^4 - 7z^2 - [2(XU+YV) + 4X]z + 4(1-U)\}/\{S^2 + C^2\}^{1/2},$$

$$\beta + \beta' = 4(\alpha + \alpha') \qquad \beta - \beta' = 2(\alpha - \alpha') - 3/\pi \,\mathrm{Arctan}\,(S/C) \tag{17}$$

with:

$$S = [(XV + YU) + 2Y]z + 2(V + XY)\,, \quad C = z^2 + [(XY - YV) + 2X]z + 2(U - Y^2),$$

$$X = \cos 2\pi\alpha\,, Y = \sin 2\pi\alpha\,, \quad U = \cos 2\pi(\alpha+\alpha') \quad V = \sin 2\pi\,(\alpha+\alpha') \tag{18}$$

Following the intuition provided by the last section, the "dynamical spectrum" is defined as the invariant set of the map $F(z,\alpha,\alpha') = (E,\beta,\beta')$ of $\boldsymbol{RxT^2}$. Since $\beta + \beta' = 4(\alpha + \alpha')$ in (17), only one of the two normalized fluxes is actually relevant, leading to an effective 2D map. *Is the dynamical spectrum equal to the actual spectrum of the original operator ? This is a question with no answer yet.* Nevertheless the numerical calculation of the dynamical spectrum given in fig.2 below [Ghez(87)], shows that it should be.

One should point out there that this calculation has been compared to an experiment performed in Grenoble, on a superconducting network designed according to fig. 1. Landau-Ginzburg's theory [de Gennes(81), Alexander(83)] shows that the transition between the normal metal and the super conducting phases occurs in the (T,B) plane (where T is the temperature) on a curve which is simply related to the edge of the dynamical spectrum as calculated above [Ghez(87)]. The comparison between theory and experiment is actually very accurate as shown in fig.3 below.

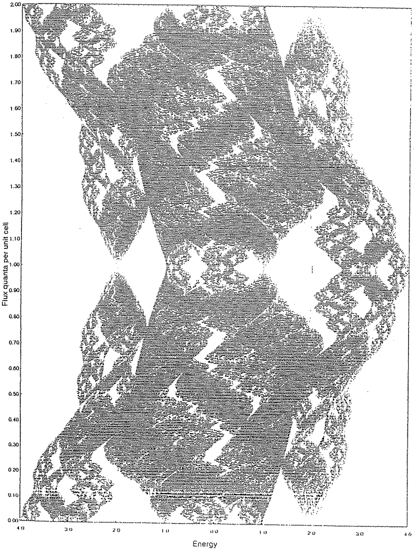

Fig.2- The dynamical spectrum of the Sierpinsky Laplacean in a magnetic field. The magnetic flux is represented on the horizontal axis, whereas the energy is represented on the vertical one. (Picture designed by C.Kreft) -

Following the intuition provided by the last section, the "dynamical spectrum" is defined as the invariant set of the map $F(z,\alpha,\alpha') = (E,\beta,\beta')$ of $\mathbf{R}xT^2$. Since $\beta + \beta' = 4(\alpha + \alpha')$ in (17), only one of the two normalized fluxes is actually relevant, leading to an effective 2D map. *Is the dynamical spectrum equal to the actual spectrum of the original operator ? This is a question with no answer yet.* Nevertheless the numerical calculation of the dynamical spectrum given in fig.2 below [Ghez(87)], shows that it should be.

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Weak Uncertainty Principle (Kasso Okoudjou, Laurent Saloff-Coste, T., 2008)

The $\mathbb{R}^1$ Heisenberg Uncertainty Principle is equivalent, if $\|f\|_{L^2} = 1$, to

$$\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |x-y|^2 |f(x)|^2\,|f(y)|^2\,dx\,dy\right)\cdot\left(\int_{\mathbb{R}} |f'(x)|^2\,dx\right) \geqslant \frac{1}{8}$$

On a metric measure space (K, d, μ) with an energy form $\mathcal{E}$

a weak uncertainty principle

$$\boxed{Var_\gamma(u)\,\mathcal{E}(u,u) \geqslant C} \tag{1}$$

holds for $u \in L^2(K) \bigcap Dom(\mathcal{E})$

$$Var_\gamma(u) = \iint_{K\times K} d(x,y)^\gamma |u(x)|^2\,|u(y)|^2\,d\mu(x)\,d\mu(y). \tag{2}$$

provided either that d is the effective resistance metric, or some of the suitable Poincare inequalities are satisfied.

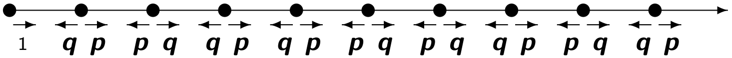
Laplacians on fractals with spectral gaps gaps have nicer Fourier series (Robert Strichartz, 2005)

If the Laplacian has an infinite sequence of exponentially large spectral gaps and the heat kernel satisfies sub-Gaussian estimates, then the partial sums of Fourier series (spectral expansions of the Laplacian) converge **uniformly** along certain special subsequences.

U.Andrews, J.P.Chen, G.Bonik, R.W.Martin, T.,
Wave equation on one-dimensional fractals with spectral decimation.
J. Fourier Anal. Appl. 23 (2017)
`http://teplyaev.math.uconn.edu/fractalwave/`

An introduction given in 2007:
`http://www.math.uconn.edu/~teplyaev/gregynog/AT.pdf`

Half-line example



Transition probabilities in the pq random walk. Here $p \in (0,1)$ and $q = 1 - p$.

$$(\Delta_p f)(x) = \begin{cases} f(0) - f(1), & \text{if } x = 0 \\ f(x) - qf(x-1) - pf(x+1), & \text{if } 3^{-m(x)}x \equiv 1 \pmod 3 \\ f(x) - pf(x-1) - qf(x+1), & \text{if } 3^{-m(x)}x \equiv 2 \pmod 3 \end{cases}$$

Theorem (J.P.Chen, T., 2016)

If $p \neq \frac{1}{2}$, the Laplacian Δ_p on $\ell^2(\mathbb{Z}_+)$ has **purely singularly continuous spectrum**. *The spectrum is the Julia set, a* **topological Cantor set of Lebesgue measure zero**, *of the polynomial* $R(z) = \dfrac{z(z^2 - 3z + (2 + pq))}{pq}$

This is a simple, possibly the simplest, quasi-periodic example related to the recent results of A.Avila, D.Damanik, A.Gorodetski, S.Jitomirskaya, Y.Last, B.Simon et al.

Bohr asymptotics

For 1D Schödinger operator

$$H\psi = -\psi'' + V(x)\psi, \qquad x \geq 0 \tag{3}$$

if $V(x) \to +\infty$ as $x \to +\infty$ then (H. Weyl), the spectrum of H in $L^2([0,\infty), dx)$ is discrete and, under some technical conditions,

$$N(\lambda, V) := \#\{\lambda_i(H) \leq \lambda\} \sim \frac{1}{\pi} \int_0^\infty \sqrt{(\lambda - V(x))_+}\, dx. \tag{4}$$

This is known as the Bohr's formula. It can be generalized for $\mathbb{R}^n$.

Theorem (Fractal Bohr's formula (Joe Chen, Stanislav Molchanov, T., J. Phys. A: Math. Theor. (2015)))

On infinite Sierpinski-type fractafolds, under mild assumptions,

$$\lim_{\lambda\to\infty} \frac{N(V,\lambda)}{g(V,\lambda)} = 1, \tag{5}$$

where

$$g(V,\lambda) := \int_{K_\infty} \left[(\lambda - V(x))_+\right]^{d_s/2} G\left(\frac{1}{2}\log(\lambda - V(x))_+\right) \mu_\infty(dx), \tag{6}$$

where G is the Kigami-Lapidus periodic function, obtained via a renewal theorem.

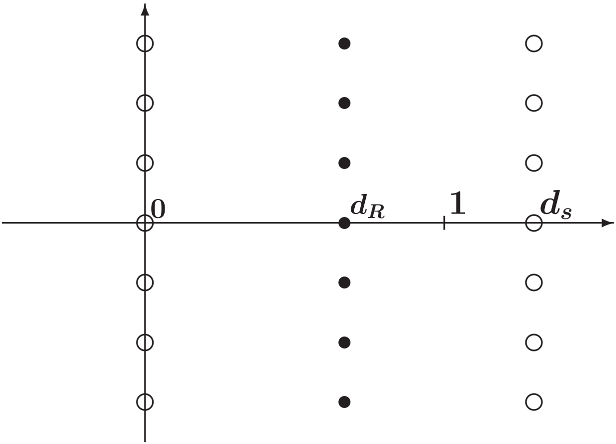
Spectral zeta function

Theorem. (Derfel-Grabner-Vogl, Steinhurst-T., Chen-T.-Tsougkas, Kajino (2007–2017)) For a large class of **finitely ramified symmetric fractals**, which includes the Sierpiński gaskets, and may include the Sierpiński carpets, the spectral zeta function

$$\zeta(s) = \sum \lambda_j^{s/2}$$

has a meromorphic continuation from the half-pain $Re(s) > d_S$ to $\mathbb{C}$. Moreover, all the poles and residues are computable from the geometric data of the fractal. Here λ_j are the eigenvalues if the unique symmetric Laplacian.

- Example: $\zeta(s)$ is the Riemann zeta function up to a trivial factor in the case when our fractal is $[0, 1]$.
- In more complicated situations, such as the Sierpiński gasket, there are infinitely many non-real poles, which can be called complex spectral dimensions, and are related to oscillations in the spectrum.



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

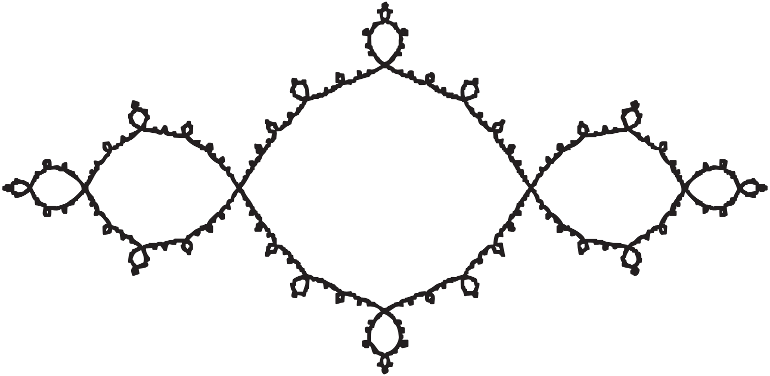
Poles (white circles) of the spectral zeta function of the Sierpiński gasket.

Spectral Analysis of the Basilica Graphs (with Luke Rogers, Toni Brzoska, Courtney George, Samantha Jarvis)

The question of existence of groups with **intermediate growth, i.e. subexponential but not polynomial**, was asked by **Milnor in 1968** and answered in the positive by **Grigorchuk in 1984**. There are still open questions in this area, and a complete picture of which orders of growth are possible, and which are not, is missing.

The Basilica group is a group generated by a finite automation acting on the binary tree in a self-similar fashion, introduced by **R. Grigorchuk and A. Zuk in 2002**, does not belong to the closure of the set of groups of subexponential growth under the operations of group extension and direct limit.

In 2005 L. Bartholdi and B. Virag further showed it to be amenable, making the Basilica group **the 1st example of an amenable but not subexponentially amenable** group (also **"Münchhausen trick" and amenability of self-similar groups by V.A. Kaimanovich**).



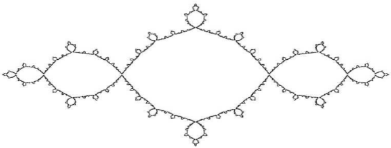
The basilica Julia set, the Julia set of $z^2 - 1$ and the limit set of the basilica group of exponential growth (Grigorchuk, Żuk, Bartholdi, Virág, Nekrashevych, Kaimanovich, Nagnibeda et al.).

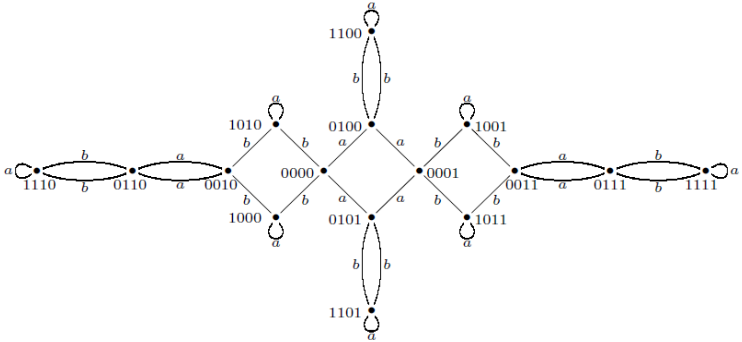
In 2005, V. Nekrashevych described the Basilica as the iterated monodromy group, and there exists a natural way to associate it to the Basilica fractal (**Nekrashevych+T., 2008**).

In **Schreier graphs of the Basilica group (2010), Nagnibeda et al.** classified up to isomorphism all possible limits of finite Schreier graphs of the Basilica group.

In **Laplacians on the Basilica Julia set (2010), L. Rogers+T.** constructed Dirichlet forms and the corresponding Laplacians on the Basilica fractal in two different ways: by imposing a self-similar harmonic structure and a graph-directed self-simliar structure on the fractal.

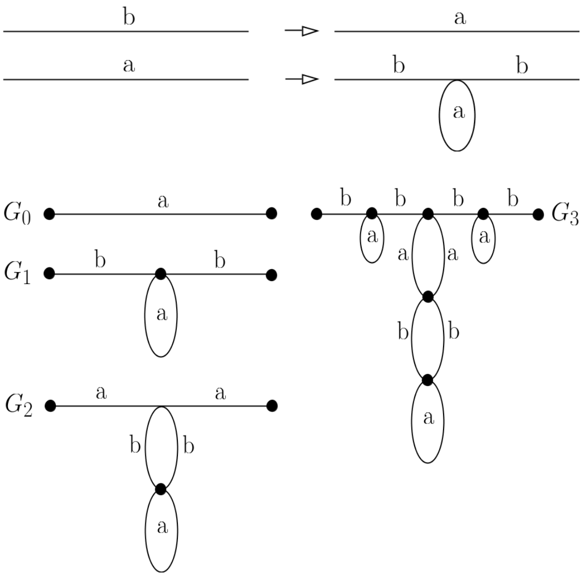
In **2012-2015, Dong, Flock, Molitor, Ott, Spicer, Totari and Strichartz** provided numerical techniques to approximate eigenvalues and eigenfunctions on families of Laplacians on the Julia sets of $z^2 + c$.



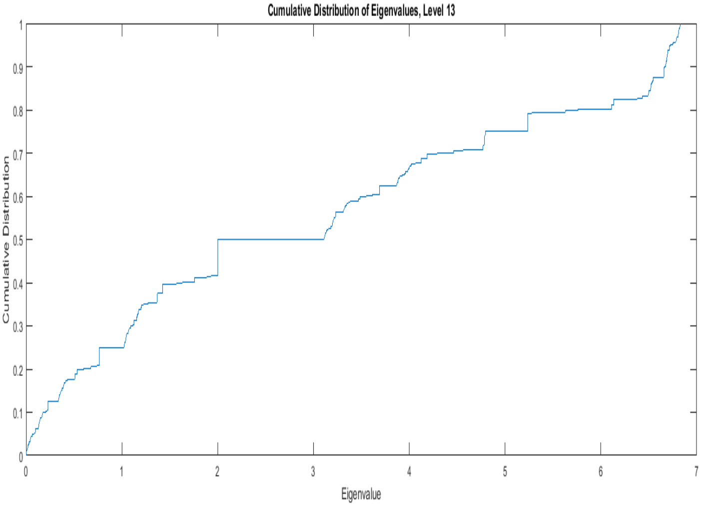


pictures taken from paper by Nagnibeda et. al.

Replacement Rule and the Graphs G_n



Distribution of Eigenvalues, Level 13



One can define a Dirichlet to Neumann map for the two boundary points of the graphs G_n. One can construct a dynamical system to determine these maps (which are two by two matrices). The dynamical system allows us to prove the following.

Theorem

In the Hausdorff metric, $\limsup_{n\to\infty} \sigma(L^{(n)})$ has a gap that contains the interval $(2.5, 2.8)$.

Theorem (arXiv:1908.10505)

In the Hausdorff metric, $\limsup_{n\to\infty} \sigma(L^{(n)})$ has infinitely many gaps.

Infinite Blow-ups of G_n

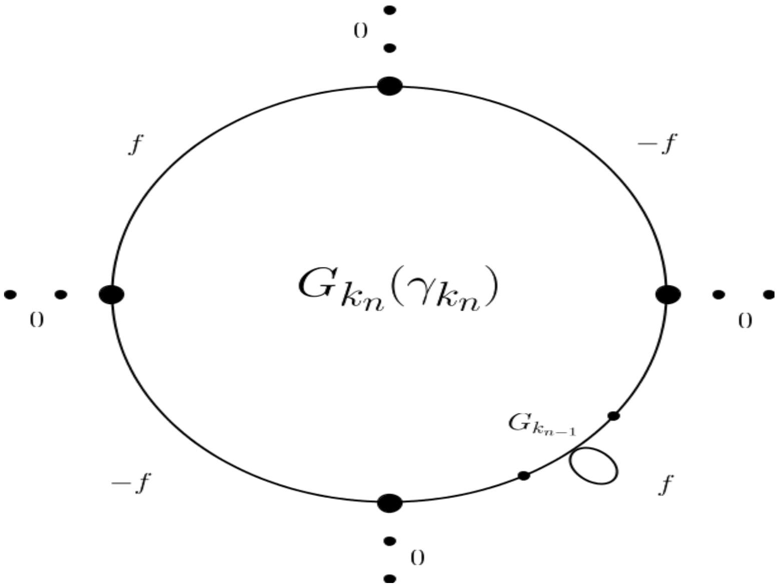
Definition

Let $\{k_n\}_{n\in\mathbb{N}}$ be a strictly increasing subsequence of the natural numbers. For each n, embed G_{k_n} in some isomorphic subgraph of $G_{k_{n+1}}$. The corresponding infinite blow-up is $G_\infty := \cup_{n\geq 0} G_{k_n}$.

Assumption

The infinite blow-up G_∞ satisfies:

- For $n \geq 1$, the long path of $G_{k_{n-1}}$ is embedded in a loop γ_n of G_{k_n}.
- Apart from $l_{k_{n-1}}$ and $r_{k_{n-1}}$, no vertex of the long path can be the $3, 6, 9$ or 12 o'clock vertex of γ_n.
- The only vertices of G_{k_n} that connect to vertices outside the graph are the boundary vertices of G_{k_n}.


0
f
$-f$
$G_{k_n}(\gamma_{k_n})$
0
0
$G_{k_{n-1}}$
$-f$
f
0

Theorem

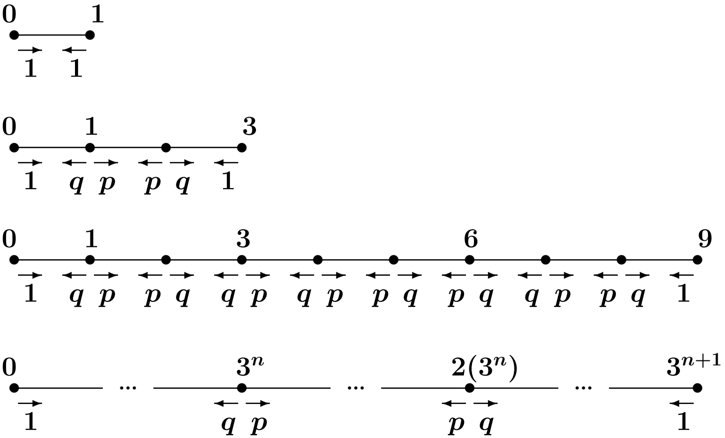
(1) $\sigma(L^{(k_n)}|_{\ell^2_{a,k_n,\gamma_n}}) = \sigma(L_0^{(j_n)})$.
(2) The spectrum of $L^{(\infty)}$ is pure point. The set of eigenvalues of $L^{(\infty)}$ is

$$\bigcup_{n\geq 0} \sigma(L_0^{(j_n)}) = \bigcup_{n\geq 0} c_{j_n}^{-1}\{0\},$$

where the polynomials c_n are the characteristic polynomials of $L_0^{(n)}$, as defined in the previous proposition.
(3) Moreover, the set of eigenfunctions of $L^{(\infty)}$ with finite support is complete in ℓ^2.

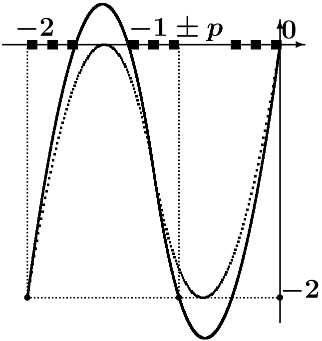
TECHNICAL DETAILS

Fix $p, q > 0$, $p + q = 1$, and define probabilistic Laplacians Δ_n on the segments $[0, 3^n]$ of $\mathbb{Z}_+$ inductively as a generator of the random walks:



and let $\Delta = \lim_{n \to \infty} \Delta_n$ be the corresponding probabilistic Laplacian on $\mathbb{Z}_+$.

If $z \neq -1 \pm p$ and $R(z) = z(z^2 + 3z + 2 + pq)/pq$, then
$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$



Theorem (Joe P. Chen and T., JMP 2016). $\sigma(\Delta) = \mathcal{J}_R$, the Julia set of $R(z)$.

If $p = q$, then $\sigma(\Delta) = [-2, 0]$, spectrum is a.c.

If $p \neq q$, then $\sigma(\Delta)$ is a Cantor set of Lebesgue measure zero, spectrum is singularly continuous.

U.Andrews, J.P.Chen, G.Bonik, R.W.Martin, A.Teplyaev, **Wave equation on one-dimensional fractals with spectral decimation**. J. Fourier Anal. Appl. 23 (2017) 994–1027. http://teplyaev.math.uconn.edu/fractalwave/

Bellissard, Geronimo, Volberg, Yuditskii, **Are they limit periodic?** Complex analysis and dynamical systems II, 43–53, Contemp. Math., 382, Israel Math. Conf. Proc., Amer. Math. Soc., Providence, RI, 2005.
(Reviewed by Maxim S. Derevyagin)

Bellissard, **Renormalization group analysis and quasicrystals**. Ideas and methods in quantum and statistical physics (Oslo, 1988), 118–148, Cambridge Univ. Press, Cambridge, 1992.

Barnsley, Geronimo and Harrington, **Almost periodic Jacobi matrices associated with Julia sets for polynomials**. Comm. Math. Phys. 99 (1985), 303–317.

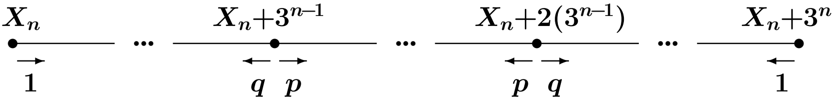
Bellissard, Bessis and Moussa, **Chaotic states of almost periodic Schrodinger operators**. Phys. Rev. Lett. 49 (1982), 701–704.

Bellissard and Simon, **Cantor spectrum for the almost Mathieu equation** J. Funct. Anal. 48 (1982), 408–419.

There are uncountably many "random" self-similar Laplacians $\boldsymbol{\Delta}$ on $\mathbb{Z}$:
For a sequence $\mathcal{K} = \{k_j\}_{j=1}^{\infty}$, $k_j \in \{0,1,2\}$, let

$$X_n = -\sum_{j=1}^{n} k_j 3^j$$

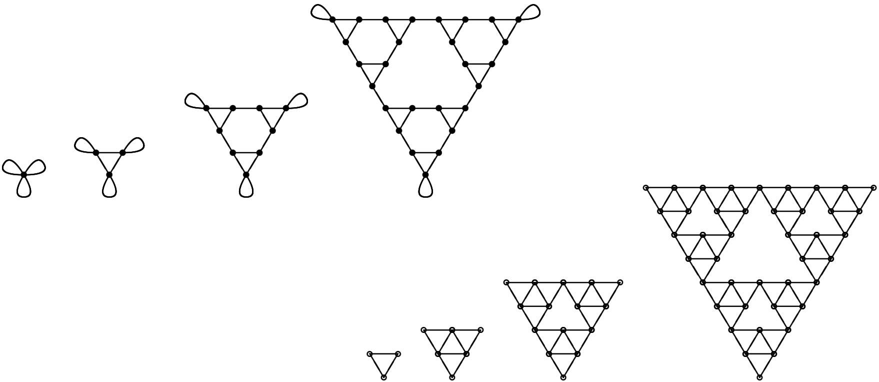
and $\boldsymbol{\Delta}_n$ is a probabilistic Laplacian on $[X_n, X_n+3^n]$:

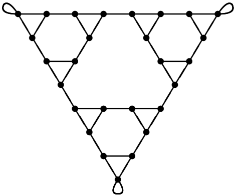


In the previous example $k_j = 0$ for all j.

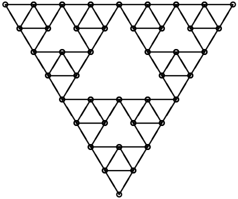
Theorem.
For any sequence $\mathcal{K}$ we have $\sigma(\boldsymbol{\Delta}) = \mathcal{J}_R$. The same is true for the Dirichlet Laplacian on $\mathbb{Z}_+$ (when $k_j \equiv 0$).

R. Grigorchuk and Z. Sunik, *Asymptotic aspects of Schreier graphs and Hanoi Towers groups.*





Sierpiński 3-graph
(Hanoi Towers-3 group)



Sierpiński 4-graph
(standard)

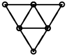
These three polynomials are conjugate:

Sierpiński 3-graph (Hanoi Towers-3 group): $f(x) = x^2 - x - 3$
$f(3) = 3$, $f'(3) = 5$

Sierpiński 4-graph, "adjacency matrix" Laplacian: $P(\lambda) = 5\lambda - \lambda^2$
$P(0) = 0$, $P'(0) = 5$

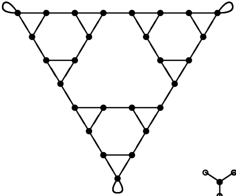
Sierpiński 4-graph, probabilistic Laplacian: $R(z) = 4z^2 + 5z$
$R(0) = 0$, $R'(0) = 5$

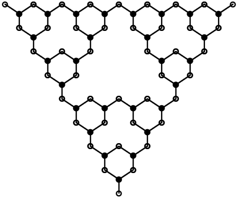
Theorem. Eigenvalues and eigenfunctions on the Sierpiński 3-graphs and Sierpiński 4-graphs are in one-to-one correspondence, with the exception of the eigenvalue $z = -\frac{3}{2}$ for the 4-graphs.

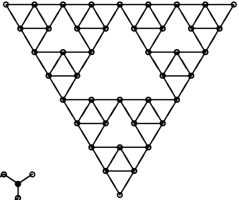
 $\xleftarrow[4z^2+5z]{}$ 

 $\xleftarrow[\frac{4}{3}z^2+\frac{8}{3}z]{}$ 

 $\xleftarrow[2z^2+4z]{}$ 







Sierpiński 3-graph
(Hanoi Towers-3 group)
$\boldsymbol{R(z) = 2z^2 + 4z}$

Sierpiński 4-graph
(standard)
$\boldsymbol{R(z) = \frac{4}{3}z^2 + \frac{8}{3}z}$

Let $\mathcal{H}$ and $\mathcal{H}_0$ be Hilbert spaces, and $U : \mathcal{H}_0 \to \mathcal{H}$ be an isometry.

Definition. We call an operator H **spectrally similar** to an operator H_0 with functions φ_0 and φ_1 if

$$U^*(H - z)^{-1}U = (\varphi_0(z)H_0 - \varphi_1(z))^{-1}$$

In particular, if $\varphi_0(z) \neq 0$ and $R(z) = \varphi_1(z)/\varphi_0(z)$, then

$$U^*(H - z)^{-1}U = \frac{1}{\varphi_0(z)}(H - R(z))^{-1}.$$

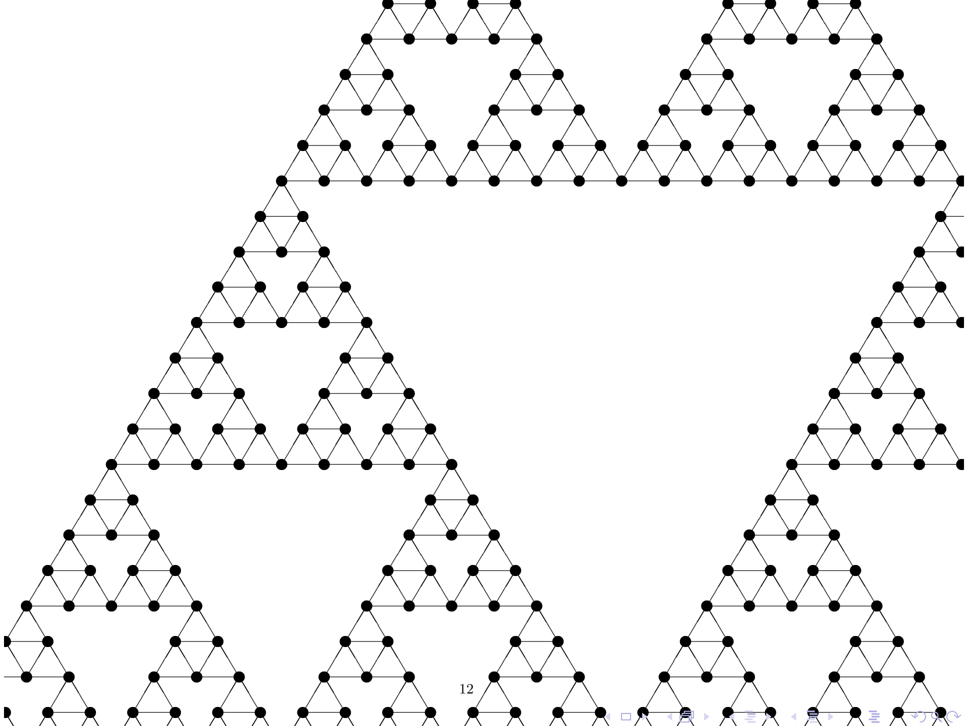
If $H = \begin{pmatrix} S & \bar{X} \\ X & Q \end{pmatrix}$ then

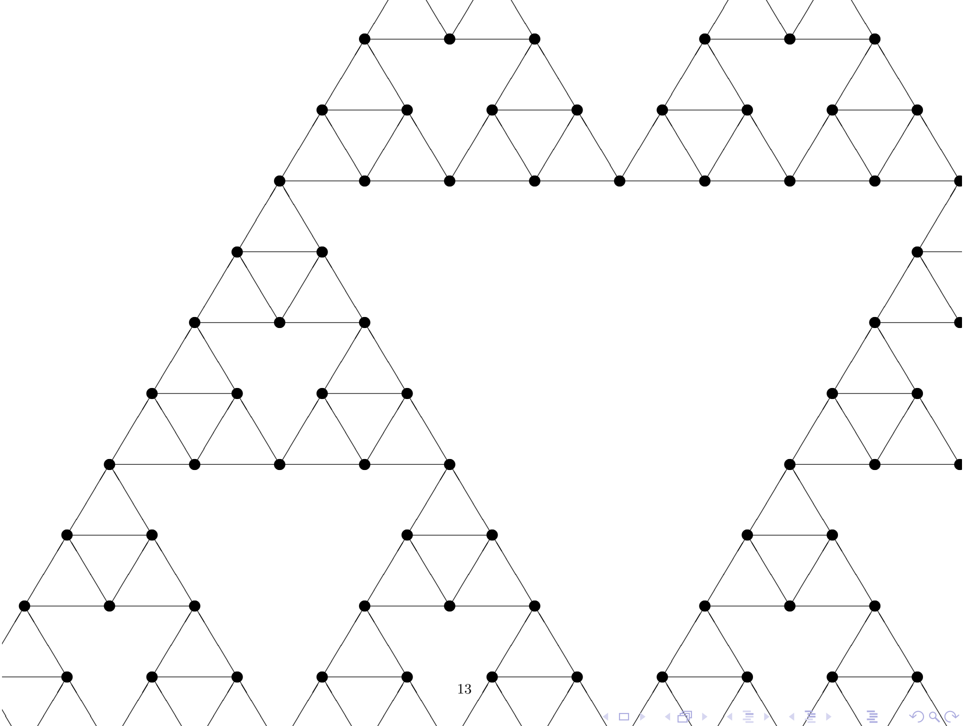
$$S - zI_0 - \bar{X}(Q - zI_1)^{-1}X = \varphi_0(z)H_0 - \varphi_1(z)I_0$$

Theorem (Malozemov and T.). If Δ is the graph Laplacian on a self-similar symmetric infinite graph, then

$$\mathcal{J}_R \subseteq \sigma(\Delta_\infty) \subseteq \mathcal{J}_R \cup \mathcal{D}_\infty$$

where $\mathcal{D}_\infty$ is a discrete set and $\mathcal{J}_R$ is the Julia set of the rational function R.





R. Rammal and G. Toulouse, *Random walks on fractal structures and percolation clusters.* J. Physique Letters **44** (1983), L13–L22.

R. Rammal, *Spectrum of harmonic excitations on fractals.* J. Physique **45** (1984).

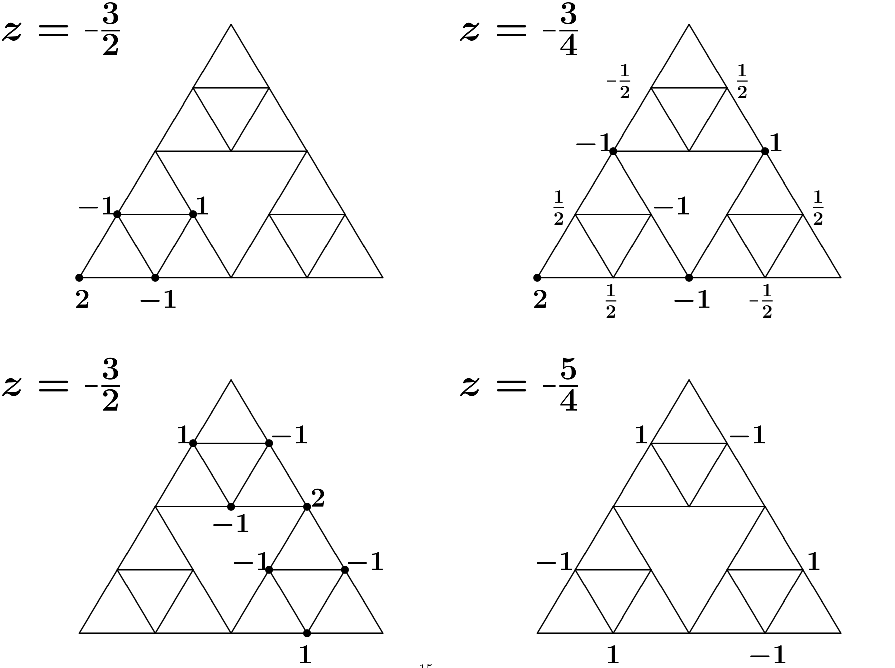
S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field.* Phys. Rev. B **29** (1984).

E. Domany, S. Alexander, D. Bensimon and L. Kadanoff, *Solutions to the Schrödinger equation on some fractal lattices.* Phys. Rev. B (3) **28** (1984).

Y. Gefen, A. Aharony and B. B. Mandelbrot, *Phase transitions on fractals. I. Quasilinear lattices. II. Sierpiński gaskets. III. Infinitely ramified lattices.* J. Phys. A **16** (1983–1984).

R. B. Stinchcombe, *Fractals, phase transitions and criticality.* Fractals in the natural sciences. Proc. Roy. Soc. London Ser. A **423** (1989), 17–33.

J. Béllissard, *Renormalization group analysis and quasicrystals,* Ideas and methods in quantum and statistical physics (Oslo, 1988). Cambridge Univ. Press, 1992.

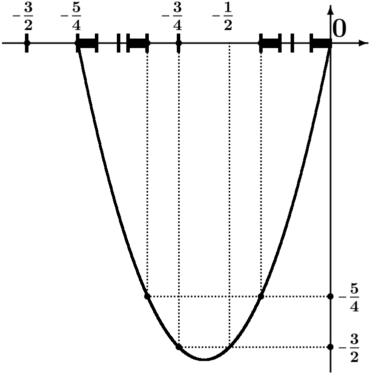

z = -3/2
−1
1
2
−1
z = -3/4
-1/2
1/2
−1
1
1/2
−1
1/2
2
1/2
−1
-1/2
z = -3/2
1
−1
2
−1
−1
−1
1
z = -5/4
1
−1
−1
1
1
−1

Let $\boldsymbol{\Delta}$ be the probabilistic Laplacian (generator of a simple random walk) on the **Sierpiński lattice**. If $z \neq -\frac{3}{2}, -\frac{5}{4}, -\frac{1}{2}$, and $R(z) = z(4z+5)$, then

$$R(z) \in \sigma(\Delta) \Longleftrightarrow z \in \sigma(\Delta)$$

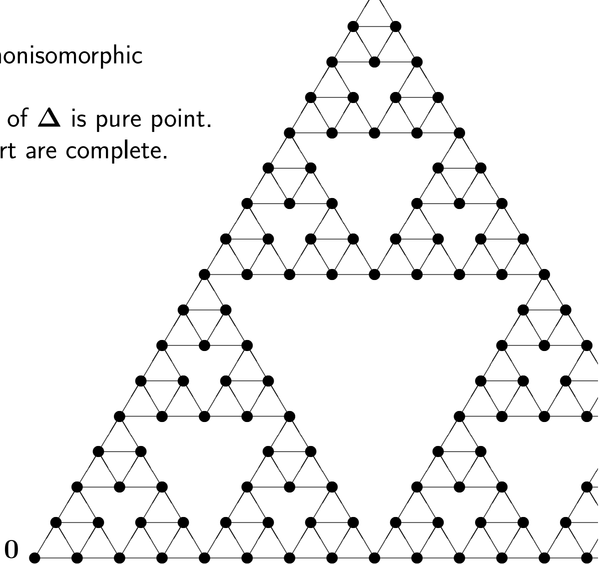
$$\sigma(\Delta) = \mathfrak{J}_R \bigcup \mathfrak{D}$$

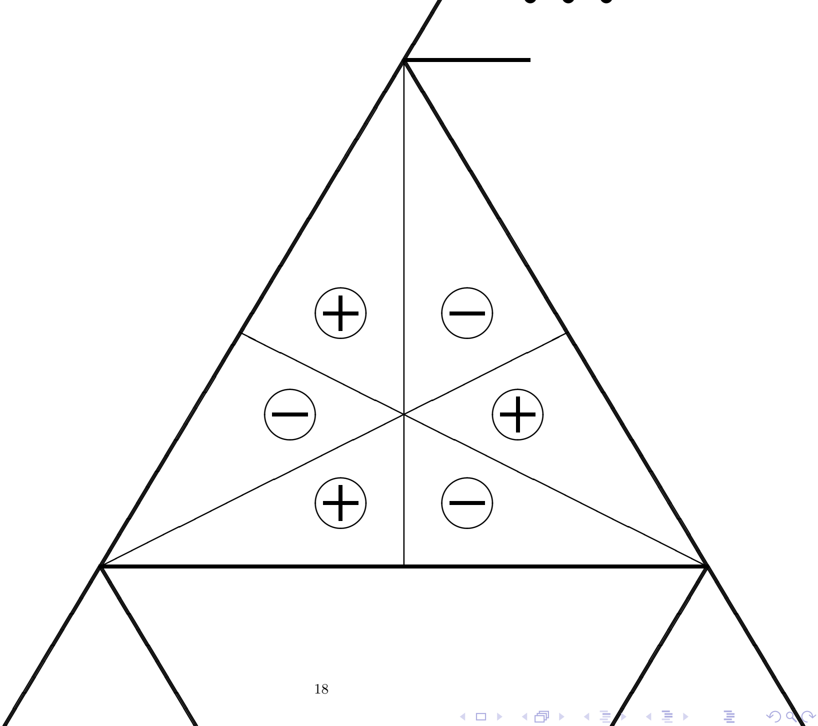
where $\mathfrak{D} \overset{\text{def}}{=} \{-\frac{3}{2}\} \bigcup \left(\bigcup_{m=0}^{\infty} R^{-m}\{-\frac{3}{4}\} \right)$
and $\mathfrak{J}_R$ is the Julia set of $R(z)$.



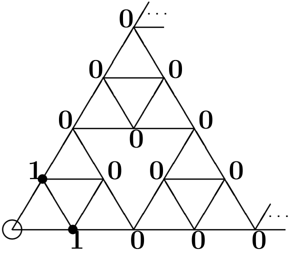
There are uncountably many nonisomorphic Sierpiński lattices.

Theorem (T). The spectrum of $\mathbf{\Delta}$ is pure point. Eigenfunctions with finite support are complete.





Let $\Delta^{(0)}$ be the Laplacian with zero (Dirichlet) boundary condition at ∂L. Then the compactly supported eigenfunctions of $\Delta^{(0)}$ are **not** complete (eigenvalues in $\mathcal{E}$ is not the whole spectrum).



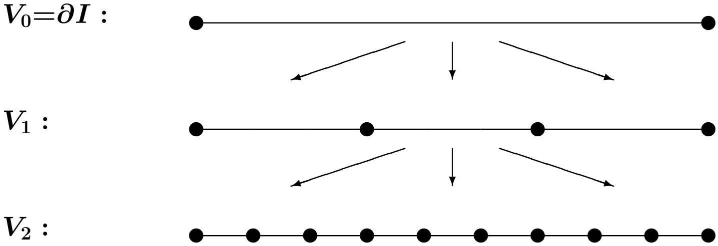
Let $\partial L^{(0)}$ be the set of two points adjacent to ∂L and $\omega_\Delta^{(0)}$ be the spectral measure of $\Delta^{(0)}$ associated with $\mathbb{1}_{\partial L^{(0)}}$. Then $\operatorname{supp}(\omega_\Delta^{(0)}) = \mathcal{J}_R$ has Lebesgue measure zero and

$$\frac{d(\omega_\Delta^{(0)} \circ R_{1,2})}{d\omega_\Delta^{(0)}}(z) = \frac{(8z+5)(2z+3)}{(2z+1)(4z+5)}$$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^1 \to \mathbb{R}^1$, $F_j(x) = \frac{1}{3}(x+p_j)$, with fixed points $p_j = 0, \frac{1}{2}, 1$. The interval $I=[0,1]$ is a unique compact set such that

$$I = \bigcup_{j=1,2,3} F_j(I)$$

The *boundary* of I is $\partial I = V_0 = \{0,1\}$ and the *discrete approximations* to I are $V_n = \bigcup_{j=1,2,3} F_j(V_{n-1}) = \left\{\frac{k}{3^n}\right\}_{k=0}^{3^n}$



Definition. The *discrete Dirichlet (energy) form* on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y\in V_n\\ y\sim x}} (f(y)-f(x))^2$$

and the *Dirichlet (energy) form* on I is $\mathcal{E}(f) = \lim_{n\to\infty} 3^n\mathcal{E}_n(f) = \int_0^1 |f'(x)|^2 dx$

Definition. A function h is *harmonic* if it minimizes the energy given the boundary values.

Proposition. $3\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$ and $3\mathcal{E}_{n+1}(h) = \mathcal{E}_n(h) = 3^{-n}\mathcal{E}(h)$ for a harmonic h.

Proposition. The Dirichlet (energy) form on I is *self-similar* in the sense that

$$\mathcal{E}(f) = 3\sum_{j=1,2,3} \mathcal{E}(f\circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{2} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on I is $\Delta f(x) = \lim_{n \to \infty} 9^n \Delta_n f(x) = f''(x)$

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_0^1 f \Delta f dx + f f' \Big|_0^1$$

Spectral asymptotics: Let $\rho(\lambda)$ be the *eigenvalue counting function* of the Dirichlet or Neumann Laplacian Δ:

$$\rho(\lambda) = \#\{j : \lambda_j < \lambda\}.$$

Then

$$\lim_{\lambda \to \infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} = \frac{1}{\pi}$$

where $d_s = 1$ is the spectral dimension.

Definition. The *spectral zeta function* is $\zeta_\Delta(s) = \sum_{\lambda_j \neq 0} (-\lambda_j)^{-s/2}$
Its poles are the *complex spectral dimensions.*

Let $R(z)$ be a polynomial of degree N such that its Julia set $\mathfrak{J}_R \subset (-\infty, 0]$, $R(0) = 0$ and $c = R'(0) > 1$.

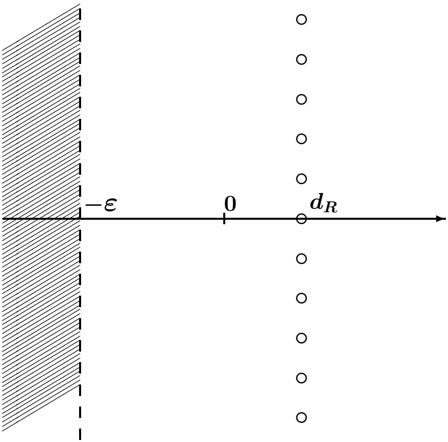
Definition. The *zeta function of* $R(z)$ for $\mathrm{Re}(s) > d_R = \frac{2\log N}{\log c}$ is

$$\zeta_R^{z_0}(s) = \lim_{n\to\infty} \sum_{z \in R^{-n}\{z_0\}} (-c^n z)^{-s/2} = \sum \lambda_j^{-s/2}$$

Theorem. $\zeta_R^{z_0}(s) = \dfrac{f_1(s)}{1 - Nc^{-s/2}} + f_2^{z_0}(s)$, where $f_1(s)$ and $f_2^{z_0}(s)$ are analytic for $\mathrm{Re}(s)>0$. If $\mathfrak{J}_R$ is totally disconnected, then this meromorphic continuation extends to $\mathrm{Re}(s)>-\varepsilon$, where $\varepsilon>0$.

In the case of polynomials this theorem has been improved by Grabner et al.

$$d_R \in \text{the poles of } \zeta_R^{z_0} \subseteq \left\{\frac{2\log N + 4in\pi}{\log c} : n \in \mathbb{Z}\right\}$$



Theorem. $\zeta_\Delta(s) = \zeta^0_R(s)$ where $R(z) = z(4z^2+12z+9)$.

The Riemann zeta function $\zeta(s)$ satisfies $\zeta(s) = \pi^s \zeta^0_R(s)$ The only complex spectral dimension is the pole at $s = 1$.

A sketch of the proof: If $z \neq -\frac{1}{2}, -\frac{3}{2}$, then

$$R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$$

and so $\zeta_\Delta(s) = \zeta^0_R(s)$ since the eigenvalues λ_j of Δ are limits of the eigenvalues of $9^n \Delta_n$.

Also $\lambda_j = -\pi^2 j^2$ and so

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \left(\pi^2 j^2\right)^{-s/2} = \pi^{-s}\zeta(s)$$

where $\zeta(s)$ is the Riemann zeta function. *Q.E.D.*

$$\zeta(s) = \pi^s \lim_{n\to\infty} \sum_{\substack{z \in R^{-n}\{0\} \\ z \neq 0}} \left(-9^n z\right)^{-s/2}$$

Definition. Δ_μ is *μ–Laplacian* if

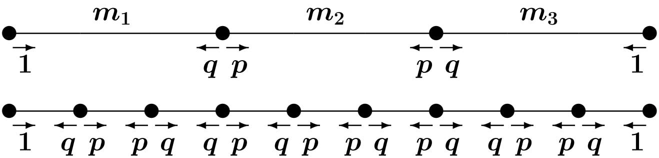
$$\mathcal{E}(f) = \int_0^1 |f'(x)|^2 dx = -\int_0^1 f \Delta_\mu f d\mu + f f' \Big|_0^1.$$

Definition. A probability measure μ is *self-similar* with weights m_1, m_2, m_3 if $\mu = \sum\limits_{j=1,2,3} m_j \mu \circ F_j$.

Proposition. $\Delta_\mu f(x) = \frac{f''}{\mu} = \lim\limits_{n\to\infty} \left(1+\frac{2}{pq}\right)^n \Delta_n f(x)$.

$$\Delta_n f\left(\tfrac{k}{3^n}\right) = \begin{cases} p f\left(\frac{k-1}{3^n}\right) + q f\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \\ q f\left(\frac{k-1}{3^n}\right) + p f\left(\frac{k+1}{3^n}\right) - f\left(\frac{k}{3^n}\right) \end{cases}$$

where $m_1 = m_3$, $p = \frac{m_2}{m_1+m_2}$, $q = \frac{m_1}{m_1+m_2}$, and



Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} \leqslant \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$d_s = \frac{\log 9}{\log(1+\frac{2}{pq})} \leqslant 1.$$

All the inequalities are strict if and only if $p \neq q$.

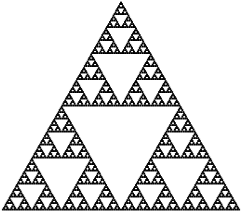
Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$

where $z \neq -1 \pm p$ and $R(z) = z(z^2+3z+2+pq)/pq$.

Note that $R'(0) = 1 + \frac{2}{pq}$, and $d_s = d_R$.

Theorem. $\zeta_{\Delta_\mu}(s) = \zeta_R^0(s)$

Three **contractions** $F_1, F_2, F_3 : \mathbb{R}^2 \to \mathbb{R}^2$,
$F_j(x) = \frac{1}{2}(x+p_j)$, with fixed points p_1, p_2, p_3.



The **Sierpiński gasket** is a unique compact set S such that

$$S = \bigcup_{j=1,2,3} F_j(S)$$

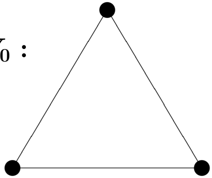
Definition. The *boundary* of S is

$$\partial S = V_0 = \{p_1, p_2, p_3\}$$

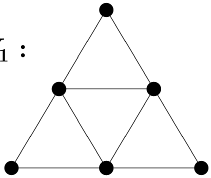
and *discrete approximations* to S are

$$V_n = \bigcup_{j=1,2,3} F_j(V_{n-1})$$

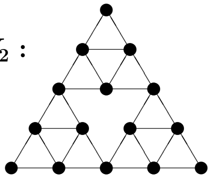
V_0 :



V_1 :



V_2 :



Definition. The *discrete Dirichlet (energy) form* on V_n is

$$\mathcal{E}_n(f) = \sum_{\substack{x,y\in V_n \\ y\sim x}} (f(y)-f(x))^2$$

and the *Dirichlet (energy) form* on S is

$$\mathcal{E}(f) = \lim_{n\to\infty} \left(\tfrac{5}{3}\right)^n \mathcal{E}_n(f)$$

Definition. A function h is *harmonic* if it minimizes the energy given the boundary values.

Proposition. $\frac{5}{3}\mathcal{E}_{n+1}(f) \geqslant \mathcal{E}_n(f)$
$\frac{5}{3}\mathcal{E}_{n+1}(h)=\mathcal{E}_n(h)=\left(\frac{5}{3}\right)^{-n}\mathcal{E}(h)$ for a harmonic h.

Theorem (Kigami). $\mathcal{E}$ is a local regular Dirichlet form on S which is self-similar in the sense that

$$\mathcal{E}(f) = \frac{5}{3} \sum_{j=1,2,3} \mathcal{E}(f\circ F_j)$$

Definition. The *discrete Laplacians* on V_n are

$$\Delta_n f(x) = \frac{1}{4} \sum_{\substack{y \in V_n \\ y \sim x}} f(y) - f(x), \quad x \in V_n \setminus V_0$$

and the Laplacian on S is

$$\Delta_\mu f(x) = \lim_{n \to \infty} 5^n \Delta_n f(x)$$

if this limit exists and $\Delta_\mu f$ is continuous.

Gauss–Green (integration by parts) formula:

$$\mathcal{E}(f) = -\int_S f \Delta_\mu f d\mu + \sum_{p \in \partial S} f(p) \partial_n f(p)$$

where μ is the normalized Hausdorff measure, which is self-similar with weights $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$:

$$\mu = \frac{1}{3} \sum_{j=1,2,3} \mu \circ F_j.$$

Spectral asymptotics: If $\rho(\lambda)$ is the eigenvalue counting function of the Dirichlet or Neumann Laplacian Δ_μ, then

$$0 < \liminf_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \limsup_{\lambda\to\infty} \frac{\rho(\lambda)}{\lambda^{d_s/2}} < \infty$$

where the spectral dimension is

$$1 < d_s = \tfrac{\log 9}{\log 5} < 2.$$

Proposition. $R(z) \in \sigma(\Delta_n) \iff z \in \sigma(\Delta_{n+1})$ where $z \neq -\frac{1}{2}, -\frac{3}{4}, -\frac{5}{4}$ and $R(z) = z(5+4z)$.

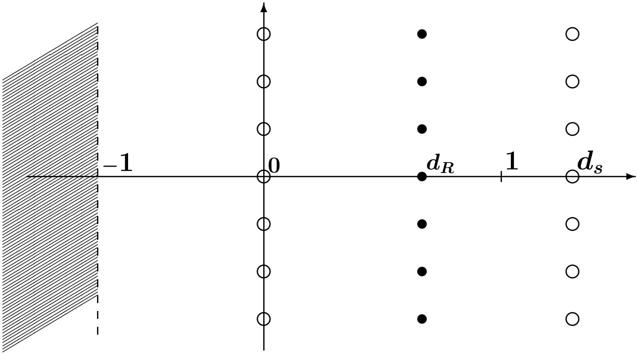
Theorem (Fukushima, Shima). Every eigenvalue of Δ_μ has a form

$$\lambda = 5^m \lim_{n\to\infty} 5^n R^{-n}(z_0)$$

where $R^{-n}(z_0)$ is a preimage of $z_0 = -\frac{3}{4}, -\frac{5}{4}$ under the n-th iteration power of the polynomial $R(z)$. The multiplicity of such an eigenvalue is $C_1 3^m + C_2$.

Theorem. Zeta function of the Laplacian on the Sierpiński gasket is

$$\zeta_{\Delta_\mu}(s) \;=\; \tfrac{1}{2}\,\zeta_R^{-\frac{3}{4}}(s)\left(\frac{1}{5^{s/2}-3}+\frac{3}{5^{s/2}-1}\right) \;+\; \tfrac{1}{2}\,\zeta_R^{-\frac{5}{4}}(s)\left(\frac{3\cdot 5^{-s/2}}{5^{s/2}-3}-\frac{5^{-s/2}}{5^{s/2}-1}\right)$$



$$d_s = \frac{\log 9}{\log 5}$$

$$d_R = \frac{\log 4}{\log 5}$$

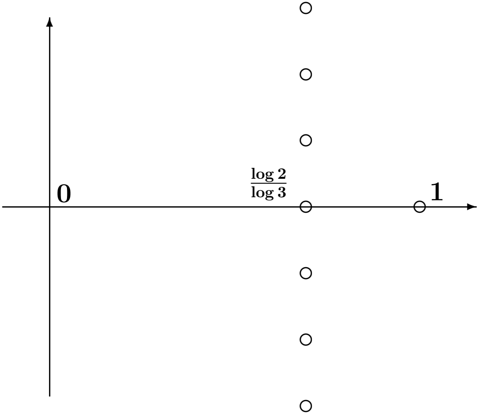
Definition. If $\mathcal{L}$ is a fractal string, that is, a disjoint collection of intervals of lengths l_j, then its *geometric zeta function* is $\zeta_{\mathcal{L}}(s) = \sum l_j^s$.

Theorem (Lapidus). If $A{=}-\frac{d^2}{dx^2}$ is a Neumann or Dirichlet Laplacian on $\mathcal{L}$, then $\zeta_A(s) = \pi^{-s}\zeta(s)\zeta_{\mathcal{L}}(s)$.

<u>**Example: Cantor self-similar fractal string.**</u>

If $\mathcal{L}$ is the complement of the middle third Cantor set in $[0,1]$, then the complex spectral dimensions are 1 and $\{\frac{\log 2+2in\pi}{\log 3}\colon n{\in}\mathbb{Z}\}$,

$$\zeta_{\mathcal{L}}(s) = \frac{1}{1-2\cdot 3^{-s}}, \quad \zeta_A(s) = \zeta(s)\frac{\pi^{-s}}{1-2\cdot 3^{-s}}$$


0
log 2
log 3
1

Definition. A post critically finite (p.c.f.) self-similar set F is a compact connected metric space with a finite boundary $\partial F \subset F$ and contractive injections $\psi_i : F \to F$ such that

$$F = \Psi(F) = \bigcup_{i=1}^{k} \psi_i(F)$$

and

$$\psi_v(F) \bigcap \psi_w(F) \subseteq \psi_v(\partial F) \bigcap \psi_w(\partial F),$$

for any two different words v and w of the same length. Here for a finite word $w \in \{1, \ldots, k\}^m$ we define $\psi_w = \psi_{w_1} \circ \ldots \circ \psi_{w_m}$.

We assume that ∂F is a minimal such subset of F. We call $\psi_w(F)$ an m-cell. *The p.c.f. assumption is that every boundary point is contained in a single 1-cell.*

Theorem (Kigami, Lapidus). The spectral dimension of the Laplacian Δ_μ is the unique solution of the equation

$$\sum_{i=1}^{k} (r_i \mu_i)^{d_s/2} = 1$$

Conjecture. On every p.c.f. fractal F there exists a local regular Dirichlet form $\mathcal{E}$ which gives positive capacity to the boundary points and is self-similar in the sense that

$$\mathcal{E}(f) = \sum_{i=1}^{k} \rho_i \mathcal{E}(f \circ \psi_i)$$

for a set of positive refinement weights $\rho = \{\rho_i\}_{i=1}^{k}$.

Definition. The group G of acts on a finitely ramified fractal F if each $g \in G$ is a homeomorphism of F such that $g(V_n) = V_n$ for all $n \geqslant 0$.

Proposition. Suppose a group G of acts on a self-similar finitely ramified fractal F and G restricted to V_0 is the whole permutation group of V_0. Then there exists a unique, up to a constant, G-invariant self-similar resistance form $\mathcal{E}$ with equal energy renormalization weights ρ_i and

$$\mathcal{E}_0(f,f) = \sum_{x,y \in V_0} \big(f(x) - f(y)\big)^2.$$

Moreover, for any G-invariant self-similar measure μ the Laplacian Δ_μ has the spectral self-similarity property (a.k.a. spectral decimation).

end of the talk :-)

Thank you!